# Closed Braids and Torsion in Khovanov Homology

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## INTRODUCTION

In 1984, knot theory was revolutionized with the discovery of the Jones polynomial [Jon]. Fifteen years later, with several questions about it still unanswered, the polynomial was categorified into what is presently known as Khovanov homology (KH) [Kho]. The idea is as follows. A bigraded chain complex is associated to a link whose homology is an invariant of the link itself. Additionally, the Euler characteristic of this chain complex, when interpreted appropriately, is the Jones polynomial.

KH is a powerful link invariant. In particular, it detects the unknot, which, for the Jones polynomial is still an open question [KM]. Further, KH has been used to prove the Milnor unknotting conjecture combinatorially among other applications [Ras].

A wealth of information, in the form of torsion subgroups, is obtained from the KH of a link which have no contribution towards the Euler characteristic of the chain complex and hence cannot be obtained from just the Jones polynomial in general. $\mathbb{Z}_2$ torsion in KH is very common. However, trying to find examples of knots and prime links with non-$\mathbb{Z}_2$ torsion in their KH is like trying to find a needle in a haystack. Up until recently, only a finite number of such examples was known.

After introducing infinite families of knots and links with $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_7$, and $\mathbb{Z}_8$ torsion in their KH, we introduce the first known examples of knots and links having $\mathbb{Z}_9$, $\mathbb{Z}_{25}$, and $\mathbb{Z}_{27}$ torsion in their KH [MPSWY]. Lastly, we resolve a part of the Przytycki-Sazdanović braid conjecture [PS].

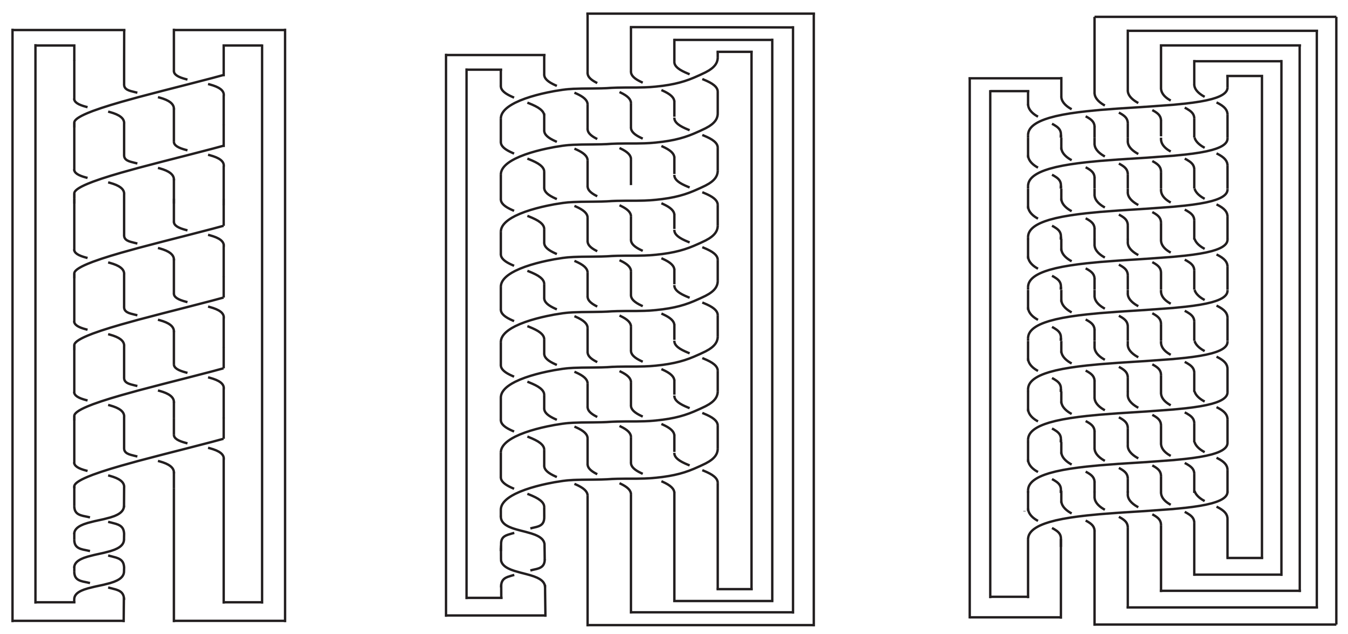

Fig. 1: The links $T^{(3)}(4,6)$, $T^{(-2)}(6,8)$ and $T(7,9)$ (from left to right)

## NON-$\mathbb{Z}_2$ TORSION IN KH

The $A$-smoothing number of a diagram $D$ is the minimal number of $A$-smoothings needed to transform $D$ into a diagram representing a trivial link. The $A$-smoothing number of a link $L$ is the minimum over all diagrams of $L$. $B$-smoothing number is defined similarly.

Let $T(m,n)$ denote the torus link of type $(m,n)$. Let $T^{(k)}(m,n)$ denote the torus link of type $(m,n)$ with $k$ twists added to the leftmost pair of strands. Figure 1 shows examples of such links. Observe that torus links of type $(m,m+2)$ have smoothing number one. Furthermore, adding twists does not change the smoothing number of a link. Also, 2-cablings of links with twists have smoothing number one.

**Theorem 1 (MPSWY)**

1. *The KH of $T^{(k)}(4,6)$ contains $\mathbb{Z}_4$ torsion for $k \geq -3$.*
2. *The KH of $T^{(k)}(5,7)$ contains $\mathbb{Z}_5$ torsion for any $k \in \mathbb{Z}$.*
3. *The KH of $T^{(k)}(6,8)$ contains $\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5$ torsion for any $k \in \mathbb{Z}$, and $\mathbb{Z}_8$ torsion for $k \leq -8$.*
4. *The KH of $T^{(k)}(7,9)$ contains $\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_5, \mathbb{Z}_7$ torsion for any $k \in \mathbb{Z}$.*

The following conjecture (verified up to $s = 23$) was proposed based on our computations for 2-cablings of certain links which equivalently can also be obtained by adding twists to torus links.

**Conjecture 2 (MPSWY)** *Let $0 < s' \leq s$. Then, the Khovanov homology of the flat 2-cabling of the torus knot $T(2, 2s+1)$ contains $\mathbb{Z}_{2^{s'}}$ torsion.*

**Theorem 3 (Muk)**

1. *The KH of the connected sum of $T(5,6)$ with itself has $\mathbb{Z}_9$ torsion.*
2. *The KH of the connected sum of $T^{-1}(6,7)$ with itself has $\mathbb{Z}_{25}$ torsion.*
3. *The KH of the closure of the braid: $(\sigma_1\sigma_2\sigma_3\sigma_4)^6(\sigma_4\sigma_5\sigma_6\sigma_7)^6(\sigma_7\sigma_8\sigma_9\sigma_{10})^6$, the overlapping connected sum of the torus knot of type $(5,6)$ with itself twice, contains $\mathbb{Z}_{27}$ torsion.*

## THE PS BRAID CONJECTURE

A part of the Przytycki-Sazdanović(PS) braid conjecture [PS] attempts to relate the order of the torsion subgroups in the KH of a closed braid with its braid index. We provide a counterexample to it in the following theorem.

**Theorem 4 (Muk)** *The part of the PS braid conjecture stating that the KH of a closed braid with braid index n cannot have p (prime) torsion for p > n is false.*

| j \| i | ⋯ | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |
|---|---|---|---|---|---|---|---|---|---|---|
| 79 | ⋯ | | | | | | | | | $1_2, 1_3$ |
| 77 | ⋯ | | | | | | | 1 | 1 | $1_2$ |
| 75 | ⋯ | | | | | | $1_2$ | $2_2$, **$1_7$** | 1 | |
| 73 | ⋯ | | | | | $1, 1_2$ | $2, 2_2$ | $1, 1_2$ | | |
| 71 | ⋯ | | | 1 | $2, 1_2$ | $2_2$ | $2_2$, **$1_7$** | 1 | | |
| 69 | ⋯ | 1 | | $3_2, 1_3$ | $2, 3_2$ | 1 | | | | |

Fig. 2: Khovanov homology of the closure of the braid: $(\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5)^7 \cdot w_{1,5}$ between $j$ gradings 69 and 79 with braid index six and $\mathbb{Z}_7$ torsion

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