Large Families of Matroids with the Same Tutte Polynomial

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Abstract

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One of the most important invariants of a matroid is the Tutte polynomial, which is defined as follows:

$$t(M; x, y) = \sum_{X \subseteq E} (x-1)^{r(E)-r(X)}(y-1)^{|X|-r(X)}.$$

In essence, it is a polynomial in two variables, x and y, that records the number of subsets X of the ground set E of the matroid M having given cardinality and rank. From its Tutte polynomial, one can deduce a great deal of information about a matroid; indeed in some cases the Tutte polynomial determines the matroid itself. In this thesis we are interested in the extreme opposite, i.e., we are interested in cases where the Tutte polynomial is the common Tutte polynomial of a very large collection of nonisomorphic matroids. Here we construct large families of matroids (superexponential in the cardinality of the ground set) having the same Tutte polynomial.

The families of matroids we construct also share another important matroid invariant, the (abstract) lattice of cyclic flats. The collection of cyclic flats of a matroid forms a lattice when ordered by inclusion. An axiom scheme for a matroid by its collection of cyclic flats and their ranks was given by Bonin and de Mier [3] (and earlier by Sims [15]), who also showed that for any lattice L, there exists a (fundamental transversal) matroid whose lattice of cyclic flats is isomorphic to L. In the theme of matching large families of matroids to a matroid invariant (where possible), we determine the possible rank functions of the fundamental transversal matroids having a given abstract lattice as its lattice of cyclic flat (Theorem 4.3.3).

Our families of matroids having the same Tutte polynomial and lattice of cyclic flats all generalize a construction of Giménez [10], who showed that all matroids in his family have the same lattice of cyclic flats, though he did not realize that they have the same Tutte

polynomial as well. Our generalizations produce yet larger families of matroids than the families of Giménez.

In the proofs of the main results of this thesis, we make extensive use of a tool that we have developed: rank sublattices of the lattice of cyclic flats. We define rank sublattices and explore some of their properties in an early chapter. To each subset X of the ground set of the matroid M, we associate a subset $R(X)$ of the collection of cyclic flats of M. The subset $R(X)$ is always a sublattice of the lattice of cyclic flats, and it determines the rank of X. Thus we may think of $R(X)$ as a refinement of the rank $r(X)$. We hope that the notion of rank sublattices will continue to serve as a useful tool in other problems in matroid theory.

Contents

List of Figures

Chapter 1

Introduction

The results in this thesis revolve around two invariants of matroids: Tutte polynomials and isomorphism types of lattices of cyclic flats. Neither of these invariants determines the other, but both invariants encode a great deal of information about the matroid. Due to the extensive amount of information encoded in the Tutte polynomial and the lattice of cyclic flats, it can be challenging to find nonisomorphic matroids that have the same Tutte polynomial and lattice of cyclic flats. This challenge is the main object of this thesis. Here we shall construct several large families of matroids having the same Tutte polynomial and isomorphic lattices of cyclic flats; the size of each family grows superexponentially with respect to the cardinality of the ground set.

The Tutte polynomial

$$t(M;x,y) = \sum_{X \subseteq E} (x-1)^{r(E)-r(X)}(y-1)^{|X|-r(X)}$$

encodes the number of subsets of the given matroid according to their ranks and cardinalities. It is one of the most important algebraic tools in the theory of matroids. Much research has been done concerning Tutte polynomials of matroids. We refer the reader to [18] for basic properties of Tutte polynomials.

The lattice of cyclic flats is a somewhat lesser-known invariant; however, we find it highly attractive for various reasons. Bonin and de Mier [3], and independently Sims [15], developed an axiom scheme for matroids by the set of cyclic flats and their ranks. Since there are often relatively few cyclic flats in a matroid, the collection of cyclic flats and their ranks is a compact way to express the matroid. The collection of all cyclic flats forms a lattice (an ordered set with meet and join operations) when ordered by set inclusion. However, the abstract lattice viewed as a lattice of cyclic flats of a matroid does not determine the matroid; this abstract lattice is the invariant that we are concerned with. To be specific, the matroid can be recovered from its collection of cyclic flats and their ranks, given as subsets of the ground set; however, if one does not see these subsets explicitly, but rather the abstract lattice formed by them under set inclusion (even including their ranks), one cannot recover the matroid.

Our constructions of large families of matroids having the same Tutte polynomial and lattice of cyclic flats begins with the family of matroids constructed by Giménez [10, 3], which is based on permutations. He showed that all matroids in his family have the same lattice of cyclic flats. Here we prove that they all have the same Tutte polynomial as well (Section 5.1). By generalizing his construction, we obtain even larger families of matroids with the same properties. In one generalization, we produce families exponentially larger than Giménez's family (Section 5.2). This is due to the fact that our construction involves the choice of several permutations per matroid (and the number of permutations may be chosen to grow linearly in the size of the ground set) whereas Giménez's family uses just one permutation per matroid in the family.

In another generalization of Giménez's construction, the matroids in the family are uniquely indexed by semi-magic squares, which are matrices with non-negative integer entries such that all row and column sums are equal to a given number (Section 5.3). Semi-magic squares generalize permutation matrices since the set of permutation matrices is the set of semi-magic squares such that the row and column sums are equal to one. Among the set of all semi-magic square matroids, there is a particular one which we prove is a lattice-path matroid (Section 5.4). This is useful because there is an efficient algorithm to compute the Tutte polynomial of a lattice-path matroid.

The key tool that we use to investigate the relationships between the above two invariants is the notion of rank sublattices of the lattice of cyclic flats. For each subset X of the ground set of the matroid, the rank sublattice $R(X)$ is the collection of cyclic flats that determine the rank of X (later we will explain exactly what this means — it is based on a formula for the rank of X given by Sims [15]). In order to show that the particular pair of matroids with isomorphic lattices of cyclic flats have the same Tutte polynomial, we establish a bijection between the power sets of the matroids such that the bijection preserves both rank and cardinality (such a bijection of subsets is not induced by a bijection of the ground sets and so is not an isomorphism, but it does imply that the Tutte polynomials are equal). The rank sublattices $R(X)$ facilitate the construction of this bijection.

Bonin and de Mier [3] showed that for any lattice L, there is a special type of matroid (called fundamental transversal) whose lattice of cyclic flats is isomorphic to L. However, there are many such fundamental transversal matroids having isomorphic lattices of cyclic flats. In this thesis, again using rank sublattices as the main tool, we characterize all possible rank functions that can be attached to the lattice, provided that the matroid is fundamental transversal (Section 4.3). The set of possible rank functions on the lattice L for fundamental transversal matroids is in a natural bijection with the set of all $(|L|-1)$-tuples of non-negative integers. One may ask the question, what ranks are possible among the class of *all* matroids? At this point it is unknown if the answer is given by a system of linear inequalities. In fact very little is known about the structure of the possible rank functions, though the axiom scheme of Bonin and de Mier provides a natural starting point.

We prove several attractive properties that relate the class of fundamental transversal matroids and rank sublattices. For example, a rank sublattice $R(X)$ is generally not an interval of the lattice of cyclic flats; however, if $R(X)$ is an interval, then we show that the interval corresponds to a fundamental transversal matroid that is a minor of the matroid (Theorem 4.2.7). Moreover, we characterize the family of fundamental transversal matroids in terms of rank sublattices (Theorem 4.2.9).

Since we have used the notion of rank sublattice throughout this thesis, we feel they are worth exploring as a subject in their own right. We give various formulations and properties of rank sublattices. Of course, one of the most fundamental properties is that it is a sublattice of the lattice of cyclic flats.

The structure of this thesis is the following. We first discuss basic definitions and properties of matroids with an emphasis on cyclic flats and lattices as well as introducing the cyclic and isthmus operators (Chapter 2). Second, we define rank sublattices $R(X)$ of the lattice of cyclic flats and develop their various properties (Chapter 3). Then we discuss some results about fundamental transversal matroids (Chapter 4). We also construct several large families of matroids with the same Tutte polynomial and isomorphic lattices of cyclic flats (Chapter 5).

Chapter 2

Background

In this chapter, we develop basic tools of matroid theory with an emphasis on cyclic flats and the lattice that they form. Another purpose of this chapter is to introduce two new operators: the cyclic and isthmus operators. We first discuss various subsets and functions of matroids, including cyclic sets, cyclic flats, and rank and nullity functions (Section 2.1). We define dual matroids (Section 2.2) and minors of matroids (Section 2.3). Then we define the closure, cyclic, and isthmus operators (Section 2.4). Finally, we introduce various lattices associated to matroids (Section 2.5).

2.1 Matroid Fundamentals

First we give the definition of matroid [14, Section 1.1].

Definition 2.1.1. A *matroid* M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E satisfying the following three conditions:
(I1) $\emptyset \in \mathcal{I}$,
(I2) if $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$, then $I_2 \in \mathcal{I}$, and
(I3) if I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

For a matroid $M = (E, \mathcal{I})$, we call E the *ground set* of M and we say that M is a *matroid on* E. A subset of E that is a member of $\mathcal{I}$ is called an *independent set* of M. We sometimes write $\mathcal{I}(M)$ for $\mathcal{I}$ and $E(M)$ for E, particularly when several matroids are being considered. The symbol E will denote the ground set of M throughout this thesis.

A subset of E that is not in $\mathcal{I}$ is called *dependent.* A *circuit* of M is a minimal dependent set. A *cyclic set* is a subset of E that is a (possibly empty) union of circuits of M. A maximal independent set is called a *basis* of M. By (I3), all bases have the same cardinality. Clearly, an independent set is a subset of a basis by (I2). A superset of a basis is called a *spanning set* of M. A *hyperplane* is a maximal non-spanning set. A *flat* is a subset of E that is a (possibly empty) intersection of hyperplanes of M. Notice that E is trivially a flat. A *cyclic flat* is a flat that is also a cyclic set of M.

The sets of all independent sets, circuits, cyclic sets, bases, spanning sets, hyperplanes, flats, and cyclic flats are denoted by $\mathcal{I}(M)$, $\mathcal{C}(M)$, $\mathcal{U}(M)$, $\mathcal{B}(M)$, $\mathcal{S}(M)$, $\mathcal{H}(M)$, $\mathcal{F}(M)$, and $\mathcal{Z}(M)$, respectively. When the matroid is clear we simply write $\mathcal{I}$, $\mathcal{C}$, $\mathcal{U}$, $\mathcal{B}$, $\mathcal{S}$, $\mathcal{H}$, $\mathcal{F}$, and $\mathcal{Z}$, respectively. Notice that $\mathcal{B} = \mathcal{I} \cap \mathcal{S}$ and $\mathcal{Z} = \mathcal{F} \cap \mathcal{U}$.

Let 2^E denote the set of all subsets of E (commonly known as the power set of E). The *rank function* $r_M : 2^E \to \mathbb{Z}^+ \cup 0$ of a matroid M is defined by

$$r_M(X) = \max\{|I| : I \subseteq X, I \in \mathcal{I}(M)\}.$$

Thus, the rank function gives the cardinality of a maximal independent set in the given subset. The *nullity function* $n_M : 2^E \to \mathbb{Z}^+ \cup 0$ of M is defined by

$$n_M(X) = |X| - r_M(X). \tag{2.1}$$

We write $r(M)$ and $n(M)$ for $r_M(E)$ and $n_M(E)$, respectively, and call them the rank and nullity of the matroid M. Notice that $r(M)$ is the cardinality of a basis of M. We omit the subscript M when the matroid is clear.

Clearly, a spanning set has rank $r(M)$. It follows that a hyperplane of M is a flat of rank $r(M) - 1$. Similarly, an independent set has nullity 0. Thus, a circuit of M is a cyclic set of nullity 1.

2.2 Duality

For a matroid M with the set $\mathcal{B}(M)$ of bases, the set $\mathcal{B}^*(M) = \{E - B : B \in \mathcal{B}(M)\}$ is the set of bases of another matroid on the same ground set E. This matroid is called the *dual* of M and is denoted by M^*. Thus, $\mathcal{B}(M^*) = \mathcal{B}^*(M)$ [14, Theorem 2.1.1]. The bases of M^* are called the *cobases* of M.

Similarly, cyclic sets, flats, and cyclic flats of M^* are called *cocyclic sets*, *coflats*, and *cocyclic coflats* of M, respectively. The set of these subsets is denoted by $\mathcal{U}^*(M)$, $\mathcal{F}^*(M)$, and $\mathcal{Z}^*(M)$, respectively. Thus, we have $\mathcal{U}(M^*) = \mathcal{U}^*(M)$, $\mathcal{F}(M^*) = \mathcal{F}^*(M)$, and $\mathcal{Z}(M^*) = \mathcal{Z}^*(M)$. The rank and nullity functions of M^* are called the *dual rank* and *dual nullity* functions of M, respectively. They are denoted by r^*_M and n^*_M. When the matroid is clear, we omit M and simply write $\mathcal{U}^*$, $\mathcal{F}^*$, $\mathcal{Z}^*$, r^*, and n^*.

For each collection of subsets we have discussed, such as $\mathcal{U}(M)$, $\mathcal{F}(M)$, etc., the set of complements of these subsets is an important collection of subsets in the dual matroid.

Lemma 2.2.1. *Let M be a matroid on E.*

(1) *A subset U is a cyclic set if and only if $E - U$ is a coflat.*

(2) *A subset F is a flat if and only if $E - F$ is a cocyclic set.*

(3) *A subset Z is a cyclic flat if and only if $E - Z$ is a cocyclic coflat.*

Proof. For statement (1), U is a cyclic set of M if and only if U is a union of circuits of M. Equivalently, $E - U$ is an intersection of cohyperplanes of M, i.e., a coflat of M. Statement (2) follow from (1). Finally, statement (3) follows from statements (1) and (2) because $\mathcal{Z} = \mathcal{F} \cap \mathcal{U}$ and $\mathcal{Z}^* = \mathcal{F}^* \cap \mathcal{U}^*$. □

The dual rank function can be expressed in terms of the rank function as follows.

$$r^*(X) = |X| - r(M) + r(E - X) \tag{2.2}$$

In particular, when $X = E$, we have the following special case.

$$r(E) + r^*(E) = |E| \tag{2.3}$$

The following is a simple yet useful lemma that gives the connection between the rank and nullity functions via duality.

Lemma 2.2.2. *Let r and n be the rank and nullity functions for the matroid M on E and suppose that $X \subseteq E$. Then we have*

$$r(E) = r(X) + n^*(E - X) = n^*(E) \quad \textit{and}$$
$$n(E) = n(X) + r^*(E - X) = r^*(E).$$

Proof. Using (2.1) and (2.3), we obtain $r^*(E) = n(E)$. Hence, $r(E) = n^*(E)$ by duality. Using equation (2.2), we have

$$r(E) = |X| - r^*(X) + r(E - X) = n^*(X) + r(E - X).$$

Dually, we have $r^*(E) = n(X) + r^*(E - X)$. □

The axiom scheme by the rank function is the following [14, Corollary 1.3.4].

Lemma 2.2.3. *Let E be a set. A function $r : 2^E \to \mathbb{Z}^+ \cup \{0\}$ is the rank function of a matroid on E if and only if r satisfies the following conditions.*

(R1) *If $X \subseteq E$, then $0 \leq r(X) \leq |X|$.*

(R2) *If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$* (monotonicity).

(R3) *If $X, Y \subseteq E$, then $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$* (upper semimodularity).

The following lemma gives the corresponding axiom scheme using the nullity function.

Lemma 2.2.4. *Let E be a set. A function $n : 2^E \to \mathbb{Z}^+ \cup \{0\}$ is the nullity function of a matroid on E if and only if n satisfies the following conditions.*

(N1) *If $X \subseteq E$, then $0 \leq n(X) \leq |X|$.*

(N2) *If $X \subseteq Y \subseteq E$, then $n(X) \leq n(Y)$* (monotonicity).

(N3) *If $X, Y \subseteq E$, then $n(X) + n(Y) \leq n(X \cup Y) + n(X \cap Y)$* (lower semimodularity).

Proof. The equivalence of (R1) and (N1), and that of (R3) and (N3), follow from equation (2.1). The equivalence of (R2) and (N2) follows from Lemma 2.2.2 since the value $r(X) + n^*(E - X)$ is constant. □

2.3 Minors and Direct Sums

There are two particularly basic ways to build new matroids from old. From a single matroid M, we may construct smaller matroids, called minors, by deletions and contractions. From two matroids M and N, we may construct a larger matroid $M \oplus N$, called the direct sum of M and N.

2.3.1 Minors

For a matroid M on E and a subset $X \subseteq E$, let $\mathcal{I}|X = \{I \subseteq X : I \in \mathcal{I}(M)\}$. The matroid $(X, \mathcal{I}|X)$ is called the *restriction of M to X* and is denoted by $M|X$. We write $M \backslash X$ for $M|(E - X)$ and call it the *deletion of X from M*. We also write M/X for $(M^* \backslash X)^*$ and call it the *contraction of X from M*. We sometimes write $M.X$ for $M/(E - X)$ and call it the *contraction of M to X*.

Any sequence of deletions and contractions from M can be written in the form $M \backslash X/Y$ for some pair of disjoint sets X and Y, either of which may be empty [14, Proposition 3.1.26]. Matroids of this form are called *minors* of M. Equivalently, a minor of M can be written in the form $M|X/Y$ using restriction rather than deletion. The next lemmas provide the expressions of $\mathcal{F}, \mathcal{U}, \mathcal{Z}, \mathcal{B}$, and the rank function for minors of M. We will use these results in Section 4.2. Recall that $\mathcal{F}, \mathcal{U}, \mathcal{Z}$, and $\mathcal{B}$ denote the sets of all flats, cyclic sets, cyclic flats, and bases, respectively.

Lemma 2.3.1. *Let M be a matroid on E and $T \subseteq E$. Then*

(1) $\mathcal{F}(M|T) = \{F \subseteq T : F \in \mathcal{F}(M)\}$ *for* $T \in \mathcal{F}(M)$,

(2) $\mathcal{F}(M/T) = \{F \subseteq E - T : F \cup T \in \mathcal{F}(M)\}$,

(3) $\mathcal{U}(M|T) = \{U \subseteq T : U \in \mathcal{U}(M)\}$,

(4) $\mathcal{U}(M/T) = \{U \subseteq E - T : U \cup T \in \mathcal{U}(M)\}$ *for* $T \in \mathcal{U}(M)$,

(5) $\mathcal{Z}(M|T) = \{Z \subseteq T : Z \in \mathcal{Z}(M)\}$ *for* $T \in \mathcal{F}(M)$, *and*

(6) $\mathcal{Z}(M/T) = \{Z \subseteq E - T : Z \cup T \in \mathcal{Z}(M)\}$ *for* $T \in \mathcal{U}(M)$.

Proof. For statement (1), we know that $\mathcal{F}(M|T) = \{F \cap T : F \in \mathcal{F}(M)\}$ by [14, Proposition 3.3.1(ii)]. Since $T \in \mathcal{F}(M)$, we have $F \cap T \in \mathcal{F}(M)$. Statement (2) is from [14, Proposition 3.3.1(i)]. Statement (3) is immediate since $\mathcal{C}(M|T) = \{C \subseteq T : C \in \mathcal{C}(M)\}$ and cyclic sets are unions of circuits. For (4), we have $\mathcal{U}(M/T) = \{U \subseteq E - T : (E - T) - U \in \mathcal{F}((M/T)^*)\}$ by the duality of $\mathcal{U}$ and $\mathcal{F}$. Since $T \in \mathcal{U}(M)$ implies $E - T \in \mathcal{F}(M^*)$, we have

$$\mathcal{F}((M/T)^*) = \mathcal{F}(M^*|(E - T)) = \{F \subseteq E - T : F \in \mathcal{F}(M^*)\}.$$

Thus,

$$\begin{aligned}\mathcal{U}(M/T) &= \{U \subseteq E - T : (E - T) - U \in \mathcal{F}(M^*)\}\\ &= \{U \subseteq E - T : U \cup T \in \mathcal{U}(M)\}.\end{aligned}$$

Clearly, statement (5) follows from statements (1) and (3), and statement (6) follows from statements (2) and (4). □

The set of bases of $M|T$ is given by the following [14, Proposition 3.1.15].

Lemma 2.3.2. *Let M be a matroid on E and $T \subseteq E$. Then $\mathcal{B}(M|T)$ is the set of maximal members of $\{B \cap T : B \in \mathcal{B}(M)\}$.*

The rank functions of $M|T$ and M/T are given below [14, Proposition 3.1.5 and 3.1.6].

Lemma 2.3.3. *For a matroid M on E and a set $T \subseteq E$,*

$$\begin{aligned}r_{M|T}(X) &= r_M(X), \quad \textit{for } X \subseteq T, \textit{and}\\ r_{M/T}(X) &= r_M(X \cup T) - r_M(T) \quad \textit{for } X \subseteq E - T.\end{aligned}$$

2.3.2 Direct Sum

Let M_1 and M_2 be matroids on disjoint sets E_1 and E_2, respectively. Let $E = E_1 \cup E_2$ and $\mathcal{I} = \{I_1 \cup I_2 : I_1 \in \mathcal{I}(M_1), I_2 \in \mathcal{I}(M_2)\}$. The *direct sum* of M_1 and M_2 is the matroid $(E, \mathcal{I})$, which is denoted by $M_1 \oplus M_2$. Clearly, $r(M_1 \oplus M_2) = r(M_1) + r(M_2)$. More generally, for

n matroids, $M_1, \ldots, M_n$, on disjoint sets $E_1, \ldots, E_n$, the direct sum $M_1 \oplus \cdots \oplus M_n$ is the pair $(E, \mathcal{I})$ where $E = E_1 \cup \cdots \cup E_n$ and $\mathcal{I} = \{I_1 \cup \cdots \cup I_n : I_i \in \mathcal{I}(M_i) \text{ for } 1 \leq i \leq n\}$.

2.4 Operators on Subsets of Matroids

The most common operator in the theory of matroids is the closure operator. For a given subset X of E, the closure operator gives the smallest flat containing X. In this section, we introduce two more operators and discuss them together with the closure operator.

Suppose M is a matroid with the ground set E. An element $e \in E$ is called a *loop* of M if $\{e\}$ is a circuit. Equivalently, a loop is an element that is in no basis. On the other hand, an element $e \in E$ is called an *isthmus* of M if e is in every basis of M. Equivalently, an isthmus is an element that is in no circuit of M. Thus, e is an isthmus of M if and only if e is a loop of M^*.

From this definition of isthmus, it follows that each subset X can be decomposed into the largest cyclic set (union of all circuits) in $M|X$ and the set of all isthmuses of $M|X$. This simple observation leads us to define two new operators: the cyclic and isthmus operators. Let us start with the definitions of all three operators. The definition of the closure operator is taken directly from [14, Definition 1.4.1].

Definition 2.4.1. Let M be a matroid on E with the rank function r. The *closure operator*, *cyclic operator*, and *isthmus operator* of M are the functions from 2^E to 2^E, denoted by cl_M, cyc_M, and isth_M, respectively, defined by

$$\begin{aligned}
\mathrm{cl}_M(X) &= \{e \in E : r(X \cup e) = r(X)\}, \\
\mathrm{cyc}_M(X) &= \{e \in X : r(X - e) = r(X)\}, \quad \text{and} \\
\mathrm{isth}_M(X) &= \{e \in E : r(X - e) < r(X)\},
\end{aligned}$$

where $X \subseteq E$. When the matroid is clear, we omit the subscripts M.

For example, let M be the matroid whose geometric representation and graphic representation is as in Figure 2.1. In geometric representations, elements in boxes denote loops. Also the set of elements on the same point, line, and plane are rank-one, -two and -three

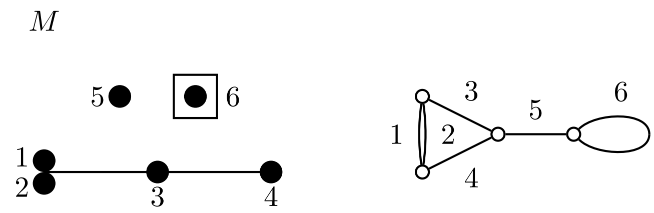

Figure 2.1: Example of the cyclic and closure operators

flats, respectively. Let $X = \{1, 2, 3, 5\}$. Then $\mathrm{cl}(X) = \{1, 2, 3, 4, 5, 6\}$, $\mathrm{cyc}(X) = \{1, 2\}$, $\mathrm{cl}(\mathrm{cyc}(X)) = \{1, 2, 6\}$, and $\mathrm{cyc}(\mathrm{cl}(X)) = \{1, 2, 3, 4, 6\}$.

Clearly, we have $X = \mathrm{cyc}(X) \dot\cup \mathrm{isth}(X)$ for any subset X, where $\dot\cup$ denote the disjoint union. Note that $e \in X$ is an isthmus of $M|X$ if and only if $r(X - e) < r(X)$ (in fact, $r(X - e) = r(X) - 1$). Thus, $\mathrm{isth}(X)$ is the set of all isthmuses of $M|X$. Also, $\mathrm{cyc}(X)$ is the union of all circuits in $M|X$, i.e., the largest cyclic subset of X. It follows that a restriction $M|X$ can be decomposed as the direct sum

$$M|X = M|\mathrm{cyc}(X) \oplus M|\mathrm{isth}(X). \tag{2.4}$$

One can also translate these definitions in terms of the nullity function n.

$$\begin{aligned}
\mathrm{cl}_M(X) &= X \cup \{e \in E : n(X \cup e) > n(X)\} \\
\mathrm{cyc}_M(X) &= \{e \in E : n(X - e) < n(X)\} \\
\mathrm{isth}_M(X) &= \{e \in X : n(X - e) = n(X)\}
\end{aligned}$$

In particular, this characterization of the closure operator is equivalent to the following well-known "completion of circuits" characterization [14, Proposition 1.4.10],

$$\mathrm{cl}_M(X) = X \cup \{e : e \in C \subseteq X \cup e \text{ for some } C \in \mathcal{C}(M)\}. \tag{2.5}$$

The next lemma is an observation from equation (2.4).

Lemma 2.4.2. *For a matroid M on E and a set $X \subseteq E$, we have*

$$r(X) = r(\mathrm{cyc}(X)) + |\mathrm{isth}(X)| \quad \textit{and}$$
$$n(X) = n(\mathrm{cyc}(X)).$$

The next well-known lemma confirms that our definition of flats (intersections of hyperplanes) and the standard definition of flats ($X = \mathrm{cl}(X)$) in [14, Corollary 1.4.6] coincide.

Lemma 2.4.3. *Let M be a matroid on E and $X \subseteq E$. Then X is a flat of M if and only if $X = \mathrm{cl}(X)$.*

Also, the following is one of the most basic properties of the closure operator.

Lemma 2.4.4. *The set $\mathrm{cl}(X)$ is the smallest flat containing X.*

In addition, the closure operator in the restriction is quite simple [14, Proposition 3.1.16].

Lemma 2.4.5. *For a matroid M on E and sets $X \subseteq T \subseteq E$, we have $\mathrm{cl}_{M|T}(X) = \mathrm{cl}_M(X) \cap T$.*

Analogous to Lemma 2.4.3 above, a subset X is a cyclic set if and only if $\mathrm{cyc}(X) = X$. Furthermore, a subset X is independent if and only if $\mathrm{isth}(X) = X$ (note that $\mathrm{isth}(X)$ is independent in M because it is in $M|X$). Thus, we have

$$\mathcal{F} = \{\mathrm{cl}(X) : X \subseteq E\},$$
$$\mathcal{U} = \{\mathrm{cyc}(X) : X \subseteq E\}, \quad \text{and}$$
$$\mathcal{I} = \{\mathrm{isth}(X) : X \subseteq E\}.$$

Analogous to the decomposition of a set X into $\mathrm{cyc}(X) \dot\cup \mathrm{isth}(X)$, we will see that the flat $\mathrm{cl}(X)$ can be expressed as a disjoint union of X and all loops in M/X (Lemma 2.4.6 (1)). Also, we can express each operator in terms of the other operators. In order to see the connections among these operators, we need to consider operators in the dual matroid M^*. Let cl^*, cyc^*, and isth^* denote cl_{M^*}, cyc_{M^*}, and isth_{M^*}, respectively. The set $\mathrm{isth}^*(X)$ is the set of all loops in $M.X$ since $(M^*|X)^* = M.X$. In particular, the set $\mathrm{isth}^*(E)$ is the set of all loops in M.

Lemma 2.4.6. *Let M be a matroid on E and $X \subseteq E$.*

(1) $\mathrm{cl}(X) = X \,\dot{\cup}\, \mathrm{isth}^*(E - X) = E - \mathrm{cyc}^*(E - X)$

(2) $\mathrm{cl}^*(X) = X \,\dot{\cup}\, \mathrm{isth}(E - X) = E - \mathrm{cyc}(E - X)$

(3) $\mathrm{cyc}(X) = X - \mathrm{isth}(X) = E - \mathrm{cl}^*(E - X)$

(4) $\mathrm{cyc}^*(X) = X - \mathrm{isth}^*(X) = E - \mathrm{cl}(E - X)$

(5) $\mathrm{isth}(X) = X - \mathrm{cyc}(X) = \mathrm{cl}^*(E - X) \cap X$

(6) $\mathrm{isth}^*(X) = X - \mathrm{cyc}^*(X) = \mathrm{cl}(E - X) \cap X$

Proof. Note that (1) and (2) are from [14, Exercise 3.1.5]. Replacing X by $E - X$ in (1), we get

$$\mathrm{cl}(E - X) = (E - X) \cup \mathrm{isth}^*(X) = E - \mathrm{cyc}^*(X).$$

Taking a complement yields (4) and taking intersection with X yields (6). Finally, (3) and (5) are the duals of (4) and (6), respectively. □

The following well-known statements describe the properties of the closure operator and the rank function.

Lemma 2.4.7. *Let $X, Y \subseteq E$.*

(1) $r(\mathrm{cl}(X)) = r(X)$

(2) *If* $\mathrm{cl}(X) = \mathrm{cl}(Y)$*, then* $r(X) = r(Y)$*.*

(3) *If* $X \subseteq Y$ *and* $r(X) = r(Y)$*, then* $\mathrm{cl}(X) = \mathrm{cl}(Y)$*.*

(4) *If* $X \subseteq Y$*, then* $r(X) = r(Y)$ *if and only if* $Y \subseteq \mathrm{cl}(X)$*.*

We have the counterparts of these statements for the cyclic operator and the nullity function.

Lemma 2.4.8. *Let $X, Y \subseteq E$.*

(1) $n(\mathrm{cyc}(X)) = n(X)$

(2) *If* $\mathrm{cyc}(X) = \mathrm{cyc}(Y)$*, then* $n(X) = n(Y)$*.*

(3) *If* $X \subseteq Y$ *and* $n(X) = n(Y)$*, then* $\mathrm{cyc}(X) = \mathrm{cyc}(Y)$*.*

(4) *If* $X \subseteq Y$*, then* $n(X) = n(Y)$ *if and only if* $\mathrm{cyc}(Y) \subseteq X$*.*

Proof. Statement (1) is clear since the isthmuses of X do not contribute to its nullity. Statement (2) is obvious from (1). For (3), let $e \in Y - X$. By monotonicity, we have $n(X) \leq n(Y - e) \leq n(Y)$. They are all equalities by our assumption. So $n(Y) = n(Y - e)$ and $e \in \mathrm{isth}(Y)$. Thus, $M|Y$ is $M|X$ with isthmuses added, so clearly $\mathrm{cyc}(X) = \mathrm{cyc}(Y)$. For (4), the forward direction is immediate from (3). The converse is obtained by taking the cyclic parts of $\mathrm{cyc}(Y) \subseteq X \subseteq Y$ and applying (2). □

Using these lemmas, we have the following strengthening of (R2) and (N2).

Corollary 2.4.9. *Let* $X, Y \subseteq E$*.*

(1) *We have that* $r(Y) - r(X) \leq |Y - X|$*. Furthermore, if* $X \subseteq Y$*, then the equality holds if and only if* $\mathrm{cyc}(Y) \subseteq X$*.*

(2) *We have that* $n(Y) - n(X) \leq |Y - X|$*. Furthermore, if* $X \subseteq Y$ *then, the equality holds if and only if* $Y \subseteq \mathrm{cl}(X)$*.*

Proof. For (1), we have

$$\begin{aligned} r(Y) - r(X) &= |Y - X| + |X \cap Y| - n(Y) - r(X) \\ &\leq |Y - X| + |X \cap Y| - n(X \cap Y) - r(X \cap Y) \\ &= |Y - X| \end{aligned}$$

by (R2) and (N2). Furthermore, suppose that $X \subseteq Y$. From the above, we have

$$\begin{aligned} r(Y) - r(X) = |Y - X| \quad &\Leftrightarrow \quad n(Y) = n(X) \\ &\Leftrightarrow \quad \mathrm{cyc}(Y) \subseteq X \end{aligned}$$

by Lemma 2.4.8. Similar arguments hold for (2) by Lemma 2.4.7. □

2.5 Lattices

We first define basic concepts related to ordered sets and lattices, and then we discuss the lattices that arise from matroids. The reader should refer to [16, Chapter 3] for detailed information about lattice theory. We restrict all sets to be finite.

An *ordered set* is a finite set P together with a binary relation, denoted by $\leq$, satisfying the following three conditions.

(P1) For all $x \in P$, $x \leq x$. [reflexivity]

(P2) If $x \leq y$ and $y \leq x$, then $x = y$. [antisymmetry]

(P3) If $x \leq y$ and $y \leq z$, then $x \leq z$. [transitivity]

For two elements x and y of an ordered set P, we say x and y are *comparable* if $x \leq y$ or $y \leq x$; otherwise x and y are *incomparable*. We say y *covers* x and write $x \lessdot y$ if $x < y$ and there is no $z \in P$ such that $x < z < y$. If P has an element x such that $x \leq y$ for all $y \in P$, then we denote x by 0_P. Similarly, if P has an element x such that $x \geq y$ for all $y \in P$, then we denote x by 1_P. An element x is an *atom* of P if x covers 0_P.

A *suborder* of P is a subset of P with the binary relation induced by the binary relation of P. An *interval* $[x, y]_P = \{z \in P : x \leq z \leq y\}$ is a suborder defined whenever $x \leq y$. We omit the subscript P when the ordered set is clear. Thus, $x \lessdot y$ if and only if $x < y$ and $[x, y] = \{x, y\}$. A *chain* in P from x_0 to x_n is a suborder $\{x_0, x_1, \ldots, x_n\}$ of P such that $x_0 < x_1 < \cdots < x_n$. The *length* of such a chain is n, and the chain is *saturated* if x_i covers x_{i-1} for $1 \leq i \leq n$. If, for every pair $\{x, y\}$ of elements of P with $x < y$, all saturated chains from a to b have the same length, then P is *graded* (or we say that P satisfies the *Jordan-Dedekind chain condition*). When 0_P exists, the *height* $h(x)$ of an element x of P is the maximal length of a chain from 0_P to x. Thus, atoms are the elements of height 1. An *antichain* of P is a suborder of P such that any two elements in the subset are incomparable. The *width* of an ordered set P is the maximal cardinality of an antichain in P. A *filter* F of P is a suborder of P such that, if $x \in F$ and $y \geq x$, then $y \in F$. For a subset X of P, the filter $\{y \in P : y \geq x$ for some $x \in X\}$ is called the *filter generated by* X.

Two ordered sets P and Q are *isomorphic* if there is an *order-preserving bijection* φ :

$P \to Q$ whose inverse is also order-preserving; that is, $x \leq y$ in P if and only if $\varphi(x) \leq \varphi(y)$ in Q. The *order dual* of the ordered set P is the ordered set P^d on the same set as P such that $x \leq y$ in P^d if and only if $y \leq x$ in P. We call P and P^d *dual ordered sets*. Clearly, $P = (P^d)^d$.

For $x, y \in P$, if there is a unique greatest element z such that $z \leq x$ and $z \leq y$, then we call z the *meet* of x and y, and denote it by $x \wedge y$. Dually, if there is a unique least element z such that $z \geq x$ and $z \geq y$, then we call z the *join* of x and y, and denote it by $x \vee y$. A *lattice* is an ordered set in which every pair of elements has a meet and a join. For a subset S of the lattice P, we write $\bigwedge(S) = \bigwedge_{x \in S} x$ and $\bigvee(S) = \bigvee_{x \in S} x$; these are the meet and the join of all elements in S.

An element x in the lattice L is called *meet-reducible* if there are y and z in L such that $y > x$, $z > x$, and $x = y \wedge z$. Similarly, x in L is called *join-reducible* if there are y and z in L such that $y < x$, $z < x$, and $x = y \vee z$. We denote the set of all meet-reducible elements of L by $\mathcal{M}_L$ and the set of all join-reducible elements of L by $\mathcal{J}_L$.

A lattice is *semimodular* if it is graded and each pair x and y of elements of L satisfies the inequality $h(x) + h(y) \geq h(x \wedge y) + h(x \vee y)$. A *geometric lattice* is a finite semimodular lattice in which every element is a join of atoms.

It is well known that, under set inclusion, the set $\mathcal{F}(M)$ of flats of M is a geometric lattice graded by the rank function r_M. We denote both the set of flats and the lattice of flats of M by $\mathcal{F}(M)$, or simply $\mathcal{F}$ when the matroid is clear. The meet and join of $F_1, F_2 \in \mathcal{F}$ are given by $F_1 \wedge F_2 = F_1 \cap F_2$ and $F_1 \vee F_2 = \text{cl}(F_1 \cup F_2)$, respectively. Also $0_{\mathcal{F}} = \text{cl}(\emptyset)$ and $1_{\mathcal{F}} = E$.

In this section, we discuss two more lattices that arise from the structure of a matroid: the lattice of cyclic sets and the lattice of cyclic flats. Both the set of cyclic sets $\mathcal{U}$ and the set of cyclic flats $\mathcal{Z}$ are lattices under set inclusion. First we discuss some connections between $\mathcal{F}$ and $\mathcal{U}$ via duality. The fact that the cyclic sets form a lattice whose order dual is geometric is known [17].

Lemma 2.5.1. *The following properties hold for the sets $\mathcal{F}$ and $\mathcal{U}$ of a matroid M.*

(1) *The set $\mathcal{U}$ of cyclic sets is a lattice under inclusion; furthermore, its order dual is*

isomorphic to $\mathcal{F}^*$, the lattice of flats of M^*; that is $\mathcal{U} \cong (\mathcal{F}^*)^d$.

(2) *The lattice $\mathcal{U}$ of cyclic sets is graded by the nullity function n. Its meet and join operations are given by $U_1 \wedge U_2 = \mathrm{cyc}(U_1 \cap U_2)$ and $U_1 \vee U_2 = U_1 \cup U_2$ for $U_1, U_2 \in \mathcal{U}$. Moreover, $0_{\mathcal{U}} = \emptyset$ and $1_{\mathcal{U}} = \mathrm{cyc}(E)$.*

Proof. For (1), notice that $\mathcal{U}$ and $\mathcal{F}^*$ are dual ordered sets by Lemma 2.2.1 (2). Since $\mathcal{F}^*$ is a lattice, so is $\mathcal{U}$. For (2), observe that $n(U) = r^*(E) - r^*(E - U)$ for $U \in \mathcal{U}$ by Lemma 2.2.2. Since the lattice $\mathcal{F}^* = \{E - U : U \in \mathcal{U}\}$ is graded by the dual rank function r^*, and $\mathcal{U} \cong (\mathcal{F}^*)^d$ by (1), the lattice $\mathcal{U}$ is graded by the nullity function n. The second sentence is obvious from the definition of cyclic sets. □

Now we show some properties of the lattice of cyclic flats.

Lemma 2.5.2. *Let M be a matroid on E.*

(1) *If U is a cyclic set, then $\mathrm{cl}(U)$ is the smallest cyclic flat containing U.*

(2) *If F is a flat, then $\mathrm{cyc}(F)$ is the largest cyclic flat contained in F.*

(3) $\mathcal{Z} = \{\mathrm{cl}(\mathrm{cyc}(X)) : X \subseteq E\} = \{\mathrm{cyc}(\mathrm{cl}(X)) : X \subseteq E\}$.

(4) *The set of cyclic flats $\mathcal{Z}$ is a lattice under inclusion. For $Z_1, Z_2 \in \mathcal{Z}$, the meet and join operations are given by $Z_1 \wedge Z_2 = \mathrm{cyc}(Z_1 \cap Z_2)$ and $Z_1 \vee Z_2 = \mathrm{cl}(Z_1 \cup Z_2)$, respectively. Moreover, we have $0_{\mathcal{Z}} = \mathrm{cl}(\emptyset) = \mathrm{isth}^*(E)$ and $1_{\mathcal{Z}} = \mathrm{cyc}(E)$.*

Proof. For (1), we have $\mathrm{cl}(U) = U \cup \{e : e \in C \subseteq U \cup e \text{ for some } C \in \mathcal{C}(M)\}$ using equation (2.5), which is clearly a cyclic set. Since $\mathrm{cl}(U)$ is the smallest flat containing U by Lemma 2.4.4, it is also the smallest cyclic flat containing U. For (2), if $F \in \mathcal{F}$, then $E - F \in \mathcal{U}^*$ by Lemma 2.2.1 (2). So $E - \mathrm{cyc}(F) = \mathrm{cl}^*(E - F) \in \mathcal{Z}^*$ by (1) and Lemma 2.4.6 (2). Thus, $\mathrm{cyc}(F) \in \mathcal{Z}$. Statement (3) is clear from (1) and (2). For (4), since $\mathcal{Z} = \mathcal{F} \cap \mathcal{U}$, where $\mathcal{F}$ and $\mathcal{U}$ are both lattices by inclusion, we only need to check that the meet and join are well-defined. Since $Z_1 \cap Z_2$ is a flat and $Z_1 \cup Z_2$ is a cyclic set, it is clear that the meet and join are $\mathrm{cyc}(Z_1 \cap Z_2)$ and $\mathrm{cl}(Z_1 \cup Z_2)$, respectively, by (2) and (1). □

Chapter 3

Rank Sublattices of the Lattice of Cyclic Flats

In this chapter, we define rank sublattices of the lattice of cyclic flats and explore some of their basic properties. In subsequent chapters, we will use rank sublattices as a basic tool in the proofs of the main theorems.

We associate a rank sublattice $R(X)$ with each subset X of the ground set of a matroid M. The lattice $R(X)$ consists of the cyclic flats that determine the rank $r(X)$ of X. In Section 3.1, we define the sets $R(X)$ and explore various formulations of $R(X)$. In Section 3.2, we see that $R(X)$ is indeed a sublattice of $\mathcal{Z}$, and that the least and greatest elements can be expressed using the closure and cyclic operators.

3.1 Definition and Equivalent Formulations of $R(X)$

It is well known that there are many equivalent formulations of matroids [14]. Any of these formulations can be taken as the starting point for matroid theory. One of the less common axiom schemes is by the set of cyclic flats and the ranks associated with them [3, 15].

Theorem 3.1.1. *Let $\mathcal{Z}$ be a collection of subsets of a set E and let r be an integer-valued function on $\mathcal{Z}$. There is a matroid for which $\mathcal{Z}$ is the collection of cyclic flats and r is the rank function restricted to the sets in $\mathcal{Z}$ if and only if*

(Z0) *$\mathcal{Z}$ is a lattice under inclusion,*

(Z1) $r(0_{\mathcal{Z}}) = 0$,

(Z2) $0 < r(Z_2) - r(Z_1) < |Z_2 - Z_1|$ *for* $Z_1, Z_2 \in \mathcal{Z}$ *with* $Z_1 \subsetneq Z_2$, *and*

(Z3) $r(Z_1) + r(Z_2) \geq r(Z_1 \vee Z_2) + r(Z_1 \wedge Z_2) + |(Z_1 \cap Z_2) - (Z_1 \wedge Z_2)|$ *for* $Z_1, Z_2 \in \mathcal{Z}$.

This rank function is then extended to all subsets of E in the following way [3].

Lemma 3.1.2. *If M is a matroid on E and $X \subseteq E$, then*

$$r(X) = \min_{Z \in \mathcal{Z}} \{r(Z) + |X - Z|\}.$$

Proof. By applying Corollary 2.4.9 (1) to $X \cap Z \subseteq X$, we obtain

$$r(X) \leq r(X \cap Z) + |X - Z| \leq r(Z) + |X - Z|.$$

It remains to show that there is a cyclic flat that satisfies both inequalities with equalities. The set $Z = \text{cl}(\text{cyc}(X))$ is a cyclic flat by Lemma 2.5.2 (3). By equation (2.4), we have, as needed,

$$r(X) = r(\text{cyc}(X)) + |X - \text{cyc}(X)| = r(Z) + |X - Z|. \qquad \square$$

This lemma motivates the following definition of $R_M(X)$.

Definition 3.1.3. Let M be a matroid on E and $X \subseteq E$. We define $R_M(X)$ as follows:

$$R_M(X) = \{Z \in \mathcal{Z}(M) : r(X) = r(Z) + |X - Z|\}.$$

When the matroid is clear, we write $R(X)$ instead of $R_M(X)$.

Thus, $R_M(X)$ is precisely the set of all cyclic flats of M that determine $r(X)$. For example, in Figure 3.1, let $X = \{1, 2, 4, 5\}$ be a subset of the matroid M. Then the set $R_M(X) = \{12, 1234, E\}$ is a sublattice of the lattice of cyclic flats $\mathcal{Z}(M)$.

For the rest of this section, I develop some basic tools of $R(X)$ that we will use throughout this thesis; these are original results. The next lemma is an easy observation that can be verified directly from the definition of $R(X)$.

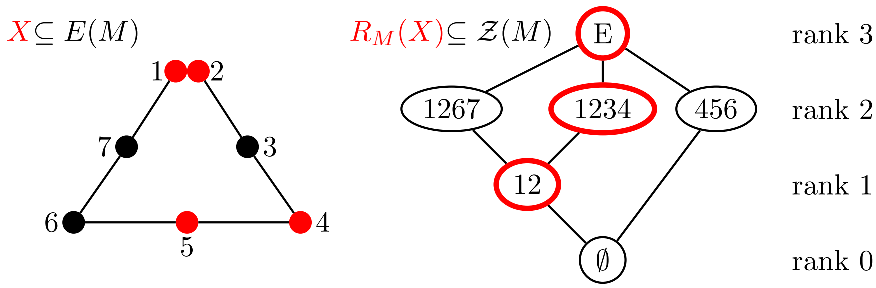

Figure 3.1: Lattice of cyclic flat $\mathcal{Z}(M)$ and its rank sublattice $R_M(X)$

Lemma 3.1.4. *Let M be a matroid on E. A set $X \subseteq E$ is independent if and only if $0_{\mathcal{Z}} \in R(X)$ and $X \cap 0_{\mathcal{Z}} = \emptyset$. Similarly, X spans M if and only if $1_{\mathcal{Z}} \in R(X)$ and $X \cup 1_{\mathcal{Z}} = E$. In particular, when M has no loops, then X is independent if and only if $\emptyset \in R(X)$; also, when M has no isthmuses, then X is spanning if and only if $E \in R(X)$.*

The set $R(X)$ plays a major role in Chapter 5 of this thesis when exploring some families of matroids with isomorphic lattices of cyclic flats and the same Tutte polynomial. Furthermore, $R(X)$ itself has some very attractive properties. For example, $R(X)$ is a sublattice of $\mathcal{Z}$ (Section 3.2) and we can characterize when $R(X)$ is an interval of the lattice $\mathcal{Z}$ (Section 4.2). I have developed six equivalent formulations of $R(X)$. For the rest of this section, we develop four of them. Two other formulations are discussed in Section 3.2.

Clearly each cyclic flat in $R(X)$ determines not only the rank of X but also the nullity of X, which gives the formulation of $R(X)$ by the nullity function.

Lemma 3.1.5. *Let M be a matroid on E with the nullity function n. For $X \subseteq E$,*

$$n(X) = \max_{Z \in \mathcal{Z}(M)} \{n(Z) - |Z - X|\}$$

and

$$R(X) = \{Z \in \mathcal{Z}(M) : n(X) = n(Z) - |Z - X|\}.$$

Proof. Since $|X| + |Z - X| = |Z| + |X - Z|$, we have $|X| - r(Z) - |X - Z| = |Z| - r(Z) - |Z - X|$. It follows that $r(X) = r(Z) + |X - Z|$ if and only if $n(X) = n(Z) - |Z - X|$; also, $r(X) < r(Z) + |X - Z|$ if and only if $n(X) > n(Z) - |Z - X|$. □

The next theorem is one of the most important formulations of $R(X)$.

Theorem 3.1.6. *For a matroid M on E, $X \subseteq E$, and $Z \in \mathcal{Z}(M)$, we have $Z \in R(X)$ if and only if $X \cap Z$ spans Z and all elements in $X - Z$ are isthmuses of $M|X$, that is,* $\mathrm{cyc}(X) \subseteq Z = \mathrm{cl}(X \cap Z)$.

Proof. If $Z \in R(X)$, then we have $r(X) = r(Z) + |X - Z|$. Also, $r(X) \leq r(X \cap Z) + |X - Z|$ by Corollary 2.4.9 (1). It follows that $r(Z) = r(X \cap Z)$ and $r(X) = r(X \cap Z) + |X - Z|$. The first implies $\mathrm{cl}(X \cap Z) = Z$ by Lemma 2.4.7. The second implies that all elements of $X - Z$ are isthmuses of $M|X$.

For the converse, using the decomposition $M|X = M|(X \cap Z) \oplus M|(X - Z)$, we have

$$r(X) = r(X \cap Z) + r(X - Z) = r(Z) + |X - Z|. \qquad \square$$

To continue our discussion about the equivalent formulations of $R(X)$, we now consider duality. Let $R^*_M(X)$, or simply $R^*(X)$ when the matroid is clear, denote $R_{M^*}(X)$. That is,

$$R^*(X) = \{Z \in \mathcal{Z}^* : r^*(X) = r^*(Z) + |X - Z|\}.$$

The next theorem shows that $R^*(X)$ is the collection of complements of the sets in $R(E-X)$.

Theorem 3.1.7. *For a matroid M on E and $X \subseteq E$, we have $R^*(X) = \{Z \subseteq E : E - Z \in R(E - X)\}$.*

Proof. We have the following equivalent conditions,

$$Z \in R^*(X) \Leftrightarrow Z \in \mathcal{Z}^* \text{ and } r^*(X) = r^*(Z) + |X - Z|$$

as well as

$$\begin{aligned} E - Z \in R(E - X) \quad \Leftrightarrow \quad & E - Z \in \mathcal{Z} \text{ and} \\ & r(E - X) = r(E - Z) + |(E - X) - (E - Z)|. \end{aligned}$$

By Lemma 2.2.1 (3), we know that $Z \in \mathcal{Z}^* \Leftrightarrow E - Z \in \mathcal{Z}$. Also we have

$$\begin{aligned} & r^*(X) = r^*(Z) + |X - Z| \\ \Leftrightarrow \quad & |X| - r(M) + r(E - X) = |Z| - r(M) + r(E - Z) + |X - Z| \\ \Leftrightarrow \quad & r(E - X) = r(E - Z) + |(E - X) - (E - Z)|. \end{aligned}$$

□

Using Theorem 3.1.7, we obtain another formulation of $R(X)$ that is dual to Theorem 3.1.6.

Corollary 3.1.8. *For $X \subseteq E$ and $Z \in \mathcal{Z}$, we have $Z \in R(X)$ if and only if $\mathrm{cyc}(X \cup Z) = Z \subseteq \mathrm{cl}(X)$.*

Proof. We have $Z \in \mathcal{Z} \Leftrightarrow E - Z \in \mathcal{Z}^*$ by Lemma 2.2.1 (3). By Lemma 2.4.6, and Theorems 3.1.6 and 3.1.7, we have

$$\begin{aligned} Z \in R(X) \quad & \Leftrightarrow \quad E - Z \in R^*(E - X) \\ & \Leftrightarrow \quad \mathrm{cyc}^*(E - X) \subseteq E - Z = \mathrm{cl}^*((E - X) \cap (E - Z)) \\ & \Leftrightarrow \quad E - \mathrm{cl}(X) \subseteq E - Z = E - \mathrm{cyc}(X \cup Z) \\ & \Leftrightarrow \quad \mathrm{cyc}(X \cup Z) = Z \subseteq \mathrm{cl}(X). \end{aligned}$$

□

Corollary 3.1.9. *For $X \subseteq E$ and $Z \in \mathcal{Z}$, we have $Z \in R(X)$ if and only if $\mathrm{cyc}(X \cup Z) = \mathrm{cl}(X \cap Z)$. Moreover, the equality $\mathrm{cyc}(X \cup Z) = \mathrm{cl}(X \cap Z)$ forces this set to be Z.*

Proof. By Theorem 3.1.6 and Corollary 3.1.8, if $Z \in R(X)$, then

$$\mathrm{cyc}(X \cup Z) = Z = \mathrm{cl}(X \cap Z).$$

For the converse, we know that

$$\mathrm{cl}(X \cap Z) \subseteq Z \subseteq \mathrm{cyc}(X \cup Z).$$

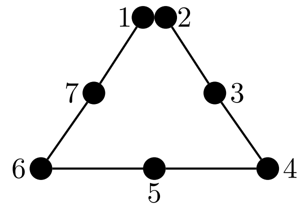

Figure 3.2: Example of a cyclic flat that is not in $R(X)$

By assumption, they are all equalities. Hence

$$\text{cyc}(X) \subseteq \text{cyc}(X \cup Z) = Z = \text{cl}(X \cap Z) \subseteq \text{cl}(X),$$

and we have $Z \in R(X)$ either by Theorem 3.1.6 or by Corollary 3.1.8. □

Remark: If $X \subseteq E$, $Z \in \mathcal{Z}$ and $\text{cyc}(X) \subseteq Z \subseteq \text{cl}(X)$, then it is not necessarily true that $Z \in R(X)$. For example let $X = \{1,2,4,5\}$ in Figure 3.2. Then $\text{cyc}(X) = \{1,2\}$ and $\text{cl}(X) = \{1,2,3,4,5,6,7\}$. Thus Z=$\{1,2,6,7\}$ is a cyclic flat such that $\text{cyc}(X) \subseteq Z \subseteq \text{cl}(X)$ but $Z \notin R(X)$.

Corollary 3.1.10. *If $Z \in \mathcal{Z}$, then $R(Z) = \{Z\}$.*

Proof. By the first part of Corollary 3.1.9, we have $Z' \in R(Z)$ if and only if $Z \in R(Z')$. By the second part of Corollary 3.1.9 applied separately to Z and Z', the equality $\text{cyc}(Z \cup Z') = \text{cl}(Z \cap Z')$ forces this set to be Z and to be Z', so $Z = Z'$. □

There are still other formulations of $R(X)$ that we will discuss in Section 3.2. We conclude this section by discussing the set $R(X)$ for the minors of a given matroid. This result is used in Section 4.2.

Lemma 3.1.11. *For a matroid M on E,*

(1) $R_{M|T}(X) = \{Z \subseteq T : Z \in R_M(X)\}$ *for* $X \subseteq T$ *and* $T \in \mathcal{F}(M)$, *and*

(2) $R_{M/T}(X) = \{Z \subseteq E - T : Z \cup T \in R_M(X \cup T)\}$ *for* $X \subseteq E - T$ *and* $T \in \mathcal{U}(M)$.

Proof. For (1), using Lemmas 2.3.1 (5) and 2.3.3, we have

$$\begin{aligned} R_{M|T}(X) &= \{Z \in \mathcal{Z}(M|T) : r_{M|T}(X) = r_{M|T}(Z) + |X - Z|\} \\ &= \{Z \subseteq T : Z \in \mathcal{Z}(M),\, r_M(X) = r_M(Z) + |X - Z|\} \\ &= \{Z \subseteq T : Z \in R_M(X)\}. \end{aligned}$$

For (2), using Lemmas 2.3.1 (6) and 2.3.3, we have

$$\begin{aligned} R_{M/T}(X) &= \{Z \in \mathcal{Z}(M/T) : r_{M/T}(X) = r_{M/T}(Z) + |X - Z|\} \\ &= \{Z \subseteq E - T : Z \cup T \in \mathcal{Z}(M), \\ &\qquad r_M(X \cup T) = r_M(Z \cup T) + |(X \cup T) - (Z \cup T)|\} \\ &= \{Z \subseteq E - T : Z \cup T \in R_M(X \cup T)\}. \end{aligned}$$

□

3.2 The Set $R(X)$ is a Sublattice

Since $R(X)$ is a subset of the lattice $\mathcal{Z}$ of cyclic flats, when ordered by inclusion it is a suborder of $\mathcal{Z}$. In this section, we show that $R(X)$ is in fact a sublattice of $\mathcal{Z}$ with $0_{R(X)} = \mathrm{cl}(\mathrm{cyc}(X))$ and $1_{R(X)} = \mathrm{cyc}(\mathrm{cl}(X))$.

Lemma 3.2.1. *For a matroid M on E and a set $X \subseteq E$, the ordered set $R(X)$ is closed under the meet and join operations of $\mathcal{Z}$, i.e., $R(X)$ is a sublattice of $\mathcal{Z}$.*

Proof. Let $\wedge$ and $\vee$ be the meet and join operations of the lattice $\mathcal{Z}$. For $Z_1, Z_2 \in R(X)$, we need to show that both $Z_1 \wedge Z_2$ and $Z_1 \vee Z_2$ are in $R(X)$. We have $Z_1 \wedge Z_2 = \mathrm{cyc}(Z_1 \cap Z_2)$ and $Z_1 \vee Z_2 = \mathrm{cl}(Z_1 \cup Z_2)$ by Lemma 2.5.2 (4).

For the meet operation, we have the following sequence of containments,

$$Z_1 \wedge Z_2 \subseteq \mathrm{cyc}(X \cup (Z_1 \wedge Z_2)) \subseteq \mathrm{cyc}(X \cup Z_1) \cap \mathrm{cyc}(X \cup Z_2) = Z_1 \cap Z_2.$$

By taking the cyclic part of each term, we get $\mathrm{cyc}(X \cup (Z_1 \wedge Z_2)) = Z_1 \wedge Z_2$. On the other hand, we have

$$Z_1 \wedge Z_2 \subseteq Z_1 \cap Z_2 \subseteq \mathrm{cl}(X).$$

Thus, $Z_1 \wedge Z_2 \in R(X)$ by Corollary 3.1.8.

Similarly, for the join operation, we have the following sequence of containments,

$$Z_1 \cup Z_2 = \mathrm{cl}(X \cap Z_1) \cup \mathrm{cl}(X \cap Z_2) \subseteq \mathrm{cl}(X \cap (Z_1 \vee Z_2)) \subseteq Z_1 \vee Z_2.$$

By taking the closure of each term, we get $Z_1 \vee Z_2 = \mathrm{cl}(X \cap (Z_1 \vee Z_2))$. On the other hand, we have

$$\mathrm{cyc}(X) \subseteq Z_1 \cup Z_2 \subseteq Z_1 \vee Z_2.$$

Thus, $Z_1 \vee Z_2 \in R(X)$ by Theorem 3.1.6.

Alternatively, we can show that closure under the meet operation implies closure under the join operation, and vise versa, by duality (Theorem 3.1.7). □

Now we find an expression for the maximum and minimum elements in the lattice $R(X)$.

Lemma 3.2.2. *For a matroid M on E and a set $X \subseteq E$, we have $0_{R(X)} = \mathrm{cl}(\mathrm{cyc}(X))$ and $1_{R(X)} = \mathrm{cyc}(\mathrm{cl}(X))$.*

Proof. We need to show that both $\mathrm{cl}(\mathrm{cyc}(X))$ and $\mathrm{cyc}(\mathrm{cl}(X))$ are in $R(X)$, and, for any $Z \in R(X)$, the containment $\mathrm{cl}(\mathrm{cyc}(X)) \subseteq Z \subseteq \mathrm{cyc}(\mathrm{cl}(X))$ holds.

The proof of Lemma 3.1.2 shows that $\mathrm{cl}(\mathrm{cyc}(X))$ is in $R(X)$. For $\mathrm{cyc}(\mathrm{cl}(X)) \in \mathcal{Z}$, we know that it is in $\mathcal{Z}$ by Lemma 2.5.2 (2). Notice that

$$\mathrm{cyc}(\mathrm{cl}(X)) \subseteq X \cup \mathrm{cyc}(\mathrm{cl}(X)) \subseteq \mathrm{cl}(X).$$

So we get $\mathrm{cyc}(X \cup \mathrm{cyc}(\mathrm{cl}(X))) = \mathrm{cyc}(\mathrm{cl}(X))$ by taking the cyclic part of each term. Hence $\mathrm{cyc}(\mathrm{cl}(X)) \in R(X)$ by Corollary 3.1.8.

For any $Z \in R(X)$, we have $\mathrm{cyc}(X) \subseteq Z$ by Theorem 3.1.6. By taking the closure of both sides of the inclusion, we get $\mathrm{cl}(\mathrm{cyc}(X)) \subseteq Z$. Similarly, we have $Z \subseteq \mathrm{cl}(X)$ by Corollary 3.1.8. By taking the cyclic part of both sides of the inclusion, we obtain $Z \subseteq \mathrm{cyc}(\mathrm{cl}(X))$. Hence $0_{R(X)} = \mathrm{cl}(\mathrm{cyc}(X))$ and $1_{R(X)} = \mathrm{cyc}(\mathrm{cl}(X))$. □

From Lemmas 3.2.1 and 3.2.2, we have the following main result of this section.

Theorem 3.2.3. *For a matroid M on E and a set $X \subseteq E$, the ordered set $R(X)$ is a sublattice of $\mathcal{Z}$ with $0_{R(X)} = \mathrm{cl}(\mathrm{cyc}(X))$ and $1_{R(X)} = \mathrm{cyc}(\mathrm{cl}(X))$.*

The next corollary follows immediately from this.

Corollary 3.2.4. *For a matroid M on E, we have $R(\mathrm{cyc}(X)) = \{0_{R(X)}\}$ and $R(\mathrm{cl}(X)) = \{1_{R(X)}\}$ for all $X \subseteq E$.*

Now we discuss some properties of $0_{R(X)}$ and $1_{R(X)}$. The first corollary follows since the closure and the cyclic operators are both monotonic.

Corollary 3.2.5. *If $X \subseteq Y \subseteq E$, then $0_{R(X)} \leq 0_{R(Y)}$ and $1_{R(X)} \leq 1_{R(Y)}$.*

Lemma 3.2.6. *For a matroid M on E and a set $X \subseteq E$,*

(1) $\mathrm{cyc}(X) \in \mathcal{Z}$ *if and only if* $0_{R(X)} \subseteq X$, *and*

(2) $\mathrm{cl}(X) \in \mathcal{Z}$ *if and only if* $X \subseteq 1_{R(X)}$.

Proof. If $\mathrm{cyc}(X) \in \mathcal{Z}$, then $0_{R(X)} = \mathrm{cyc}(X) \subseteq X$. Conversely, if $0_{R(X)} \subseteq X$, then we have $0_{R(X)} = \mathrm{cyc}(0_{R(X)}) \subseteq \mathrm{cyc}(X)$. Hence $\mathrm{cyc}(X) = 0_{R(X)} \in \mathcal{Z}$. Similarly, if $\mathrm{cl}(X) \in \mathcal{Z}$, then $X \subseteq \mathrm{cl}(X) = 1_{R(X)}$. Conversely, if $X \subseteq 1_{R(X)}$, then $\mathrm{cl}(X) \subseteq \mathrm{cl}(1_{R(X)}) = 1_{R(X)}$. Hence $\mathrm{cl}(X) = 1_{R(X)} \in \mathcal{Z}$. □

Lemma 3.2.7. *For a matroid M on E and a set $X \subseteq E$,*

(1) $X \cap 0_{R(X)} = \mathrm{cyc}(X)$, *and*

(2) $X \cup 1_{R(X)} = \mathrm{cl}(X)$.

The next lemma identifies some cases in which the rank sublattice remains the same when we modify the subset X of E.

Lemma 3.2.8. *For a matroid M on E and a set $X \subseteq E$,*

(1) $R(X) = R(X \cup 0_{R(X)})$,

(2) $R(X) = R(X \cap 1_{R(X)})$, *and*

(3) $R(X) = R((X \cap 1_{R(X)}) \cup 0_{R(X)})$.

Proof. For (1), we take the closure of both sides of the inclusion $\mathrm{cl}(X) \supseteq X \cup 0_{R(X)}$ to obtain $\mathrm{cl}(X) = \mathrm{cl}(X \cup 0_{R(X)})$. Then we have $1_{R(X)} = 1_{R(X \cup 0_{R(X)})}$ by taking the cyclic parts of each side. Also we have $0_{R(X)} = 0_{R(X \cup 0_{R(X)})}$ because $0_{R(X)} = \mathrm{cyc}(X \cup 0_{R(X)})$ by Corollary 3.1.8. For $Z \in \mathcal{Z}$ with $0_{R(X)} \leq Z \leq 1_{R(X)}$, we have $\mathrm{cyc}(X \cup Z) = \mathrm{cyc}(X \cup 0_{R(X)} \cup Z)$. Hence $R(X) = R(X \cup 0_{R(X)})$ by Corollary 3.1.8.

Statement (2) follows from Theorem 3.1.7 using duality.

Statement (3) follows from (1) and (2) because

$$\begin{aligned} R(X) &= R(X \cap 1_{R(X)}) \\ &= R((X \cap 1_{R(X)}) \cup 0_{R(X \cap 1_{R(X)})}) \\ &= R((X \cap 1_{R(X)}) \cup 0_{R(X)}). \end{aligned}$$ □

From Theorem 3.1.7, the sets $0_{R(X)}$ and $1_{R^*(E-X)}$ are related by duality as the next lemma states.

Lemma 3.2.9. *For a matroid M on E and a set $X \subseteq E$,*

(1) $0_{R(X)} = E - 1_{R^*(E-X)}$, *and*

(2) $1_{R(X)} = E - 0_{R^*(E-X)}$.

Now we introduce two more formulations of $R(X)$ using $0_{R(Y)}$ and $1_{R(Y)}$, where Y is a superset of X and a subset of X respectively.

Lemma 3.2.10. *For a matroid M on E and a set $X \subseteq E$, we have*

$$R(X) = \{1_{R(Y)} : \mathrm{cyc}(X) \subseteq Y \subseteq X\}.$$

Proof. We show the equivalence with the formulation of $R(X)$ in Theorem 3.1.6,

$$R(X) = \{Z \in \mathcal{Z} : \mathrm{cyc}(X) \subseteq Z = \mathrm{cl}(X \cap Z)\}.$$

Suppose $Z \in R(X)$. Let $Y = X \cap Z$. Then we have $\mathrm{cyc}(X) \subseteq Y \subseteq X$ and

$$Z = \mathrm{cyc}(Z) = \mathrm{cyc}(\mathrm{cl}(X \cap Z)) = \mathrm{cyc}(\mathrm{cl}(Y)) = 1_{R(Y)}.$$

On the other hand, if $\mathrm{cyc}(X) \subseteq Y \subseteq X$, then $X-Y \subseteq \mathrm{isth}(X)$, so $r(X) = r(Y)+|X-Y|$. Therefore $r(X) = r(1_{R(Y)}) + |Y - 1_{R(Y)}| + |X - Y|$. Thus, $r(X) \geq r(1_{R(Y)}) + |X - 1_{R(Y)}|$ since $X - 1_{R(Y)} \subseteq (X - Y) \cup (Y - 1_{R(Y)})$. The opposite inequality holds by Corollary 2.4.9, so equality holds, that is, $1_{R(Y)} \in R(X)$. □

The following corollary is the dual of Lemma 3.2.10.

Corollary 3.2.11. *For a matroid M on E and a set $X \subseteq E$, we have*

$$R(X) = \{0_{R(Y)} : X \subseteq Y \subseteq \mathrm{cl}(X)\}.$$

Proof. We prove this by duality. In Lemma 3.2.10, replace M, X, and Y by M^*, $E - X$, and $E - Y$, respectively. We obtain

$$R^*(E - X) = \{1_{R^*(E-Y)} : \mathrm{cyc}^*(E - X) \subseteq E - Y \subseteq E - X\}.$$

Then we have the following series of equalities,

$$\begin{aligned} R(X) &= \{E - Z : Z \in R^*(E - X)\} \quad \text{(by Theorem 3.1.7)} \\ &= \{E - 1_{R^*(E-Y)} : \mathrm{cyc}^*(E - X) \subseteq E - Y \subseteq E - X\} \\ &= \{0_{R(Y)} : X \subseteq Y \subseteq \mathrm{cl}(X)\}. \quad \text{(by Lemma 2.4.6 and 3.2.9)} \end{aligned}$$

□

The following theorem shows a few connections among $R(X \cup e)$, $R(X - e)$ and $R(X)$ for $e \in E(M)$. It will be useful to note that for $F \in \mathcal{F}$ and $U \in \mathcal{U}$, we have $n(F) = n(F \cup e)$ and $r(U) = r(U - e)$ for each $e \in E(M)$.

Theorem 3.2.12. *For a matroid M on E, a set $X \subseteq E$, and an element $e \in E$, we have that $0_{R(X \cup e)}$ and $1_{R(X-e)}$ are in $R(X)$.*

Proof. We first show that $0_{R(X \cup e)} \in R(X)$. The case that $e \in X$ is trivial, so assume that $e \notin X$. Let $Y = \mathrm{cyc}(X \cup e) - e$. By Lemma 3.2.10, it suffices to show that $0_{R(X \cup e)} = 1_{R(Y)}$ and $\mathrm{cyc}(X) \subseteq Y \subseteq X$. Since $r(\mathrm{cyc}(X \cup e)) = r(Y)$, we have $\mathrm{cl}(\mathrm{cyc}(X \cup e)) = \mathrm{cl}(Y)$. By taking the cyclic part of each term, we obtain $0_{R(X \cup e)} = 1_{R(Y)}$. Furthermore, since

$e \notin \mathrm{cyc}(X)$, we have

$$\mathrm{cyc}(X) \subseteq Y \subseteq (X \cup e) - e = X.$$

Similarly, we show that $1_{R(X-e)} \in R(X)$. The case that $e \notin X$ is trivial, so assume that $e \in X$. Let $Y = \mathrm{cl}(X-e) \cup e$. By Corollary 3.2.11, it suffices to show that $1_{R(X-e)} = 0_{R(Y)}$ and $X \subseteq Y \subseteq \mathrm{cl}(X)$. Since $n(\mathrm{cl}(X-e)) = n(Y)$, we have $\mathrm{cyc}(\mathrm{cl}(X-e)) = \mathrm{cyc}(Y)$. By taking the closure of both sides, we obtain $1_{R(X-e)} = 0_{R(Y)}$. Furthermore, since $e \in \mathrm{cl}(X)$, we have

$$X = (X-e) \cup e \subseteq Y \subseteq \mathrm{cl}(X). \qquad \square$$

Chapter 4

Fundamental Transversal Matroids and Lattices of Cyclic Flats

In this chapter, we cover two topics about fundamental transversal matroids as applications of rank sublattices $R(X)$. In Section 4.1, we discuss the definition and some basic properties of fundamental transversal matroids. They are mostly well-known background. In Section 4.2, we determine when $R(X)$ is an interval in $\mathcal{Z}(M)$ (Corollary 4.2.8 and Theorem 4.2.9). In Section 4.3, using the properties of rank sublattices, we characterize which functions on a lattice L arise from the rank functions of fundamental transversal matroids whose lattices of cyclic flats are isomorphic to L (Theorem 4.3.3).

4.1 Fundamental Transversal Matroids

Fundamental transversal matroids have many attractive properties. We focus on the geometric view of transversal matroids, which is largely due to Tom Brylawski [7]. We first define transversal matroids [14, 11] and then discuss fundamental transversal matroids, which is the special case of interest.

Let $\mathcal{A} = (A_j : j \in J)$ be a set system, that is, a multiset of subsets of a finite set S. A *transversal* (or system of distinct representatives) of $\mathcal{A}$ is a set $\{x_j : j \in J\}$ of $|J|$ distinct elements such that $x_j \in A_j$ for all j in J. A *partial transversal* of $\mathcal{A}$ is a transversal of a set system of the form $(A_k : k \in K)$ with K a subset of J. It is known that the partial transversals of a set system $\mathcal{A} = (A_j : j \in J)$ are the independent sets of a matroid on S. A *transversal matroid* is a matroid whose independent sets are the partial transversals of some set system $\mathcal{A} = (A_j : j \in J)$. We say that $\mathcal{A}$ is a *presentation* of the transversal matroid. The presentation is not generally unique. Notice that the bases of a transversal matroids are the maximal partial transversals of $\mathcal{A}$.

We now define fundamental transversal matroids in terms of cyclic flats [3].

Definition 4.1.1. A matroid M is a *fundamental transversal matroid* if there is a basis B of M such that each cyclic flat of M is spanned by a subset of B. We call such a basis B a *fundamental basis* of M.

As the name suggests, fundamental transversal matroids are special types of transversal matroids. The connection comes from the following geometric view point due to Brylawski [7]. A matroid is transversal if and only if it has a geometric representation on a simplex where all dependencies arise from the faces of the simplex. Thus, parallel points can occur only at vertices, lines with more than two points must be on edges of the simplex; etc.

For example, let $\mathcal{A} = (A_1, A_2, A_3, A_4)$ be the set system with $A_1 = \{1, 2, 3\}$, $A_2 = \{4, 5, 6, 9\}$, $A_3 = \{6, 7, 8, 9\}$, and $A_4 = \{2, 3, 8, 9\}$. To obtain a geometric representation of the matroid, we first consider the simplex with four vertices, A_1 through A_4. Then we put each element of matroid on the face spanned by the vertices that the element is in. We obtain the matroid M_1 in Figure 4.1. It shows the dependencies of each element in terms of vertices.

A fundamental transversal matroid is a transversal matroid such that there is at least one element of the matroid at each vertex in some simplex representation. In this case, a set of elements, one from each vertex, is a fundamental basis. Note that a basis of a fundamental transversal matroid is not necessarily fundamental.

For example, in Figure 4.1, both matroids M_1 and M_2 are transversal, but only M_2 is fundamental transversal. In M_2, since 4 and 5 are parallel, the four-element bases $\{1, 4, 7, 10\}$ and $\{1, 5, 7, 10\}$ are both fundamental; however, the basis $\{1, 3, 6, 7\}$ is not. Each cyclic flat

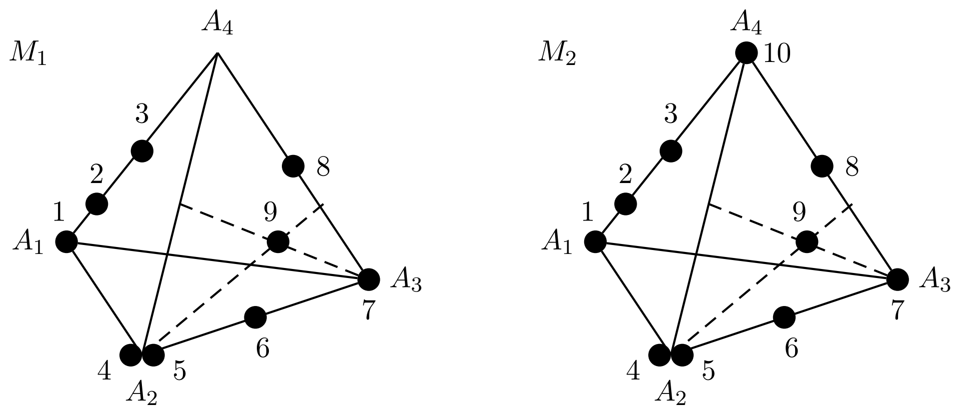

Figure 4.1: Transversal and fundamental transversal matroids

of M_2 is spanned by a subset of the fundamental basis $\{1,4,7,10\}$, namely, $\emptyset$, $\{4\}$, $\{1,10\}$, $\{4,7\}$, $\{7,10\}$, $\{1,4,10\}$, $\{1,7,10\}$, $\{4,7,10\}$, and $\{1,4,7,10\}$.

For a matroid M and a basis B, observe that M is a fundamental transversal matroid and B is a fundamental basis if and only if $Z = \mathrm{cl}(B \cap Z)$ for every cyclic flat Z of M (equivalently, $Z = \mathrm{cyc}(B \cup Z)$ for every cyclic flat Z of M). The next two lemmas discuss basic properties of fundamental transversal matroids.

The first lemma shows the result by Las Vergnas [12] that duality preserves a matroid being fundamental transversal. Moreover, the fundamental bases of the dual matroid are the complements of those in the original matroid.

Lemma 4.1.2. *If M is a fundamental transversal matroid with a fundamental basis B, then M^* is a fundamental transversal matroid with a fundamental basis $E - B$.*

Proof. By Lemma 2.2.1 (3), any cyclic flat of M^* has the form $E - Z$ where Z is some cyclic flat of M. Since $Z = \mathrm{cyc}(B \cup Z)$, we have

$$\begin{aligned} E - Z &= E - \mathrm{cyc}(B \cup Z) \\ &= \mathrm{cl}^*(E - (B \cup Z)) \\ &= \mathrm{cl}^*((E - B) \cap (E - Z)). \end{aligned}$$

Hence $E - B$ is a fundamental basis of M^*. □

The second lemma shows that being fundamental transversal is preserved when taking certain minors. Furthermore, some fundamental bases of these minors are subsets of a fundamental basis of the original matroid.

Lemma 4.1.3. *For a fundamental transversal matroid M with a fundamental basis B and cyclic flats Z_0 and Z_1 of M, the minors $M|Z_1$ and M/Z_0 are fundamental transversal matroids with fundamental bases $B \cap Z_1$ and $B - Z_0$, respectively. Furthermore, if $Z_0 \subseteq Z_1$, then $(M|Z_1)/Z_0$ is a fundamental transversal matroid with a fundamental basis $B \cap (Z_1 - Z_0)$.*

Proof. Since $B \cap Z_1$ is independent and spans Z_1, it is a basis of $M|Z_1$. Let $Z \in \mathcal{Z}(M|Z_1)$. Then $Z \in \mathcal{Z}(M)$ with $Z \subseteq Z_1$ by Lemma 2.3.1 (5). By Lemma 2.4.5, we have

$$\mathrm{cl}_{M|Z_1}((B \cap Z_1) \cap Z) = \mathrm{cl}_{M|Z_1}(B \cap Z) = \mathrm{cl}_M(B \cap Z) \cap Z_1 = Z \cap Z_1 = Z.$$

Hence $B \cap Z_1$ is a fundamental basis of $M|Z_1$.

Using this result, we now show that $B - Z_0$ is a fundamental basis of M/Z_0. By Lemma 4.1.2, we know that M^* is a fundamental transversal matroid with fundamental basis $E - B$. Since $E - Z_0 \in \mathcal{Z}^*$, the restriction $M^*|(E - Z_0)$ is a fundamental transversal matroid with a fundamental basis $(E - B) - Z_0$ on the ground set $E - Z_0$. By Lemma 4.1.2 again, $M/Z_0 = (M^*|(E - Z_0))^*$ is a fundamental transversal matroid with a fundamental basis $B - Z_0$.

The last part about $(M|Z_1)/Z_0$ is clear using the results above. □

4.2 Intervals in the Lattice of Cyclic Flats

As we have seen in Section 3.2, the suborder $R_M(X)$ is indeed a sublattice of $\mathcal{Z}(M)$ for any matroid M and subset $X \subseteq E(M)$; hence, we are justified in calling it a rank sublattice. For a fixed matroid M, one natural question is how to characterize the sublattices of $\mathcal{Z}$ that arise as the rank sublattices $R(X)$ for some X. I have not been able to answer this question fully; it appears to be quite difficult. However this question can be answered in the special case that the rank sublattice $R(X)$ is an interval.

Our first theorem characterizes when $R(X)$ is the entire lattice $\mathcal{Z}$, which is the largest interval of $\mathcal{Z}$. Later, we investigate when $R(X)$ is a proper interval of $\mathcal{Z}$.

Theorem 4.2.1. *For a matroid M on E and $X \subseteq E$, we have $R(X) = \mathcal{Z}$ if and only if* $\mathrm{cyc}(X) \subseteq \mathrm{cl}(\emptyset)$, $\mathrm{cl}(X) \supseteq \mathrm{cyc}(E)$, *and M is a fundamental transversal matroid such that* $\mathrm{isth}(X)$ *is contained in some fundamental basis of M. Moreover, if X is a basis of M, then $R(X) = \mathcal{Z}$ if and only if X is a fundamental basis of M.*

Proof. Observe that $0_{R(X)} = 0_{\mathcal{Z}}$ if and only if $\mathrm{cyc}(X) \subseteq \mathrm{cl}(\emptyset)$; likewise, $1_{R(X)} = 1_{\mathcal{Z}}$ if and only if $\mathrm{cl}(X) \supseteq \mathrm{cyc}(E)$. If both $0_{R(X)} = 0_{\mathcal{Z}}$ and $1_{R(X)} = 1_{\mathcal{Z}}$, then

$$\mathrm{cl}(\mathrm{isth}(X)) \supseteq \mathrm{cyc}(E) \tag{4.1}$$

because $\mathrm{cyc}(X)$ is a set of loops and $\mathrm{isth}(X)$ spans $\mathrm{cl}(X)$.

For the forward direction, suppose that $R(X) = \mathcal{Z}$. Let $B = \mathrm{isth}(X) \cup \mathrm{isth}(E)$. Evidently $\mathrm{isth}(X) \subseteq B$. Also B spans E by (4.1). Moreover, B is independent. Hence B is a basis. It remains to show that $\mathrm{cyc}(B \cup Z) \subseteq Z$ for every cyclic flat Z; then B is a fundamental basis of M. Since elements in $\mathrm{isth}(E)$ cannot be in any circuits, we have

$$\mathrm{cyc}(B \cup Z) = \mathrm{cyc}(\mathrm{isth}(X) \cup Z) \subseteq \mathrm{cyc}(X \cup Z) = Z.$$

(The last equality holds by Corollary 3.1.8.)

For the converse, assume that (a) $\mathrm{cyc}(X) \subseteq \mathrm{cl}(\emptyset)$, (b) $\mathrm{cl}(X) \supseteq \mathrm{cyc}(E)$, and (c) $\mathrm{isth}(X)$ is contained in some fundamental basis of M. Let $X' = \mathrm{isth}(X) = X - \mathrm{cl}(\emptyset)$. Assumptions (b) and (c) give $r(Z) = |X' \cap Z|$ for each $Z \in \mathcal{Z}(M)$. Thus each $Z \in \mathcal{Z}(M)$ is in $R(X)$ since $r(X) = |X'| = |X' \cap Z| + |X' - Z| = r(Z) + |X - Z|$.

If X is a basis of M, then we have $\mathrm{cyc}(X) = \emptyset$, $\mathrm{cl}(X) = E$, and $\mathrm{isth}(X) = X$. Thus, $R(X) = \mathcal{Z}$ if and only if X is a fundamental basis of M. □

In particular, when M has no loops and no isthmuses, we have the following result.

Corollary 4.2.2. *Let M be a matroid on E with no loops and no isthmuses and $X \subseteq E$. Then $R(X) = \mathcal{Z}$ if and only if M is a fundamental transversal matroid and X is a fundamental basis.*

It is known that every lattice is isomorphic to $\mathcal{Z}(M)$ for some fundamental transversal matroid M [3, Theorem 2.1]. Combining this result with Corollary 4.2.2, we have a stronger statement; that is, for a given lattice L, there is a fundamental transversal matroid M and a subset X of $E(M)$ such that $\mathcal{Z}(M) = R_M(X) \cong L$. In general, there are many possibilities for such fundamental transversal matroids.

Now we would like to characterize all subsets X of E for which the rank sublattice $R(X)$ is an interval of the lattice $\mathcal{Z}$. Clearly, this interval has to be of the form $[0_{R(X)}, 1_{R(X)}]$. Furthermore, for a given pair of comparable cyclic flats Z_0 and Z_1, we would like to characterize all subsets X of E such that $R(X) = [Z_0, Z_1]$. This is a generalization of Theorem 4.2.1, which is the case where $Z_0 = 0_{\mathcal{Z}}$ and $Z_1 = 1_{\mathcal{Z}}$. The minor $(M|Z_1)/Z_0$ of M is key for this investigation.

Lemma 4.2.3. *Let M be a matroid, $U \in \mathcal{U}(M)$, and $F \in \mathcal{F}(M)$. The matroid $M|U$ has no isthmuses and M/F has no loops. In particular, if Z_0 and Z_1 are cyclic flats of M with $Z_0 \leq Z_1$, then the minor $(M|Z_1)/Z_0$ has no loops and no isthmuses.*

The next lemma gives an isomorphism between the lattice of cyclic flats of the minor $(M|Z_1)/Z_0$ and the interval $[Z_0, Z_1]$ in the lattice of cyclic flats of M.

Lemma 4.2.4. *Let M be a matroid and $Z_0, Z_1 \in \mathcal{Z}(M)$ with $Z_0 \leq Z_1$. Let M' be the minor $(M|Z_1)/Z_0$ of M. The map $\varphi : \mathcal{Z}(M') \to \mathcal{Z}(M)$ given by $\varphi(Z) = Z \cup Z_0$ is a lattice embedding onto the interval $[Z_0, Z_1]$ in $\mathcal{Z}(M)$. Furthermore, for $X \subseteq Z_1 - Z_0$, the restriction of φ to the sublattice $R_{M'}(X)$ is a lattice isomorphism from $R_{M'}(X)$ to $R_M(X \cup Z_0)$.*

Proof. By Lemma 2.3.1 (5) and (6), we have

$$\begin{aligned}
\mathcal{Z}(M') &= \mathcal{Z}((M|Z_1)/Z_0) \\
&= \{Z \subseteq E - Z_0 : Z \cup Z_0 \in \mathcal{Z}(M|Z_1)\} \\
&= \{Z \subseteq E - Z_0 : Z \cup Z_0 \subseteq Z_1 \text{ and } Z \cup Z_0 \in \mathcal{Z}(M)\} \\
&= \{Z \subseteq Z_1 - Z_0 : Z \cup Z_0 \in \mathcal{Z}(M)\} \\
&= \{Z \subseteq Z_1 - Z_0 : Z \cup Z_0 \in [Z_0, Z_1]_{\mathcal{Z}(M)}\}.
\end{aligned}$$

Therefore, we have

$$Z \in \mathcal{Z}(M') \;\Leftrightarrow\; Z \subseteq Z_1 - Z_0 \text{ and } Z \cup Z_0 \in [Z_0, Z_1]_{\mathcal{Z}(M)}.$$

Hence φ is an isomorphism between $\mathcal{Z}(M')$ and $[Z_0, Z_1]_{\mathcal{Z}(M)}$.

For $X \subseteq Z_1 - Z_0$, we use Lemma 3.1.11 (1) and (2) to get

$$\begin{aligned} R_{M'}(X) &= R_{(M|Z_1)/Z_0}(X) \\ &= \{Z \subseteq E - Z_0 : Z \cup Z_0 \in R_{M|Z_1}(X \cup Z_0)\} \\ &= \{Z \subseteq E - Z_0 : Z \cup Z_0 \subseteq Z_1 \text{ and } Z \cup Z_0 \in R_M(X \cup Z_0)\} \\ &= \{Z \subseteq Z_1 - Z_0 : Z \cup Z_0 \in R_M(X \cup Z_0)\}. \end{aligned}$$

This means

$$Z \in R_{M'}(X) \;\Leftrightarrow\; Z \subseteq Z_1 - Z_0 \text{ and } Z \cup Z_0 \in R_M(X \cup Z_0)$$

i.e., φ is a bijection. Clearly, φ is order-preserving. Hence it is a lattice isomorphism between $R_{M'}(X)$ and $R_M(X \cup Z_0)$. □

Using Theorem 4.2.1 and 4.2.4, we will characterize when $R(X)$ is the interval $[Z_0, Z_1]$ of $\mathcal{Z}$. This result is comprised of two parts. First, we characterize when $Z_0 = 0_{R(X)}$ and $Z_1 = 1_{R(X)}$ for a given pair of cyclic flats $Z_0 \leq Z_1$ (Theorem 4.2.6). Second, we characterize when $R(X)$ is an interval $[0_{R(X)}, 1_{R(X)}]$ of $\mathcal{Z}$ (Theorem 4.2.7).

When $Z_0 = 0_{R(X)}$ and $Z_1 = 1_{R(X)}$ for some X, the next lemma identifies a basis of $(M|Z_1)/Z_0$.

Lemma 4.2.5. *If M is a matroid on E, $X \subseteq E$, $Z_0 = 0_{R(X)}$, and $Z_1 = 1_{R(X)}$, then $B' = X \cap (Z_1 - Z_0)$ is a basis of the minor $M' = (M|Z_1)/Z_0$ of M.*

Proof. Since

$$R(B' \cup Z_0) = R((X \cap Z_1) \cup Z_0) = R(X)$$

by Lemma 3.2.8 (3), we have $Z_0, Z_1 \in R(B' \cup Z_0)$. By the lattice isomorphism in Lemma 4.2.4, we have $\emptyset, Z_1 - Z_0 \in R_{M'}(B')$. Hence B' is a basis of M' by Lemma 3.1.4. □

For a set X, we characterize when two given cyclic flats are $0_{R(X)}$ and $1_{R(X)}$.

Theorem 4.2.6. *Let M be a matroid on E, $X \subseteq E$, and $Z_0, Z_1 \in \mathcal{Z}(M)$ with $Z_0 \leq Z_1$. Also let $M' = (M|Z_1)/Z_0$ and $B' = X \cap (Z_1 - Z_0)$. Both $0_{R(X)} = Z_0$ and $1_{R(X)} = Z_1$ hold if and only if*

(i) *B' is a basis of M',*

(ii) *$X \cap Z_0 \in \mathcal{U}(M)$,*

(iii) *$\mathrm{cl}(X \cap Z_0) = Z_0$,*

(iv) *$X \cup Z_1 \in \mathcal{F}(M)$, and*

(v) *$\mathrm{cyc}(X \cup Z_1) = Z_1$.*

In particular, if $X = B' \cup Z_0$, then both $0_{R(X)} = Z_0$ and $1_{R(X)} = Z_1$ hold if and only if B' is a basis of M'.

Proof. Suppose $0_{R(X)} = Z_0$ and $1_{R(X)} = Z_1$. By Lemma 4.2.5, we have that B' is a basis of M', i.e., condition (i) holds. Conditions (ii) and (iv) hold since Lemma 3.2.7 gives $X \cap Z_0 = \mathrm{cyc}(X)$ and $X \cup Z_1 = \mathrm{cl}(X)$. By taking the closure and the cyclic parts of these equalities, we obtain conditions (iii) and (v).

For the converse, suppose that conditions (i)–(v) hold. Notice that M' has no loops and no isthmuses by Lemma 4.2.3. It follows that $\emptyset, Z_1 - Z_0 \in R_{M'}(B')$ by condition (i) and Lemma 3.1.4. By the lattice isomorphism $\varphi : R_{M'}(B') \to R(B' \cup Z_0)$ given by $\varphi(Z) = Z \cup Z_0$ from Lemma 4.2.4, we have $Z_0, Z_1 \in R(B' \cup Z_0)$. By Corollary 3.1.9, we have

$$\mathrm{cyc}(B' \cup Z_0) = Z_0 \quad \text{and} \tag{4.2}$$

$$\mathrm{cl}(B' \cup Z_0) = Z_1. \tag{4.3}$$

Using these equalities, we will show that $\mathrm{cyc}(X) = X \cap Z_0$ and $\mathrm{cl}(X) = X \cup Z_1$. Then by (iii) and (v), we obtain $0_{R(X)} = Z_0$ and $1_{R(X)} = Z_1$. For $\mathrm{cyc}(X) = X \cap Z_0$, it suffices to show that $\mathrm{cyc}(X) \subseteq Z_0$ by assumption (ii). We have

$$\mathrm{cyc}(X) \subseteq X \cap \mathrm{cyc}(X \cup Z_1) = X \cap Z_1 \subseteq B' \cup Z_0$$

by (v). By taking the cyclic sets of both sides of the inclusion and using (4.2), we obtain

$$\mathrm{cyc}(X) \subseteq \mathrm{cyc}(B' \cup Z_0) = Z_0.$$

Similarly, for $\mathrm{cl}(X) = X \cup Z_1$, it suffices to show that $\mathrm{cl}(X) \supseteq Z_1$ by assumption (iv). We have

$$\mathrm{cl}(X) \supseteq X \cup \mathrm{cl}(X \cap Z_0) = X \cup Z_0 \supseteq B' \cup Z_0$$

by (iii). By taking the closure of both sides of the inclusion and using (4.3), we get

$$\mathrm{cl}(X) \supseteq \mathrm{cl}(B' \cup Z_0) = Z_1. \qquad \square$$

Now we characterize the sets X for which $R(X)$ is an interval.

Theorem 4.2.7. *Let M be a matroid on E and $X \subseteq E$. Then $R(X)$ is an interval in $\mathcal{Z}(M)$ if and only if $(M|1_{R(X)})/0_{R(X)}$ is a fundamental transversal matroid with a fundamental basis $X \cap (1_{R(X)} - 0_{R(X)})$.*

Proof. Let $Z_0 = 0_{R(X)}$, $Z_1 = 1_{R(X)}$, $M' = (M|Z_1)/Z_0$, and $B' = X \cap (Z_1 - Z_0)$. (Then B' is a basis of M' by Lemma 4.2.5.) For the lattice isomorphism φ in Lemma 4.2.4, we have $\varphi(\mathcal{Z}(M')) = [Z_0, Z_1]_{\mathcal{Z}(M)}$ and $\varphi(R_{M'}(B')) = R_M(B' \cup Z_0)$. Since

$$R_M(B' \cup Z_0) = R_M((X \cap Z_1) \cup Z_0) = R_M(X)$$

by Lemma 3.2.8 (3), we have

$$R_M(X) = [Z_0, Z_1]_{\mathcal{Z}(M)} \Leftrightarrow R_{M'}(B') = \mathcal{Z}(M').$$

Since M' has no loops and no isthmuses by Lemma 4.2.3, it follows from Corollary 4.2.2 that $R_{M'}(B') = \mathcal{Z}(M')$ if and only if M' is a fundamental transversal matroid and B' is a fundamental basis of M'. $\square$

Combining Theorems 4.2.6 and 4.2.7, we have the following main result of the section. This result tells us when $R(X)$ is an interval in the lattice of cyclic flats.

Corollary 4.2.8. *Let M be a matroid on E, X be a subset of E, and $Z_0, Z_1 \in \mathcal{Z}(M)$ with $Z_0 \leq Z_1$. Also let $M' = (M|Z_1)/Z_0$ and $B' = X \cap (Z_1 - Z_0)$. Then $R(X) = [Z_0, Z_1]_{\mathcal{Z}(M)}$ if and only if:*

(i) *M' is a fundamental transversal matroid with a fundamental basis B',*

(ii) *$X \cap Z_0 \in \mathcal{U}(M)$,*

(iii) *$\mathrm{cl}(X \cap Z_0) = Z_0$,*

(iv) *$X \cup Z_1 \in \mathcal{F}(M)$, and*

(v) *$\mathrm{cyc}(X \cup Z_1) = Z_1$.*

From this corollary, we can deduce a stronger result about fundamental transversal matroids. In Theorem 4.2.1, we saw that if M is fundamental transversal, then there is a subset X such that $R(X) = \mathcal{Z}$. Now we show that, for any interval in $\mathcal{Z}$, there is a subset X such that $R(X)$ is the interval.

Theorem 4.2.9. *Let M be a matroid. The following are equivalent:*

(1) *M is a fundamental transversal matroid,*

(2) *$R(X) = \mathcal{Z}(M)$ for some $X \subseteq E$, and*

(3) *for each pair of comparable cyclic flats $Z_0 \leq Z_1$, there is a subset $X \subseteq E$ such that $R(X) = [Z_0, Z_1]$ in $\mathcal{Z}(M)$.*

Proof. Evidently (3) implies (2). Also (2) implies (1) by Theorem 4.2.1. It remains to show that (1) implies (3).

Let B be a fundamental basis of M. For a given pair of cyclic flats $Z_0 \leq Z_1$, let $M' = (M|Z_1)/Z_0$, $B' = B \cap (Z_1 - Z_0)$, and $X = (B \cap Z_1) \cup Z_0$. Then M' is a fundamental transversal matroid with a fundamental basis B' by Lemma 4.1.3, where B' can be written as $B' = X \cap (Z_1 - Z_0)$. Since $X \cap Z_0 = Z_0$ and $X \cup Z_1 = Z_1$, the conditions (i)–(iv) in Corollary 4.2.8 hold. Hence $R(X) = [Z_0, Z_1]$ in $\mathcal{Z}(M)$. □

In the rest of this section, we prove various results concerning the cases where $|R(X)| = 1$ or $|R(X)| = 2$. For the case $|R(X)| = 1$, we have the following easy characterization of X.

Lemma 4.2.10. *Let M be a matroid on E and $X \subseteq E$. Then $|R(X)| = 1$ if and only if* $\mathrm{isth}(X) = \mathrm{isth}(\mathrm{cl}(X))$.

Proof. We have

$$\begin{aligned} r(\mathrm{cl}(\mathrm{cyc}(X))) + |\mathrm{isth}(X)| &= r(\mathrm{cyc}(X)) + |\mathrm{isth}(X)| \\ &= r(X) \\ &= r(\mathrm{cl}(X)) \\ &= r(\mathrm{cyc}(\mathrm{cl}(X))) + |\mathrm{isth}(\mathrm{cl}(X))|. \end{aligned}$$

It follows that $|\mathrm{isth}(X)| = |\mathrm{isth}(\mathrm{cl}(X))|$ if and only if $r(\mathrm{cl}(\mathrm{cyc}(X))) = r(\mathrm{cyc}(\mathrm{cl}(X)))$. Now $\mathrm{isth}(\mathrm{cl}(X)) \subseteq \mathrm{isth}(X)$, so $\mathrm{isth}(X) = \mathrm{isth}(\mathrm{cl}(X))$ if and only if $0_{R(X)} = 1_{R(X)}$, that is, $|R(X)| = 1$. □

Two special cases are worth mentioning.

Corollary 4.2.11. *Let $X \subseteq E$. Then $R(X) = \{\emptyset\}$ if and only if M has no loops and X is an independent flat. Also $R(X) = \{E\}$ if and only if M has no isthmuses and X is a spanning cyclic set.*

Proof. Suppose $0_{R(X)} = 1_{R(X)} = \emptyset$. Then $\mathrm{cl}(\mathrm{cyc}(X)) = \emptyset$ implies $\mathrm{cyc}(X) = \emptyset$. So M has no loops and X is independent. Also $\mathrm{cyc}(\mathrm{cl}(X)) = \emptyset$ implies

$$\mathrm{cl}(X) = \mathrm{isth}(\mathrm{cl}(X)) = \mathrm{isth}(X) = X$$

by Lemma 4.2.10. Hence X is a flat. For the converse, if X is an independent flat in the loopless matroid M, then

$$1_{R(X)} = \mathrm{cyc}(\mathrm{cl}(X)) = \mathrm{cyc}(X) = \emptyset.$$

The second statement is the dual of the first. □

The case $|R(X)| = 2$ is related to the family of uniform matroids, which we now define using spanning circuits.

Definition 4.2.12. A matroid M is *uniform* if and only if every circuit in M is spanning.

For each rank r and cardinality k, there is a unique uniform matroid $U_{r,k}$ up to isomorphism. The uniform matroid $U_{0,k}$ consists entirely of loops, whereas the uniform matroid $U_{k,k}$ consists entirely of isthmuses. The following result is clear from Lemma 2.5.2 (3).

Lemma 4.2.13. *A matroid M on E is uniform if and only if $\mathcal{Z}(M) \subseteq \{\emptyset, E\}$. Furthermore, $\mathcal{Z}(U_{0,k}) = \{E\}$, $\mathcal{Z}(U_{k,k}) = \{\emptyset\}$, and $\mathcal{Z}(U_{r,k}) = \{\emptyset, E\}$ for $0 < r < k$.*

It is easy to verify that uniform matroids are fundamental transversal matroids and all their bases are fundamental. However, the converse doesn't hold.

We first consider the special case that $R(X)$ is a two-element interval in $\mathcal{Z}(M)$, i.e., $R(X) = \{Z_0, Z_1\}$ with $Z_0 \lessdot Z_1$. Using the lattice isomorphism in Lemma 4.2.4 for uniform matroids, we have the following result.

Corollary 4.2.14. *For a matroid M, two cyclic flats Z_0 and Z_1 with $Z_0 < Z_1$, and the minor $M' = (M|Z_1)/Z_0$, we have that M' is uniform of positive rank and nullity if and only if $Z_0 \lessdot Z_1$ in $\mathcal{Z}(M)$.*

Proof. By Lemma 4.2.3, M' has no loops and no isthmuses. By Lemma 4.2.4, we have $\mathcal{Z}(M') \cong [Z_0, Z_1]$ given by the isomorphism $Z \mapsto Z \cup Z_0$. Hence, by Lemma 4.2.13, we have

$$\begin{aligned} \mathcal{Z}(M') = \{\emptyset, Z_1 - Z_0\} \quad &\Leftrightarrow \quad [Z_0, Z_1] = \{Z_0, Z_1\} \\ &\Leftrightarrow \quad Z_0 \lessdot Z_1. \end{aligned}$$ □

Corollary 4.2.15. *Let M be a matroid and Z_0, $Z_1 \in \mathcal{Z}(M)$ with $Z_0 \lessdot Z_1$. Then we have $R(X) = \{Z_0, Z_1\}$ if and only if $X \cap Z_0 \in \mathcal{U}(M)$, $\mathrm{cl}(X \cap Z_0) = Z_0$, $X \cup Z_1 \in \mathcal{F}(M)$, and $\mathrm{cyc}(X \cup Z_1) = Z_1$.*

Proof. Since $Z_0 \lessdot Z_1$, we know that $(M|Z_1)/Z_0$ is uniform by Corollary 4.2.14. Hence $(M|Z_1)/Z_0$ is fundamental transversal and each basis, in particular $X \cap (Z_1 - Z_0)$, is fundamental. Apply Corollary 4.2.8 to obtain the result. □

Next, we consider the general case of $|R(X)| = 2$. The next lemma follows from Lemmas 3.1.2 and 3.1.4. Note that it is just a mild reformulation of the definition.

Lemma 4.2.16. *Let M be a matroid on E with no loops and no isthmuses. Let B be a basis of M. Then $R(B) = \{\emptyset, E\}$ if and only if $|B - Z| > r(M) - r(Z)$ for all $Z \in \mathcal{Z}(M)$ with $\emptyset < Z < E$.*

Corollary 4.2.17. *For a matroid M on E and $X \subseteq E$, let $M' = (M|1_{R(X)})/0_{R(X)}$ and $B = X \cap (1_{R(X)} - 0_{R(X)})$ and assume that $0_{R(X)} \neq 1_{R(X)}$. We have $|R(X)| = 2$ if and only if $|B - Z| > r(M') - r_{M'}(Z)$ for all $Z \in \mathcal{Z}(M')$ with $\emptyset < Z < 1_{R(X)} - 0_{R(X)}$.*

Proof. Recall that

$$R(X) = R((X \cap 1_{R(X)}) \cup 0_{R(X)}) = R(B \cup 0_{R(X)})$$

by Lemma 3.2.8 (3). So we have the following series of equivalent statements:

$$\begin{aligned} |R(X)| = 2 \quad &\Leftrightarrow \quad R(B \cup 0_{R(X)}) = \{0_{R(X)}, 1_{R(X)}\} \\ &\Leftrightarrow \quad R_{M'}(B) = \{\emptyset, 1_{R(X)} - 0_{R(X)}\} \quad \text{(by Lemma 4.2.4)} \\ &\Leftrightarrow \quad |B - Z| > r(M') - r_{M'}(Z) \text{ for all } Z \in \mathcal{Z}(M') \\ &\qquad \text{with } \emptyset < Z < 1_{R(X)} - 0_{R(X)}. \end{aligned}$$

□

4.3 Rank Functions on Lattices of Cyclic Flats

In this section, given a lattice L, we characterize the functions ρ on L that arise from fundamental transversal matroids in the sense that there is a fundamental transversal matroid M and an isomorphism $\varphi : L \to \mathcal{Z}(M)$ with $\rho(x) = r(\varphi(x))$ for all $x \in L$.

In Theorem 4.3.3, we will show that such functions are precisely $\mathcal{Z}$-rank functions that we define in Definition 4.3.1. In order to show this theorem, we define another function d_ρ (Definition 4.3.4) and show that ρ and d_ρ can be recovered from one another (Theorem 4.3.6).

We restrict our attention to fundamental transversal matroids because there is a bijection between the set of $\mathcal{Z}$-rank functions and the set of $(|L|-1)$-tuples of positive integers. Thus it is relatively easy to express the set of all possible rank functions for a given fundamental transversal matroid. Two of the major accomplishments of this section using Theorem 4.3.3

are to identify the minimal value of such a rank function (Corollary 4.3.12) and the minimal cardinality of the ground set (Corollary 4.3.13).

Let us start with the definition of $\mathcal{Z}$-rank functions. Recall that $\bigwedge(S) = \bigwedge_{x\in S} x$ and $\bigvee(S) = \bigvee_{x\in S} x$.

Definition 4.3.1. Let L be a lattice. A function $\rho : L \to \mathbb{Z}$ is a *$\mathcal{Z}$-rank function* if

(ZR1) $\rho(0_L) = 0$,

(ZR2) $\rho(x) < \rho(y)$ for all $x < y$ in L, and

(ZR3) $\rho\left(\bigwedge(S)\right) \le \sum_{T\subseteq S}(-1)^{|T|+1}\rho\left(\bigvee(T)\right)$, for all nonempty subsets $S \subseteq L$.

The inequality in (ZR3) holds trivially with equality when $|S| = 1$ and it reduces to $\rho(x) + \rho(y) \ge \rho(x \vee y) + \rho(x \wedge y)$ when $|S| = 2$; that is, ρ is semi-modular. We call the inequality in (ZR2) *monotonicity*, and the inequality in (ZR3) the *Mason-Ingleton inequality.*

The original Mason-Ingleton inequality [11][6, Theorem 1.1] is the following theorem, which is Ingleton's refinement of Mason's characterization of transversal matroids. The different context should prevent any possible confusion.

Theorem 4.3.2. *A matroid is transversal if and only if for all nonempty sets $\mathcal{A}$ of cyclic flats,*

$$r\left(\bigcap(\mathcal{A})\right) \le \sum_{\mathcal{A}'\subseteq\mathcal{A}}(-1)^{|\mathcal{A}'|+1}r\left(\bigcup(\mathcal{A}')\right).$$

Our main result of this section is the following theorem.

Theorem 4.3.3. *Let L be a lattice and $\rho : L \to \mathbb{Z}$ be a function. Then ρ is a $\mathcal{Z}$-rank function if and only if there is a fundamental transversal matroid M with a lattice isomorphism $\varphi : L \to \mathcal{Z}(M)$ satisfying $\rho = r\varphi$, where r is the rank function of M.*

In order to prove this theorem, we will define a function d_ρ (Definition 4.3.4) that captures the "slack" of the rank difference between an element and its cover(s). We show that ρ and d_ρ determine one another (Theorem 4.3.5). Also we show that ρ is a $\mathcal{Z}$-rank function if and only if d_ρ is nonnegative (Theorem 4.3.6). These results help us to prove Theorem 4.3.3 in terms of d_ρ instead of ρ.

Before we define the function d_ρ, we introduce some useful notation. For a lattice L, recall that $\mathcal{M}_L$ and $\mathcal{J}_L$ are the sets of meet-reducible and join-reducible elements of L. Let L_B be the set of all meet-irreducible elements except 1_L, i.e.,

$$L_B = (L - \mathcal{M}_L) - \{1_L\}.$$

For $x \in L$, let C_x be the set of all elements that cover x, i.e.,

$$C_x = \{y \in L : y \gtrdot x\}.$$

Thus, $|C_x| > 1$ if and only if $x \in \mathcal{M}_L$, and $|C_x| = 1$ if and only if $x \in L_B$.

Now we define a function d_ρ in terms of ρ.

Definition 4.3.4. *For a lattice L and a function $\rho : L \to \mathbb{Z}$, define $d_\rho : L - \{1_L\} \to \mathbb{Z}$ by*

$$d_\rho(x) = \begin{cases} \rho(y) - \rho(x), & \text{if } C_x = \{y\} \\ \left(\displaystyle\sum_{T \subseteq C_x} (-1)^{|T|+1} \rho\left(\bigvee(T)\right) \right) - \rho(x), & \text{if } |C_x| > 1. \end{cases}$$

The function ρ can be recovered from d_ρ, as the next theorem shows.

Theorem 4.3.5. *Let L be a lattice and $\rho : L \to \mathbb{Z}$ be a function with $\rho(0_L) = 0$. Then we have*

$$\rho(x) = |\{w \in L_B : w \ngeq x\}| + \sum_{\substack{w \in L - \{1_L\} \\ w \ngeq x}} d_\rho(w)$$

for all $x \in L$.

Proof. For each $x \in L$, define

$$\lambda(x) = |\{w \in L_B : w \geq x\}| + \sum_{\substack{w \in L - \{1_L\} \\ w \geq x}} d_\rho(w).$$

We will show $\lambda(x) = \rho(1_L) - \rho(x)$ for all $x \in L$. Using this, we can easily find $\rho(x)$ for all $x \in L$.

We first express $\lambda(x)$ in terms of other $\lambda(w)$'s, where $w > x$, and $d_\rho(x)$. If $C_x = \{y\}$, then we have

$$\begin{aligned}
\lambda(x) &= |\{w \in L_B : w \geq x\}| + \sum_{\substack{w \in L - \{1_L\} \\ w \geq x}} d_\rho(w) \\
&= |\{w \in L_B : w \geq y\}| + \left(\sum_{\substack{w \in L - \{1_L\} \\ w \geq y}} d_\rho(w) \right) + 1 + d_\rho(x) \\
&= \lambda(y) + 1 + d_\rho(x). \qquad (4.4)
\end{aligned}$$

If $|C_x| > 1$, then we find the expression for $\lambda(x)$ by the inclusion-exclusion principle. For $T \subseteq C_x$, define

$$\begin{aligned}
L_{\geq}(T) &= \{w \in L : w \geq y \text{ for all } y \in T\}, \\
L_{=}(T) &= \{w \in L : w \geq y \text{ for all } y \in T \text{ and } w \ngeq y \text{ for all } y \in C_x - T\}, \\
S_{\geq}(T) &= |L_B \cap L_{\geq}(T)| + \sum_{w \in (L - \{1_L\}) \cap L_{\geq}(T)} d_\rho(w), \quad \text{and} \\
S_{=}(T) &= |L_B \cap L_{=}(T)| + \sum_{w \in (L - \{1_L\}) \cap L_{=}(T)} d_\rho(w).
\end{aligned}$$

Since we have

$$S_{\geq}(T') = \sum_{T : T' \subseteq T \subseteq C_x} S_{=}(T) \quad \text{for all } T' \subseteq C_x,$$

we obtain

$$S_{=}(T') = \sum_{T : T' \subseteq T \subseteq C_x} (-1)^{|T - T'|} S_{\geq}(T) \quad \text{for all } T' \subseteq C_x$$

by inclusion-exclusion. In particular, for $T' = \emptyset$, we have

$$S_{=}(\emptyset) = \sum_{T \subseteq C_x} (-1)^{|T|} S_{\geq}(T). \qquad (4.5)$$

Moreover, notice that we have

$$L_{\geq}(\emptyset) - L_{=}(\emptyset) = |L| - |\{w \in L : w \ngeq x\}| = |\{w \in L : w > x\}|.$$

It follows that

$$
\begin{aligned}
S_{\geq}(\emptyset) - S_{=}(\emptyset) &= |\{w \in L_B : w > x\}| + \sum_{\substack{w \in L - \{1_L\} \\ w > x}} d_\rho(w) \\
&= |\{w \in L_B : w \geq x\}| + \left(\sum_{\substack{w \in L - \{1_L\} \\ w \geq x}} d_\rho(w) \right) - d_\rho(x) \\
&= \lambda(x) - d_\rho(x). \qquad (4.6)
\end{aligned}
$$

Thus, we obtain the expression of $\lambda(x)$:

$$
\begin{aligned}
\lambda(x) &= S_{\geq}(\emptyset) - S_{=}(\emptyset) + d_\rho(x) \quad \text{(by (4.6))} \\
&= \left(\sum_{\substack{T \subseteq C_x \\ T \neq \emptyset}} (-1)^{|T|+1} S_{\geq}(T) \right) + d_\rho(x) \quad \text{(by (4.5))} \\
&= \left(\sum_{\substack{T \subseteq C_x \\ T \neq \emptyset}} (-1)^{|T|+1} \lambda\left(\bigvee(T) \right) \right) + d_\rho(x). \qquad (4.7)
\end{aligned}
$$

The last equality holds because $S_{\geq}(T) = \lambda\left(\bigvee(T)\right)$.

Now we show $\lambda(x) = \rho(1_L) - \rho(x)$ by induction. Evidently $\lambda(1_L) = 0$. Assume that the equation $\lambda(w) = \rho(1_L) - \rho(w)$ holds for all $w > x$. For the case $C_x = \{y\}$, we have

$$
\begin{aligned}
\lambda(x) &= \lambda(y) + 1 + d_\rho(x) \quad \text{(by (4.4))} \\
&= \rho(1_L) - \rho(y) + 1 + d_\rho(x) \quad \text{(by the inductive hypothesis)} \\
&= \rho(1_L) - \rho(x).
\end{aligned}
$$

Also, for the case $|C_x| > 1$, we have

$$\begin{aligned}
\lambda(x) &= \sum_{\substack{T\subseteq C_x\\ T\neq\emptyset}} (-1)^{|T|+1}\lambda\left(\bigvee(T)\right) + d_\rho(x) \quad \text{(by (4.7))}\\
&= \sum_{\substack{T\subseteq C_x\\ T\neq\emptyset}} (-1)^{|T|+1}\left(\rho(1_L) - \rho\left(\bigvee(T)\right)\right) + d_\rho(x) \quad \text{(by the inductive hypothesis)}\\
&= \rho(1_L) - \sum_{\substack{T\subseteq C_x\\ T\neq\emptyset}} (-1)^{|T|+1}\rho\left(\bigvee(T)\right) + d_\rho(x)\\
&= \rho(1_L) - \rho(x).
\end{aligned}$$

Hence

$$\lambda(x) = \rho(1_L) - \rho(x) \tag{4.8}$$

for all $x \in L$.

We determine the values of $\rho(x)$. For $\rho(1_L)$, let $x = 0_L$ in (4.8). Since $\rho(0_L) = 0$ by our assumption, we get

$$\rho(1_L) = \lambda(0_L) + \rho(0_L) = \lambda(0_L) = |L_B| + \sum_{w\in L-\{1_L\}} d_\rho(w).$$

Thus, for each $x \in L$, we have

$$\begin{aligned}
\rho(x) &= \rho(1_L) - \lambda(x)\\
&= |L_B| + \sum_{w\in L-\{1_L\}} d_\rho(w) - |\{w \in L_B : w \geq x\}| - \sum_{\substack{w\in L-\{1_L\}\\ w\geq x}} d_\rho(w)\\
&= |\{w \in L_B : w \not\geq x\}| + \sum_{\substack{w\in L-\{1_L\}\\ w\not\geq x}} d_\rho(w).
\end{aligned}$$

□

Some cases of conditions (ZR2) and (ZR3) are easily seen to hold when d_ρ is nonnegative: monotonicity in (ZR2) is satisfied when x is meet-irreducible, and the Mason-Ingleton inequality (ZR3) is satisfied when S is the set C_x. Indeed, the nonnegative condition on d_ρ together with the condition $\rho(0_L) = 0$ in (ZR1) is enough to characterize all $\mathcal{Z}$-rank functions.

Theorem 4.3.6. *Let L be a lattice. A function $\rho : L \to \mathbb{Z}$ is a $\mathcal{Z}$-rank function if and only if $\rho(0_L) = 0$ and d_ρ is nonnegative.*

By Theorems 4.3.5 and 4.3.6, it is clear that values of a $\mathcal{Z}$-rank function ρ are minimized for all $x \in L$ if and only if $d_\rho(x) = 0$ for all $x \in L - \{1_L\}$.

The proof of Theorem 4.3.6 requires the following series of lemmas. The idea is to show that if the Mason-Ingleton inequality holds for a subset of L, then it also holds after particular elements are added to or deleted from the subset, or replaced by other elements in the lattice L. Lemmas 4.3.7 and 4.3.10 are the two main tools that justify, respectively, adding or deleting element, and exchanging elements.

Lemma 4.3.7. *Let L be a lattice and $\rho : L \to \mathbb{Z}$ be a function. Suppose that $S \subseteq L$ and y is a non-minimal element of S. The Mason-Ingleton inequality holds for $S - y$ if and only if it does for S.*

Proof. By assumption, we know that $\bigwedge(S) = \bigwedge(S - y)$. Thus, we have

$$\rho\left(\bigwedge(S)\right) = \rho\left(\bigwedge(S - y)\right).$$

Also we have

$$\sum_{T \subseteq S} (-1)^{|T|+1} \rho\left(\bigvee(T)\right) = \sum_{T \subseteq S-y} (-1)^{|T|+1} \rho\left(\bigvee(T)\right) + \sum_{T \subseteq S-y} (-1)^{|T|} \rho\left(\bigvee(T) \vee y\right).$$

Note that the last sum is 0 because it can be rewritten as

$$\sum_{T \subseteq S-x-y} (-1)^{|T|+1} \left(\rho\left(\bigvee(T) \vee x \vee y\right) - \rho\left(\bigvee(T) \vee y\right)\right) = 0$$

where $x \in S$ is any element such that $x < y$. □

If a subset $S \subseteq L$ has a minimum element, then we can use Lemma 4.3.7 recursively to obtain the following corollary. Recall that if $|S| = 1$, then the Mason-Ingleton inquality holds with equality.

Corollary 4.3.8. *Let L be a lattice and $\rho : L \to \mathbb{Z}$ be a function. If $S \subseteq L$ has a minimum element, then the Mason-Ingleton inequality trivially holds with equality for S.*

The next technical lemma is used in Lemma 4.3.10.

Lemma 4.3.9. *Let L be a lattice and $\rho : L \to \mathbb{Z}$ be a function. Let $f : L \to L$ be a function such that $f(\bigvee(T)) = \bigvee(f(T))$ for all $T \subseteq L$. Then for $A \subseteq L$,*

$$\sum_{T\subseteq A}(-1)^{|T|+1}\rho\left(f\left(\bigvee(T)\right)\right) = \sum_{T'\subseteq f(A)}(-1)^{|T'|+1}\rho\left(\bigvee(T')\right).$$

Proof. Let A_0 be a subset of A such that $f|_{A_0}$ is a bijection onto $f(A)$, i.e., $f(A_0) = f(A)$ and $|A_0| = |f(A_0)|$. Let $A_1 = A - A_0$. Then A is a disjoint union of A_0 and A_1. The left side can be written as two sums as follows:

$$\begin{aligned}
&\sum_{T\subseteq A}(-1)^{|T|+1}\rho\left(f\left(\bigvee(T)\right)\right)\\
&= \sum_{T\subseteq A_0}(-1)^{|T|+1}\rho\left(f\left(\bigvee(T)\right)\right) + \sum_{\substack{T_0\subseteq A_0\\ T_1\subseteq A_1\\ T_1\neq\emptyset}}(-1)^{|T_0\cup T_1|+1}\rho\left(f\left(\bigvee(T_0\cup T_1)\right)\right). \qquad (4.9)
\end{aligned}$$

For the first term of (4.9), notice that there is a bijection between $T \subseteq A_0$ and $T' \subseteq f(A)$ given by $T' = f(T)$. Thus,

$$\begin{aligned}
\sum_{T\subseteq A_0}(-1)^{|T|+1}\rho\left(f\left(\bigvee(T)\right)\right) &= \sum_{T\subseteq A_0}(-1)^{|T|+1}\rho\left(\bigvee(f(T))\right)\\
&= \sum_{T'\subseteq f(A)}(-1)^{|T'|+1}\rho\left(\bigvee(T')\right),
\end{aligned}$$

which is the right side of the desired equation. It remains to show that the second term of equation (4.9) is zero. We show this by induction on $|A_1|$. For the base case, $|A_1| = 0$, the sum is vacuously zero. Let $a_1 \in A_1$. Then there is a unique $a_0 \in A_0$ such that $f(a_0) = f(a_1)$.

Thus, we have

$$\begin{aligned}
&\sum_{\substack{T_0\subseteq A_0\\ T_1\subseteq A_1\\ T_1\neq\emptyset}} (-1)^{|T_0\cup T_1|+1}\rho\left(f\left(\bigvee(T_0\cup T_1)\right)\right)\\
&\quad= \sum_{\substack{T_0\subseteq A_0\\ T_1\subseteq A_1-\{a_1\}\\ T_1\neq\emptyset}} (-1)^{|T_0\cup T_1|+1}\rho\left(f\left(\bigvee(T_0\cup T_1)\right)\right)\\
&\quad\quad+ \sum_{\substack{T_0\subseteq A_0-\{a_0\}\\ T_1\subseteq A_1-\{a_1\}}} (-1)^{|T_0\cup T_1|}\rho\left(f\left(\bigvee(T_0\cup T_1\cup\{a_1\})\right)\right)\\
&\quad\quad+ \sum_{\substack{T_0\subseteq A_0-\{a_0\}\\ T_1\subseteq A_1-\{a_1\}}} (-1)^{|T_0\cup T_1|+1}\rho\left(f\left(\bigvee(T_0\cup T_1\cup\{a_0,a_1\})\right)\right)
\end{aligned}$$

The first sum on the right is zero by the inductive hypothesis. The second and third sums cancel because f and $\bigvee$ commute, and $f(T_0 \cup T_1 \cup \{a_1\}) = f(T_0 \cup T_1 \cup \{a_0, a_1\})$. □

The next lemma shows that the Mason-Ingleton inequality still holds if an element of the subset of L is replaced by a greater element, as long as the meet remains the same.

Lemma 4.3.10. *Let L be a lattice and $\rho : L \to \mathbb{Z}$ be a non-negative function. Suppose $S \subseteq L$, $x \in S$, $y \notin S$ and $\bigwedge((S - x) \cup y) = \bigwedge(S) < x < y$. Also suppose that the Mason-Ingleton inequality holds on ρ for every subset of L whose meet is strictly greater than $\bigwedge(S)$. If the Mason-Ingleton inequality holds for S, then it holds for $(S - x) \cup y$.*

Proof. By the assumption, we have

$$\rho\left(\bigwedge((S-x)\cup y)\right) = \rho\left(\bigwedge(S)\right) \leq \sum_{T\subseteq S}(-1)^{|T|+1}\rho\left(\bigvee(T)\right).$$

So our goal is to show that

$$\sum_{T\subseteq S}(-1)^{|T|+1}\rho\left(\bigvee(T)\right) \leq \sum_{T\subseteq(S-x)\cup y}(-1)^{|T|+1}\rho\left(\bigvee(T)\right),$$

i.e., the right side of the Mason-Ingleton inequality for $(S - x) \cup y$ is no smaller than that for S. We have the following series of equalities:

$$
\begin{aligned}
&\sum_{T\subseteq (S-x)\cup y} (-1)^{|T|+1}\rho\left(\bigvee(T)\right)\\
&= \sum_{T\subseteq S-x} (-1)^{|T|+1}\rho\left(\bigvee(T)\right) + \sum_{T\subseteq S-x} (-1)^{|T|}\rho\left(\bigvee(T)\vee y\right)\\
&= \sum_{T\subseteq S} (-1)^{|T|+1}\rho\left(\bigvee(T)\right) - \sum_{T\subseteq S-x} (-1)^{|T|}\rho\left(\bigvee(T)\vee x\right) + \sum_{T\subseteq S-x} (-1)^{|T|}\rho\left(\bigvee(T)\vee y\right)\\
&= \sum_{T\subseteq S} (-1)^{|T|+1}\rho\left(\bigvee(T)\right) + \sum_{T\subseteq (S-x)\cup y} (-1)^{|T|+1}\rho\left(\bigvee(T)\vee x\right).
\end{aligned}
$$

The last equality holds because $\bigvee(T)\vee y = \bigvee(T)\vee x\vee y$.

Now it remains to show that the second sum is nonnegative. Let $f : L \to L$ be the function given by $f(w) = w \vee x$. Clearly, f and $\bigvee$ are commutative operations on L. Thus, we have

$$
\begin{aligned}
\sum_{T\subseteq (S-x)\cup y} (-1)^{|T|+1}\rho\left(\bigvee(T)\vee x\right) &= \sum_{T\subseteq (S-x)\cup y} (-1)^{|T|+1}\rho\left(f\left(\bigvee(T)\right)\right)\\
&= \sum_{T'\subseteq f((S-x)\cup y)} (-1)^{|T'|+1}\rho\left(\bigvee(T')\right) && (4.10)\\
&\geq \rho\left(\bigwedge(f((S-x)\cup y))\right) && (4.11)\\
&\geq 0,
\end{aligned}
$$

where the equality (4.10) holds by Lemma 4.3.9. Also, the Mason-Ingleton inequality (4.11) hold by our assumption because

$$\bigwedge(f((S-x)\cup y)) \geq x > \bigwedge(S).$$

□

Recall that C_x is the set of covers of x. By using Lemmas 4.3.7 and 4.3.10, we show that each proper subset of C_x satisfies the Mason-Ingleton inequality if C_x does.

Corollary 4.3.11. *Let L be a lattice and $\rho : L \to \mathbb{Z}$ be a function. For $x \in L$, suppose $|C_x| > 1$ and C is a nonempty subset of C_x. Suppose that monotonicity holds for every pair of comparable elements strictly greater than x. If the Mason-Ingleton inequality holds for*

C_x and every subset of L whose meet is strictly greater than x, then it holds for C.

Proof. We show this by induction on $|C|$. Suppose the Mason-Ingleton inequality holds for $C \subseteq C_x$. We show that it holds for $C - y$ for $y \in C$. When $|C| = 2$, then $|C - y| = 1$ and the Mason-Ingleton inequality trivially holds. So assume that $|C| \geq 3$. Let $y' \in C$ with $y' \neq y$. Since $\bigwedge((C - y) \cup (y \vee y')) = \bigwedge(C)$, the Mason-Ingleton inequality holds for $(C - y) \cup (y \vee y')$ by Lemma 4.3.10, and it holds for $C - y$ by Lemma 4.3.7. □

Now we are ready to see the proof of Theorem 4.3.6, the characterization of $\mathcal{Z}$-rank functions ρ using nonnegativity of d_ρ.

Proof of Theorem 4.3.6. For the forward direction, assume that (ZR1), (ZR2), and (ZR3) hold. Evidently $\rho(0_L) = 0$ by (ZR1). If $C_x = \{y\}$, then $d_\rho(x) = \rho(y) - \rho(x) - 1 \geq 0$ by (ZR2). If $|C_x| > 1$, then $\rho(\bigwedge(C_x)) = \rho(x)$ by letting $S = C_x$ in (ZR3). Thus we have $d_\rho(x) \geq 0$.

For the converse, assume that $\rho(0_L) = 0$ and $d_\rho(x) \geq 0$ for all $x \in L - \{1_L\}$. Evidently (ZR1) holds. For (ZR2), it suffices to show monotonicity for all pairs of elements $x \lessdot y$. Furthermore, since $d_\rho(x) \geq 0$, we only need to show monotonicity for the case of $|C_x| > 1$ and $y \in C_x$. For (ZR3), we may assume that a subset S of L has no unique minimum element; otherwise the inequality holds automatically by Corollary 4.3.8. Thus, we only need to show the Mason-Ingleton inequality for the case of a subset S with $\bigwedge(S) \in \mathcal{M}_L$.

We show both monotonicity (ZR2) and the Mason-Ingleton inequality (ZR3) together by induction on elements in $\mathcal{M}_L$ in the reverse order. Let $x \in \mathcal{M}_L$. Assume that monotonicity holds for all pairs (v, w) with $x < v < w$. Also, assume that the Mason-Ingleton inequality holds for all subsets T of L whose meet is strictly greater than x, i.e., $x < \bigwedge(T)$. The base case is when x is a maximal element of $\mathcal{M}_L$. In this case, these assumptions trivially hold.

Now we show the Mason-Ingleton inequality for all subsets S with $x = \bigwedge(S)$. Since $d_\rho(x) \geq 0$, the Mason-Ingleton inequality holds for the set C_x of all covers of x. Our goal is to obtain S from C_x while satisfying the Mason-Ingleton inequality. Let $C = \{c_1, \ldots, c_n\}$ be a minimal subset of C_x such that S is contained in the filter generated by C (generally, the choice of C is not unique). This set C satisfies the Mason-Ingleton inequality by Corollary 4.3.11. Partition S into n parts $S_1, \ldots, S_n$ so that $\bigwedge(S_i) \geq c_i$ for $1 \leq i \leq n$. By the minimality of C, there is an element $y_i \in S_i$, such that $y_i \geq c_j$ implies $j = i$. To get S from C by a sequence of exchanges and to ensure that the Mason-Ingleton inequality holds at each stage, proceed as follows. Start with $A_0 = C$; the Mason-Ingleton inequality holds for A_0 by Corollary 4.3.11. For i from 1 to n, set $A'_i = A_{i-1} \cup (S_i - y_i)$ and $A_i = (A'_i - c_i) \cup y_i$. Thus, $A_n = S$. By Lemma 4.3.7, the Mason-Ingleton inequality holds for A'_i; by Lemma 4.3.10, it holds for A_i.

Finally, we show monotonicity for all pairs $x \lessdot y$ with $y \in C_x$. Let $y' \in C_x$ with $y' \neq y$. Since the Mason-Ingleton inequality holds for the two-element set $\{y, y'\} \subseteq C_x$, we have

$$\rho(x) \leq \rho(y) + \rho(y') - \rho(y \vee y') < \rho(y)$$

because $\rho(y') < \rho(y \vee y')$ by the inductive hypothesis. □

Now we are ready to present the proof of Theorem 4.3.3, the main result of this section. The forward direction of the proof constructs the fundamental transversal matroid M from the lattice L and its $\mathcal{Z}$-rank function ρ so that $L \cong \mathcal{Z}(M)$ and the rank function r of M produces the same values as ρ. Note that we will automatically have $L \cong \mathcal{Z}(M) = R(B)$ for a basis B with this construction because of Theorem 4.2.1.

Proof of Theorem 4.3.3. We prove the converse of the theorem first. For a given lattice L and a function $\rho : L \to \mathbb{Z}$, suppose that M is a fundamental transversal matroid such that $\mathcal{Z}(M)$ is isomorphic to L, and $\varphi : L \to \mathcal{Z}(M)$ is the lattice isomorphism. We need to show that $\rho = r\varphi$ satisfies (ZR1)–(ZR3). Notice that $r|_{\mathcal{Z}(M)}$ satisfies (Z1) and (Z2). For (ZR1), we have $\rho(0_L) = r\varphi(0_L) = r(0_{\mathcal{Z}(M)}) = 0$ by (Z1). Also, (ZR2) holds because r is monotonic by (Z2) and φ is order-preserving.

Finally for (ZR3), let $S \subseteq L$. Then $\varphi(S) \subseteq \mathcal{Z}(M)$, and φ is a bijection between $T \subseteq S$ and $\varphi(T) \subseteq \varphi(S)$. By Theorem 4.3.2, we have

$$r\left(\bigcap(\varphi(S))\right) \leq \sum_{T \subseteq S} (-1)^{|T|+1} r\left(\bigcup(\varphi(T))\right).$$

Since

$$\rho(\bigwedge(S)) = r\varphi(\bigwedge(S)) = r\left(\bigwedge(\varphi(S))\right) \leq r\left(\bigcap(\varphi(S))\right) \quad \text{and}$$
$$\rho(\bigvee(T)) = r\varphi(\bigvee(T)) = r\left(\bigvee(\varphi(T))\right) = r\left(\bigcup(\varphi(T))\right),$$

the Mason-Ingleton inequality holds for S. Hence (ZR3) holds.

For the forward direction, let L be a lattice and $\rho : L \to \mathbb{Z}$ be a $\mathcal{Z}$-rank function. We construct a fundamental transversal matroid M with a fundamental basis B as follows. Recall that

$$L_B = (L - \mathcal{M}_L) - \{1_L\}.$$

Since ρ is a $\mathcal{Z}$-rank function, we have $d_\rho(x) \geq 0$ (Theorem 4.3.6). For each $x \in L - \{1_L\}$, let $S_x = \{x_i : 1 \leq i \leq d_\rho(x)\}$ be a set of $d_\rho(x)$ elements that are disjoint from L and such that $S_x \cap S_y = \emptyset$ when $x \neq y$. Let S_B be the collection of such elements, i.e.,

$$S_B = \bigcup_{x \in L - \{1_L\}} S_x = \bigcup_{x \in L - \{1_L\}} \{x_i : 1 \leq i \leq d_\rho(x)\}.$$

Our fundamental basis B is the disjoint union $L_B \dot{\cup} S_B$ where $S_B = \emptyset$ is the case with the minimal rank. We now define the set of elements N that are not in B. For each $x \in L - \mathcal{J}_L - \{0_L\}$, let $\bar{x}$ be an element that is in neither L nor S_B. Let N be the collection of such elements, i.e.,

$$N = \{\bar{x} : x \in L - \mathcal{J}_L - \{0_L\}\}.$$

Thus, our matroid M has a ground set E that is the disjoint union $B \dot{\cup} N$. It is clear that we have

$$r(M) = |B| = |L_B| + |S_B| = |L| - |\mathcal{M}_L| - 1 + \sum_{x \in L - \{1_L\}} d_\rho(x),$$
$$n(M) = |N| = |L| - |\mathcal{J}_L| - 1, \quad \text{and}$$
$$|E| = |B| + |N| = 2|L| - |\mathcal{M}_L| - |\mathcal{J}_L| - 2 + \sum_{x \in L - \{1_L\}} d_\rho(x).$$

Place the elements of B at the vertices of the $|B|$-simplex. For $x \in L$, define $B_x \subseteq B$ and $N_x \subseteq N$ by

$$B_x = \{w \in L_B : w \ngeq x\} \cup \bigcup_{w \ngeq x} S_w \quad \text{and}$$

$$N_x = \{\bar{w} \in N : w \leq x\}.$$

Put each element $\bar{x} \in N$ freely in the face spanned by the set of vertices B_x, i.e., $B_x \cup \bar{x}$ is a circuit. By the construction, the resulting matroid M is fundamental transversal with a fundamental basis B.

For each $x \in L$, let $Z_x = \text{cl}(B_x)$. We show that $Z_x = B_x \dot{\cup} N_x$. If $\bar{y} \in N_x$, then $y \leq x$. It follows that we have $B_y \subseteq B_x$, for otherwise, there is an element w in $B_y - B_x$, which gives two contradictory statements, $w \ngeq y$ and $w \geq x$. Hence, $\bar{y} \in \text{cl}(B_y) \subseteq \text{cl}(B_x) = Z_x$. On the other hand, suppose $\bar{y} \in Z_x \cap N$. Then, $\bar{y} \in \text{cl}(B_x) - B_x$. So $\bar{y} \in C \subseteq B_x \cup \bar{y}$ for some circuit C. Then C must be the fundamental circuit of $\bar{y}$ with respect to B, i.e., $C = B_y \cup \bar{y}$. Hence, $B_y \subseteq B_x$. It follows that $y \leq x$, for otherwise, we have $y \nleq x$, which gives two contradictory statements, $x \in B_y$ and $x \notin B_x$. Since $y \leq x$, we have $\bar{y} \in N_x$.

In order to show that $\mathcal{Z}(M) = \{Z_x : x \in L\}$, we first show that the flat Z_x is cyclic for all $x \in L$. If $x \notin \mathcal{J}_L$, then Z_x is a cyclic flat since it is the closure of the circuit $B_x \cup \bar{x}$. Also for $x = 0_L$, we have $Z_{0_L} = \emptyset$, which is a cyclic flat of rank-0. For $x \in \mathcal{J}_L$, we apply induction. Suppose that Z_w are cyclic sets for all $w < x$. Then we have

$$N_x = \{\bar{w} \in N : w < x\} = \bigcup_{y \lessdot x} \{\bar{w} \in N : w \leq y\} = \bigcup_{y \lessdot x} N_y.$$

Also $B_x = \bigcup_{y \lessdot x} B_y$ because $w \geq x \Leftrightarrow w \geq y$ for all $y \lessdot x$. So we have

$$Z_x = B_x \cup N_x = \big(\bigcup_{y \lessdot x} B_y\big) \cup \big(\bigcup_{y \lessdot x} N_y\big) = \bigcup_{y \lessdot x} (B_y \cup N_y) = \bigcup_{y \lessdot x} Z_y$$

By the inductive hypothesis, Z_x is a cyclic set. Hence $Z_x \in \mathcal{Z}(M)$ for all $x \in L$.

We now show that any cyclic flat has the form Z_x for some $x \in L$. Since M is a fundamental transversal matroid, each cyclic flat is spanned by a subset of B. Assume that

Z is a cyclic flat and $B' \subseteq B$ such that $\mathrm{cl}(B') = Z$. For $\bar{x} \in Z - B'$, there is the fundamental circuit $C(\bar{x}, B) = B_x \cup \bar{x} \subseteq Z$ by construction. Also for $v \in B'$, there is a $\bar{x} \in Z - B'$ such that $v \in C(\bar{x}, B)$. If not, we have $Z - v = \mathrm{cl}(B' - v)$ and hence $Z - v$ would be a flat. This contradicts the fact that Z is a cyclic flat. So we have

$$Z = \bigcup_{\bar{x} \in Z - B'} C(\bar{x}, B) = \bigcup_{\bar{x} \in Z - B'} (B_x \cup \bar{x}).$$

Since $B_x \cup \bar{x}$ and B_x span Z_x, the unions of each, $\bigcup_{\bar{x} \in Z - B'} (B_x \cup \bar{x})$ and $\bigcup_{\bar{x} \in Z - B'} B_x$, also span $\mathrm{cl}\big(\bigcup_{\bar{x} \in Z - B'} Z_x\big) = \bigvee_{\bar{x} \in Z - B'} Z_x$. Also $\bigcup_{\bar{x} \in Z - B'} B_x = B_y$ where $y = \bigvee_{\bar{x} \in Z - B'} x$ because $w \geq x$ for all x with $\bar{x} \in Z - B'$ if and only if $w \geq\ y$. So we have

$$Z = \mathrm{cl}(Z) = \mathrm{cl}\big(\bigcup_{\bar{x} \in Z - B'} (B_x \cup \bar{x})\big) = \mathrm{cl}\big(\bigcup_{\bar{x} \in Z - B'} B_x\big) = \mathrm{cl}(B_y) = Z_y.$$

Hence $\mathcal{Z}(M) = \{Z_x : x \in L\}$.

Define $\varphi : L \to \mathcal{Z}(M)$ by $\varphi(x) = Z_x$. Then φ is a lattice isomorphism because $x < y$ in L if and only if $Z_x < Z_y$ in $\mathcal{Z}(M)$. In order to show $\rho = r\varphi$, we have

$$\begin{aligned}
r\varphi(x) &= r(Z_x) \\
&= |B_x| \\
&= |\{w \in L_B : w \not\geq x\} \cup \bigcup_{w \not\geq x} S_w| \\
&= |\{w \in L_B : w \not\geq x\}| + \sum_{\substack{w \in L - \{1_L\} \\ w \not\geq x}} d_\rho(w) \\
&= \rho(x) \quad \text{(by Theorem 4.3.5)}
\end{aligned}$$

for all $x \in L$. This completes the forward direction. □

Figure 4.2 and 4.3 illustrate this construction.

In Figure 4.2, we have $d_\rho(0) = d_\rho(a) = 1$ and $d_\rho(b) = 0$ from the lattice L and a function ρ. Thus, the fundamental basis is $B = L_B \cup S_B = \{a, b, 0_1, a_1\}$ and the set of elements in $E(M) - B$ is $N = \{\bar{a}, \bar{b}\}$. So we have $B_0 = \emptyset$, $B_a = \{0_1, b\}$, $B_b = \{0_1, a, a_1\}$, $B_1 = B$,

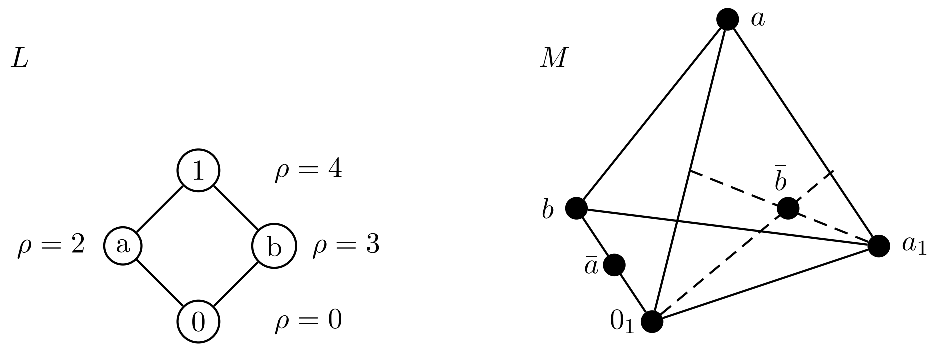

Figure 4.2: Construction of a fundamental transversal matroid M from a lattice L when $d_\rho(0) = d_\rho(a) = 1$

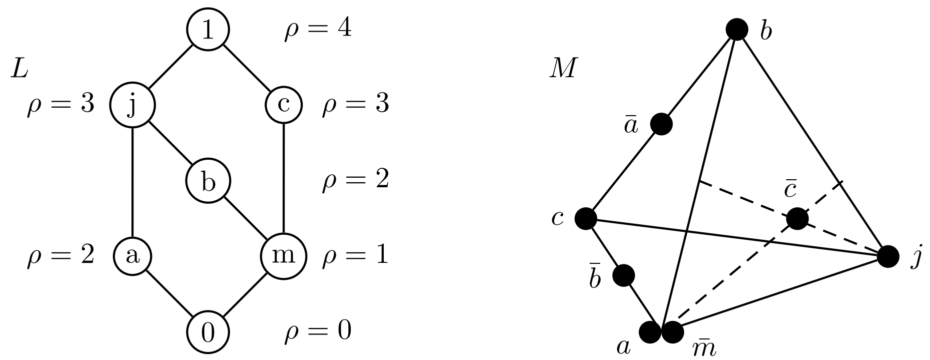

Figure 4.3: Construction of a fundamental transversal matroid M from a lattice L when $d_\rho \equiv 0$

$N_0 = \emptyset$, $N_a = \{\bar{a}\}$, $N_b = \{\bar{b}\}$, and $N_1 = N$. Hence the cyclic flats of M are $Z_0 = \emptyset$, $Z_a = \{0_1, b, \bar{a}\}$, $Z_b = \{0_1, a, a_1, \bar{b}\}$, and $Z_1 = E(M)$.

In Figure 4.3, we have $d_\rho(x) = 0$ for all $x \in L - \{1\}$. Thus, $S_B = \emptyset$, and the fundamental basis is $B = L_B = \{a, b, c, j\}$. The set of elements in $E(M) - B$ is $N = \{\bar{a}, \bar{b}, \bar{c}, \bar{m}\}$. Thus, $B_0 = \emptyset$, $B_m = \{a\}$, $B_a = \{b, c\}$, $B_b = \{a, c\}$, $B_c = \{a, b, j\}$, $B_j = \{a, b, c\}$, $B_1 = \{a, b, c, j\}$, $N_0 = \emptyset$, $N_m = \{\bar{m}\}$, $N_a = \{\bar{a}\}$, $N_b = \{\bar{m}, \bar{b}\}$, $N_c = \{\bar{m}, \bar{c}\}$, $N_j = \{\bar{m}, \bar{a}, \bar{b}\}$, and $N_1 = N$. Hence we have the cyclic flats: $Z_0 = \emptyset$, $Z_m = \{a, \bar{m}\}$, $Z_a = \{b, c, \bar{a}\}$, $Z_b = \{a, c, \bar{m}, \bar{b}\}$, $Z_c = \{a, b, j, \bar{m}, \bar{c}\}$, $Z_j = \{a, b, c, \bar{m}, \bar{a}, \bar{b}\}$, and $Z_1 = E(M)$.

In the rest of this section, we will determine the minimum rank and cardinality of the fundamental transversal matroid M whose lattice of cyclic flats is isomorphic to a given lattice L.

Corollary 4.3.12. *Let L be a lattice. Then*

$$\min\{r(M) : M \text{ is fundamental transversal and } \mathcal{Z}(M) \cong L\} = |L_B|.$$

Proof. For a fundamental transversal matroid M on E with $\mathcal{Z}(M) \cong L$, let $\varphi : L \to \mathcal{Z}(M)$ be a lattice isomorphism and $\rho = r\varphi$. Then ρ is a $\mathcal{Z}$-rank function by Theorem 4.3.3. Since

$$r(\text{cyc}(E)) = r(1_{\mathcal{Z}(M)}) = r\varphi(1_L) = \rho(1_L),$$

we have

$$r(M) = r(\text{cyc}(E)) + |\text{isth}(E)| = \rho(1_L) + |\text{isth}(E)|.$$

By Theorems 4.3.5 and 4.3.6, a function ρ attains a minimum value $|L_B|$ at 1_L if and only if $d_\rho(x) = 0$ for all $x \in L - \{1_L\}$. Thus, the value of $r(M)$ is minimum if and only if $d_\rho(x) = 0$ for all $x \in L - \{1_L\}$ and $\text{isth}(E) = \emptyset$. In this case, we have $r(M) = |L_B|$. □

Corollary 4.3.13. *Let L be a lattice. Then*

$$\begin{aligned}&\min\{|E(M)| : M \text{ is fundamental transversal and } \mathcal{Z}(M) \cong L\}\\ &\quad = 2|L| - |\mathcal{M}_L| - |\mathcal{J}_L| - 2.\end{aligned}$$

Proof. Recall that $\mathcal{Z}(M) \cong L$ if and only if $\mathcal{Z}(M^*) \cong L^d$, where L^d is the dual lattice of L. Also recall that the dual of a fundamental transversal matroid is fundamental transversal. Since $n(M) = r^*(M)$ by Lemma 2.2.2, we have

$$\begin{aligned}&\min\{n(M) : M \text{ is fundamental transversal and } \mathcal{Z}(M) \cong L\}\\ &\quad = \min\{r^*(M) : M^* \text{ is fundamental transversal and } \mathcal{Z}((M)^*) \cong L^d\}\\ &\quad = |L^d_B|\\ &\quad = |L^d - \mathcal{M}_{L^d} - \{1_{L^d}\}|\\ &\quad = |L - \mathcal{J}_L - \{0_L\}|\\ &\quad = |L| - |\mathcal{J}_L| - 1.\end{aligned}$$

Since $|E(M)| = r(M) + n(M)$, we have

$$\begin{aligned} &\min\{|E(M)| : M \text{ is fundamental transversal and } \mathcal{Z}(M) \cong L\} \\ &\quad \geq 2|L| - |\mathcal{M}_L| - |\mathcal{J}_L| - 2. \end{aligned} \tag{4.12}$$

By the construction in the proof of Theorem 4.3.3, we know that there is a matroid that satisfies (4.12) with the equality. □

Thus, the minimum rank and cardinality of a fundamental transversal matroid are $|L_B|$ and $2|L| - |\mathcal{M}_L| - |\mathcal{J}_L| - 2$ when the lattice of cyclic flats is isomorphic to L.

Chapter 5

Superexponential Families of Matroids That Have the Same Tutte Polynomial

One of the most important algebraic tools in the theory of matroids is the Tutte polynomial, which is defined as follows:

$$t(M;x,y) = \sum_{X \subseteq E} (x-1)^{r(E)-r(X)}(y-1)^{n(X)}.$$

In essence, the Tutte polynomial is a generating function that encodes the number of subsets of the ground set of the matroid according to their ranks and cardinalities. The Tutte polynomial plays the central role in matroid invariant theory. For example, both the chromatic and flow polynomials are evaluations of the Tutte polynomial. Also, the Tutte polynomial is used in coding theory, percolation theory, electrical network theory, and statistical mechanics [18].

In the next three sections, we construct several superexponential families of nonisomorphic matroids having two important characteristics in common, namely, isomorphic lattices of cyclic flats and the same Tutte polynomial.

For each family, we start with a sequence of lattices, each with a rank function. For each lattice L in the sequence, all matroids in the associated family will have the same ground set, which depends only on L. We construct each matroid in the family by choosing the set of cyclic flats. We then verify that the cyclic flat axioms (Theorem 3.1.1) hold. We then check that the family consists of nonisomorphic matroids. Finally, we prove that all matroids in the family have the same Tutte polynomial.

Recall that the width of a lattice is the maximal cardinality among the antichains of the lattice. When L is a lattice whose width is k, we call L a *width-k lattice*. Obviously, there are at least k saturated chains from 0_L to 1_L in a width-k lattice. When L has precisely k saturated chains with no pair of elements, except 0_L and 1_L, from distinct chains being comparable, we call L a *primitive width-k lattice*.

Let M be a matroid with no loops and no isthmuses. When $\mathcal{Z}(M)$ is a width-k lattice, we call M a *cyclic width-k matroid*; the class of all matroids whose cyclic width is at most k is denoted by $\mathrm{CW}(k)$ [3, Section 5]. Likewise, when $\mathcal{Z}(M)$ is a primitive width-k lattice, we call M a *primitive cyclic width-k matroid*; the class of all such matroids whose cyclic width is precisely k is denoted by $\mathrm{PCW}(k)$. The class $\mathrm{PCW}(k)$ is a subclass of $\mathrm{CW}(k)$.

Let n and m be positive integers. We denote a primitive width-2 lattice L with saturated chains $0_L \lessdot x_1 \lessdot \cdots \lessdot x_n \lessdot 1_L$ and $0_L \lessdot y_1 \lessdot \cdots \lessdot y_m \lessdot 1_L$ by

$$L = \{0_L : x_1, \ldots, x_n : y_1, \ldots, y_m : 1_L\}.$$

In particular, if M is in the class $\mathrm{PCW}(2)$ with saturated chains $\emptyset \lessdot A_0 \lessdot \cdots \lessdot A_n \lessdot E$ and $\emptyset \lessdot B_0 \lessdot \cdots \lessdot B_m \lessdot E$, then we can write

$$\mathcal{Z}(M) = \{\emptyset : A_0, \ldots, A_n : B_0, \ldots, B_m : E\}.$$

5.1 The Family Constructed by Giménez

In this section, we consider a family of matroids in $\mathrm{PCW}(2)$ defined by Giménez [10] in unpublished work; see the proof of Theorem 5.7 in [3]. For a fixed positive integer n, Giménez constructed a family of $n!$ matroids on $4n+5$ elements having isomorphic lattices of cyclic flats. Obviously, the sequence of this family grows superexponentially with respect to the cardinality of the ground set. We will see that all matroids in this family have the same Tutte polynomial; the proof uses the rank sublattices $R(X)$ of $\mathcal{Z}(M)$.

In Giménez's construction, each matroid is associated with a permutation σ on n elements. Thus, the family of matroid can be written as $\mathcal{G}_n = \{M_\sigma : \sigma \in S_n\}$, where S_n is the symmetric group on n elements. We use $\mathcal{Z}_\sigma$, r_σ, and R_σ for $\mathcal{Z}(M_\sigma)$, r_{M_σ}, and R_{M_σ}, respectively. We omit the subscript σ when σ is the identity permutation.

In order to describe Giménez's construction, let n be a fixed positive integer. All matroids in $\mathcal{G}_n$ have isomorphic lattices of cyclic flats

$$\mathcal{Z}_\sigma = \{\emptyset : A_0, \ldots, A_n : B_0, \ldots, B_n : E\}$$

with two saturated chains $C_A = \{\emptyset, A_0, \ldots, A_n, E\}$ and $C_B^\sigma = \{\emptyset, B_0, \ldots, B_n, E\}$. The superscript σ indicates that the cyclic flats $B_1, \ldots, B_n$ depend on the choice of σ. Their largest cyclic flat, which is the common ground set, is

$$E = \{a_1, \ldots, a_{n+2}, b_1, \ldots, b_{n+3}, x_1, \ldots, x_n, y_1, \ldots, y_n\},$$

hence $|E| = 4n + 5$. The other nontrivial cyclic flats of M_σ are

$$\begin{aligned} A_0 &= \{a_1, \ldots, a_{n+2}\}, \\ B_0 &= \{b_1, \ldots, b_{n+3}\}, \\ A_i &= A_{i-1} \cup \{x_i, y_i\}, \text{ for } 1 \le i \le n, \text{ and} \\ B_i &= B_{i-1} \cup \{x_{\sigma(i)}, y_i\}, \text{ for } 1 \le i \le n. \end{aligned}$$

Also, the ranks of the cyclic flats are

$$\begin{aligned} r_\sigma(\emptyset) &= 0, \\ r_\sigma(A_i) = r_\sigma(B_i) &= n + 1 + i \quad \text{for } 0 \le i \le n, \text{ and} \\ r_\sigma(E) &= 2n + 2. \end{aligned}$$

Figure 5.1 shows the structure of the lattice of cyclic flats $\mathcal{Z}_\sigma$; the elements shown between two cyclic flats are the difference between them. Giménez showed that this construction satisfies properties (Z0)–(Z3) in Theorem 3.1.1; thus, each M_σ is indeed a matroid. Moreover, he showed that no two matroids in this family are isomorphic [3].

In order to show that all matroids in $\mathcal{G}_n$ have the same Tutte polynomial, we establish a bijection between the subsets of the ground set of M (that is, M_{id}) and those of any other matroid M_σ in $\mathcal{G}_n$ with this bijection preserving both the rank and the cardinality of each set.

By Lemma 3.1.2, the rank function is extended from $\mathcal{Z}_\sigma$ to all subsets of E by

$$r_\sigma(X) = \min_{Z \in \mathcal{Z}_\sigma} \{r_\sigma(Z) + |X - Z|\}.$$

Also, recall the sublattice $R_\sigma(X)$ of $\mathcal{Z}_\sigma$ from Definition 3.1.3:

$$R_\sigma(X) = \{Z \in \mathcal{Z}_\sigma : r_\sigma(X) = r_\sigma(Z) + |X - Z|\}.$$

Thus, investigating $R_\sigma(X)$ is the key to establishing the desired bijection. For a nonspanning and dependent subset X, we will show that $R_\sigma(X)$ is contained in a single saturated chain of $\mathcal{Z}_\sigma$, either C_A or C_B^σ, and which chain it is is determined merely by the cardinalities of $X \cap A_0$ and $X \cap B_0$, and so is independent of σ. Thus, we can partition the subsets of the ground set by comparing $|X \cap A_0|$ and $|X \cap B_0|$, and establish a bijection on each block.

Lemma 5.1.1. *Let $X \subseteq E$ in $M_\sigma(E)$.*

(1) *If $|X \cap A_0| > |X \cap B_0|$, then $R_\sigma(X) \subseteq C_A$.*

(2) *If $|X \cap A_0| < |X \cap B_0|$, then $R_\sigma(X) \subseteq C_B^\sigma$.*

(3) *If $-1 \leq |X \cap A_0| - |X \cap B_0| \leq 1$, then X is either spanning or independent.*

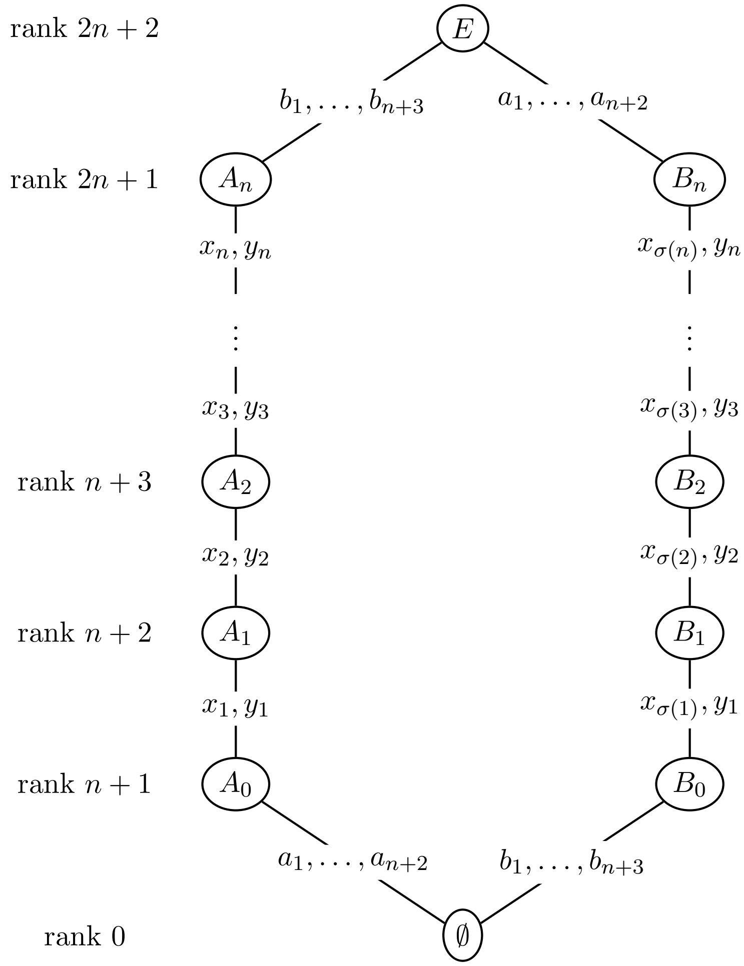

Figure 5.1: The lattice of cyclic flats $\mathcal{Z}_\sigma$

Proof. For (1), suppose that $B_i \in R_\sigma(X)$. Then we have

$$\begin{aligned} n+i+1 &= r_\sigma(B_i) \\ &= r_\sigma(X) - |X - B_i| \\ &\leq |X| - |X - B_i| \qquad (5.1) \\ &= |X \cap B_i| \\ &\leq |X \cap B_0| + |B_i - B_0| \\ &= |X \cap B_0| + 2i. \end{aligned}$$

Thus,

$$n - i + 1 \leq |X \cap B_0|.$$

On the other hand, since $X \cap A_0 \subseteq X - B_i$, we have

$$\begin{aligned} |X \cap A_0| &\leq |X - B_i| \\ &= r_\sigma(X) - r_\sigma(B_i) \\ &\leq (2n+2) - (n+i+1) \\ &= n - i + 1. \end{aligned} \tag{5.2}$$

Hence $|X \cap A_0| \leq |X \cap B_0|$. By taking the contrapositive, we have the desired statement. Statement (2) can be proven in the same way. For (3), suppose that $X \subseteq E$ is nonspanning and dependent. Then $\emptyset, E \notin R_\sigma(X)$. So we have either $A_i \in R_\sigma(X)$ for some i with $0 \leq i \leq n$ or $B_j \in R_\sigma(X)$ for some j with $0 \leq j \leq n$. Since the inequalities in (5.1) and (5.2) are strict, we have $|X \cap A_0| - |X \cap B_0| \geq 2$ or $|X \cap B_0| - |X \cap A_0| \geq 2$. Taking the contrapositive gives the desired statement. □

If $R_\sigma(X)$ doesn't lie in one of the saturated chains, then we can find $R_\sigma(X)$.

Corollary 5.1.2. *If $A_i, B_j \in R_\sigma(X)$, then $i = j$ and $R_\sigma(X) = \{\emptyset, A_i, B_i, E\}$. Furthermore, X is a basis.*

Proof. From the proof of Lemma 5.1.1, we have $|X \cap B_0| \leq n - i + 1 \leq |X \cap A_0|$ and $|X \cap A_0| \leq n - j + 1 \leq |X \cap B_0|$. It follows that $i = j$. Thus, no other A_i's and B_j's are in $R_\sigma(X)$. Since $R_\sigma(X)$ is a lattice, we know that $\emptyset, E \in R_\sigma(X)$, i.e., X is a basis. □

Now we show the main result of this section.

Theorem 5.1.3. *All matroids $M_\sigma(E)$ with $\sigma \in S_n$ have the same Tutte polynomial.*

Proof. Recall that $M = M_{id}$. It suffices to show that M and M_σ with $\sigma \neq id$ have the same Tutte polynomial. We need to show that there is a bijection $\varphi : 2^E \to 2^E$ that, when viewed as a map from the subsets of $M(E)$ to the subsets of $M_\sigma(E)$, preserves the cardinality and rank of each subset, i.e., $|X| = |\varphi(X)|$ and $r(X) = r_\sigma(\varphi(X))$ for all $X \subseteq E$. (Note that we cannot have a bijection $\varphi : E \to E$ that preserves the rank and cardinality of all subsets since the matroids are not isomorphic.)

Partition 2^E into

$$S_A = \{X \subseteq E : |X \cap A_0| > |X \cap B_0|\},$$
$$S_B = \{X \subseteq E : |X \cap A_0| < |X \cap B_0|\}, \quad \text{and}$$
$$S_C = \{X \subseteq E : |X \cap A_0| = |X \cap B_0|\}.$$

Notice that the partition does not depend on σ since A_0 and B_0 do not depend on σ. Now we define φ on each block separately.

On S_A, define φ to be the identity map. For $X \in S_A$, we have $R(X) \subseteq C_A$ and $R_\sigma(X) \subseteq C_A$ by Lemma 5.1.1. Since the chain C_A is independent of the choice of σ, we have $R(X) = R_\sigma(X)$. Hence $r(X) = r_\sigma(X) = r_\sigma(\varphi(X))$.

On S_B, define

$$\varphi(X) = (X - \{x_1, \ldots, x_n\}) \cup \{x_{\sigma(i)} : x_i \in X\}.$$

Clearly, φ is a bijection of S_B that preserves cardinality. For $X \in S_B$, we have $R(X) \subseteq C_B$ and $R_\sigma(X) \subseteq C_B^\sigma$ by Lemma 5.1.1. Each $x_{\sigma(i)}$ in C_B^σ plays the same role as x_i in C_B because it is the unique element paired up with y_i. So we have $r(X) = r_\sigma(\varphi(X))$. Hence φ is the desired bijection on S_B.

On S_C, define φ to be the identity map. By Lemma 5.1.1, $X \in S_C$ is spanning or independent in both $M(E)$ and $M_\sigma(E)$. Hence $r(X) = \min\{|X|, 2n+2\} = r_\sigma(\varphi(X))$. □

5.2 A Generalization of Giménez's Construction

We generalize Giménez's construction to obtain a larger family of nonisomorphic matroids with isomorphic lattices of cyclic flats and the same Tutte polynomial. To achieve this goal, we again define these matroids in terms of their lattices of cyclic flats.

Let n, s, and t be positive integers. Let Σ be an $s \times t$ array of permutations on n

elements

$$\Sigma = \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} & \dots & \sigma_{1,t} \\ \sigma_{2,1} & \sigma_{2,2} & \dots & \sigma_{2,t} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{s,1} & \sigma_{s,2} & \dots & \sigma_{s,t} \end{pmatrix}.$$

Thus, there are $(n!)^{st}$ choices for Σ. Let $S_n^{s,t}$ denote the set of arrays Σ. We construct a family $\{M_\Sigma : \Sigma \in S_n^{s,t}\}$ of $(n!)^{st}$ nonisomorphic matroids. Clearly, the sequence of these families grows superexponentially with respect to the cardinality of the ground set. They all have isomorphic lattices of cyclic flats in $\mathrm{PCW}(s+t)$.

To orient the reader before we address the details, we describe the global structure of $\mathcal{Z}_\Sigma$, the lattice of cyclic flats of M_Σ. Figure 5.2 may help the reader to understand this construction. The lattice $\mathcal{Z}_\Sigma$ will be a primitive width-$(s+t)$ lattice of sets under inclusion and will have $s+t$ saturated chains $C_0, C_1^\Sigma, \dots, C_s^\Sigma, C_1', \dots, C_{t-1}'$, where the superscript Σ indicates that some sets in the chain depend on Σ. The $s+1$ chains $C_0, C_1^\Sigma, \dots, C_s^\Sigma$ will consist of $n+3$ sets each and will have the form $\emptyset \lessdot A_{i,0} \lessdot \cdots \lessdot A_{i,n} \lessdot E$. The other $t-1$ chains $C_1', \dots, C_{t-1}'$ will consist of three sets each and will have the form $\emptyset \lessdot H_k \lessdot E$. Thus, the proper nontrivial cyclic flats of M_Σ are $(s+1)(n+1)$ sets $A_{i,j}$, with $0 \le i \le s$ and $0 \le j \le n$, and $t-1$ sets H_k, with $1 \le k \le t-1$. Clearly, all choices of $\Sigma \in S_n^{s,t}$ yield isomorphic lattices.

We first construct $s+t$ sets that are the atoms of the lattice $\mathcal{Z}_\Sigma$, namely, $s+1$ sets $A_{i,0}$, with $0 \le i \le s$, and $t-1$ sets H_k, with $1 \le k \le t-1$. Let $A_{0,0}, A_{1,0}, \dots, A_{s,0}$ be disjoint sets with $|A_{i,0}| = nt+2+i$. Let D be an $n \times (t+1)$ array of distinct elements, none of which is in any set $A_{i,0}$, say

$$D = \begin{pmatrix} x_{1,1} & \dots & x_{1,t} & y_1 \\ x_{2,1} & \dots & x_{2,t} & y_2 \\ \vdots & \ddots & \vdots & \vdots \\ x_{n,1} & \dots & x_{n,t} & y_n \end{pmatrix}.$$

Let H_k consist of all entries in D except those in the k-th column (i.e., $x_{1,k}, \dots, x_{n,k}$) and

$nt+1+k$ additional elements, none of which is in any set $A_{i,0}$ or $H_{k'}$ ($k' \neq k$). Thus, $|H_k| = 2nt+1+k$. These $t-1$ sets H_k enable us to distinguish between the entries in D up to columns except the last two columns. It is important to note that the atoms, $A_{i,0}$ and H_k, of $\mathcal{Z}_\Sigma$ do not depend on Σ. The set E, which is the largest element in $\mathcal{Z}_\Sigma$, is defined to be the union of all elements in the sets $A_{i,0}$ and H_k.

Using the array of permutations $\Sigma \in S_n^{s,t}$, we next construct the rest of the sets $A_{i,j}$ for $j \neq 0$. For $1 \leq i \leq s$, let Σ_i be the i-th row of Σ, i.e., $\Sigma_i = (\sigma_{i,1}, \sigma_{i,2}, \ldots, \sigma_{i,t})$. Except for the sets $A_{0,j}$ in the saturated chain C_0, all sets $A_{i,j}$ with $j \neq 0$ depend on Σ_i. We define $s+1$ matrices, $\Sigma_0(D), \ldots, \Sigma_s(D)$ as follows. Let $\Sigma_0(D) = D$. For $1 \leq i \leq s$, let $\Sigma_i(D)$ be formed from D by permuting the entries of the k-th column of D as dictated by $\sigma_{i,k}$ in Σ_i for $1 \leq k \leq t$, and fixing the last column of D. So we have

$$\Sigma_i(D) = \begin{pmatrix} x_{\sigma_{i,1}(1),1} & \cdots & x_{\sigma_{i,t}(1),t} & y_1 \\ x_{\sigma_{i,1}(2),1} & \cdots & x_{\sigma_{i,t}(2),t} & y_2 \\ \vdots & \ddots & \vdots & \vdots \\ x_{\sigma_{i,1}(n),1} & \cdots & x_{\sigma_{i,t}(n),t} & y_n \end{pmatrix}.$$

For $1 \leq j \leq n$, let $A_{i,j}$ be the union of $A_{i,0}$ and the set of entries in the first j rows of $\Sigma_i(D)$. Thus, $|A_{i,j} - A_{i,j-1}| = t+1$.

In summary, the cardinalities of the sets in $\mathcal{Z}_\Sigma$ are

$$\begin{aligned}
|A_{i,j}| &= nt + j(t+1) + 2 + i, \\
|H_k| &= 2nt + 1 + k, \text{ and} \\
|E| &= |D| + \sum_{i=0}^{s} |A_{i,0}| + \sum_{k=1}^{t-1} |H_k - D| \\
&= n(t+1) + \sum_{i=0}^{s} (nt+2+i) + \sum_{k=1}^{t-1} (nt+1+k) \\
&= n(t+1) + (nt+2)(s+1) + \binom{s+1}{2} + (nt+1)(t-1) + \binom{t}{2} \\
&= n(t+1) + (nt+1)(s+1) + \binom{s+2}{2} + nt^2 - nt - 1 + \binom{t+1}{2} \\
&= nt(s+t+1) + n + s + \binom{s+2}{2} + \binom{t+1}{2}.
\end{aligned}$$

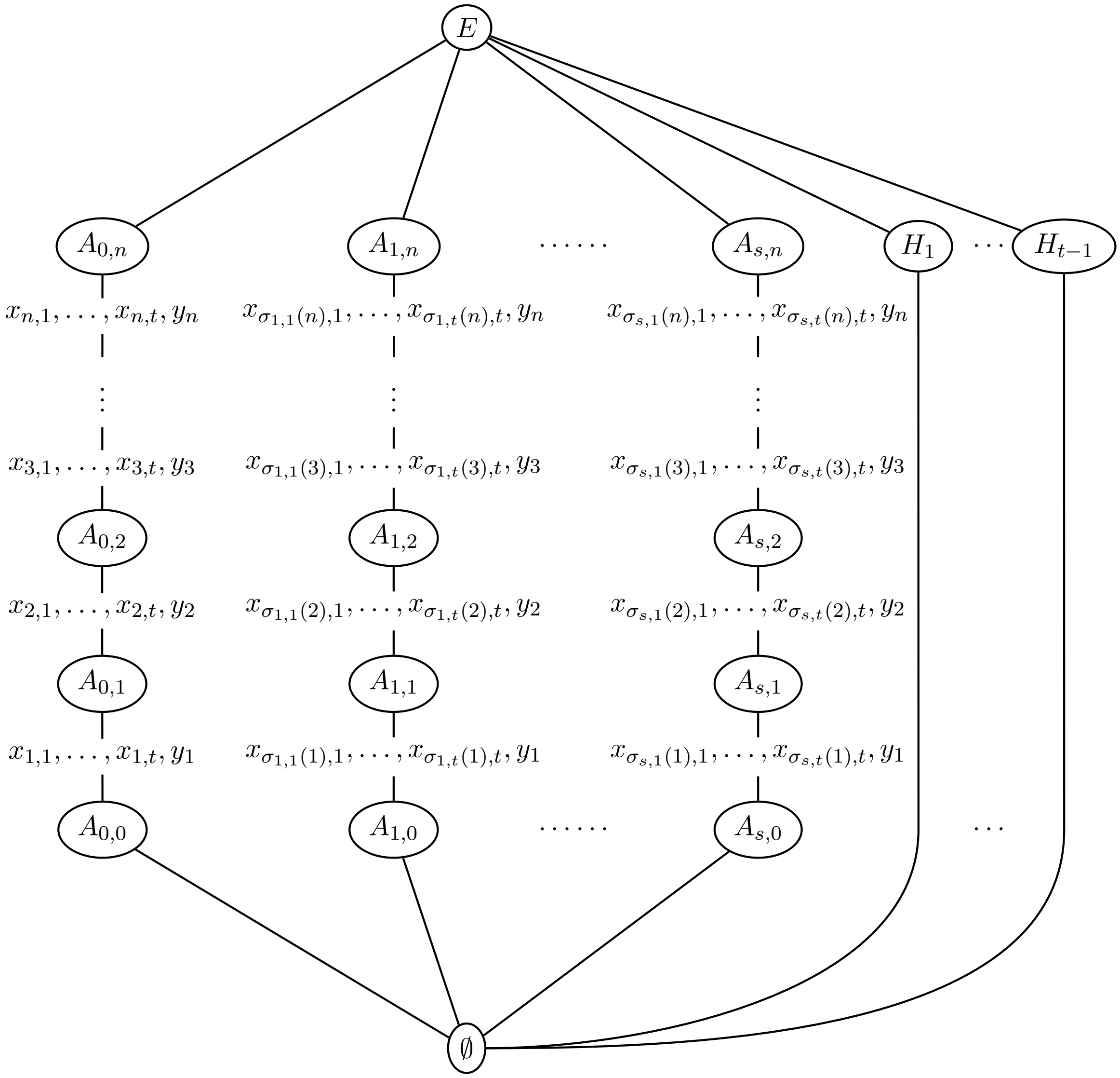

Figure 5.2: The lattice of cyclic flats $\mathcal{Z}_\Sigma$

Finally, we define a function $r_\Sigma : \mathcal{Z}_\Sigma \to \mathbb{Z}$ by

$$\begin{aligned} r_\Sigma(\emptyset) &= 0, \\ r_\Sigma(A_{i,j}) &= (n+j)t+1 \quad \text{for } 0 \le i \le s \text{ and } 0 \le j \le n, \\ r_\Sigma(H_k) &= 2nt+1 \quad \text{for } 1 \le k \le t-1, \text{ and} \\ r_\Sigma(E) &= 2nt+2. \end{aligned}$$

We show that this construction, for each choice of Σ, defines a matroid M_Σ. Note that the case $s = t = 1$ yields Giménez's construction, discussed in Section 5.1.

Lemma 5.2.1. *The lattice $\mathcal{Z}_\Sigma$ and the function r_Σ define a matroid M_Σ on E for which $\mathcal{Z}_\Sigma$ is the lattice of cyclic flats of M_Σ and r_Σ is the rank function of M_Σ restricted to $\mathcal{Z}_\Sigma$. Moreover, no two of the $(n!)^{st}$ matroids M_Σ are isomorphic.*

Proof. We show that $\mathcal{Z}_\Sigma$ and r_Σ satisfy conditions (Z0)–(Z3) in Theorem 3.1.1. By construction, (Z0), (Z1), and (Z2) hold.

The inequality in (Z3) automatically holds for comparable sets, so we need only check this inequality for pairs of sets Z_1 and Z_2 that are in different saturated chains; as a result, their join is E and their meet is $\emptyset$, so the required inequality simplifies to

$$r_\Sigma(Z_1) + r_\Sigma(Z_2) \geq 2nt + 2 + |Z_1 \cap Z_2|.$$

We treat this in three cases according to the number of sets Z_1 and Z_2 that are among $H_1, \ldots, H_{t-1}$. For $A_{i,j}$ and $A_{i',j'}$ with $i \neq i'$, we have $|A_{i,j} \cap A_{i',j'}| \leq (t+1)\min\{j, j'\}$. So the required inequality follows since

$$\begin{aligned} r_\Sigma(A_{i,j}) + r_\Sigma(A_{i',j'}) &= (n+j)t + 1 + (n+j')t + 1 \\ &= 2nt + 2 + t(j + j') \\ &\geq 2nt + 2 + 2t\min\{j, j'\} \\ &\geq 2nt + 2 + (t+1)\min\{j, j'\} \\ &\geq 2nt + 2 + |A_{i,j} \cap A_{i',j'}|. \end{aligned}$$

For H_k and $A_{i,j}$, we have $|H_k \cap A_{i,j}| = jt$. So the required inequality follows since

$$\begin{aligned} r_\Sigma(H_k) + r_\Sigma(A_{i,j}) &= 2nt + 1 + (n+j)t + 1 \\ &> 2nt + 2 + jt \\ &= 2nt + 2 + |A_{i,j} \cap H_k|. \end{aligned}$$

For H_k and $H_{k'}$ with $k \neq k'$, we have $|H_k \cap H_{k'}| = n(t-1)$. So the required inequality follows since

$$\begin{aligned} r_\Sigma(H_k) + r_\Sigma(H_{k'}) &= 2nt + 1 + 2nt + 1 \\ &> 2nt + 2 + n(t-1) \\ &= 2nt + 2 + |H_k \cap H_{k'}|. \end{aligned}$$

Finally, we show that the matrix $\Sigma = (\sigma_{i,k})$ of permutations can be recovered from $\mathcal{Z}(M)$ for any matroid M isomophic to M_Σ. To recover $\sigma_{i,k}$, we first identify the set of $n(t+1)$ elements that correspond to the entries of D. Notice that the chain C_0 can be easily identified by the cardinality of $A_{0,0}$. The elements in the j-th row of D are $\{x_{j,1}, \ldots, x_{j,t}, y_j\}$; these are precisely the elements in $A_{0,j} - A_{0,j-1}$ in the chain C_0. For $1 \leq k \leq t-1$, the elements in the k-th column of D are $\{x_{1,k}, \ldots, x_{n,k}\}$; these are precisely the elements in all $H_{k'}$ with $k' \neq k$ but not in H_k. Notice that the set of elements in the t-th and $(t+1)$-st columns of D, $\{x_{1,t}, \ldots, x_{n,t}, y_1, \ldots, y_n\}$, is precisely the set of elements in every H_k. Two distinct elements of a matroid M are called *clones* if the map that interchanges the two but fixes every other element of the ground set is an isomorphism of M. If $x_{j,t}$ and y_j are clones, then $\sigma_{i,t}(j) = j$ for all i. Otherwise it is easy to distinguish $x_{j,t}$ from y_j because only y_j is in $A_{i,j} - A_{i,j-1}$ for all i. Now we can recover the permutation $\sigma_{i,k}$ by comparing the arrangement of elements $\{x_{1,k}, \ldots, x_{n,k}\}$ in chain C_0 and C_i^Σ. Hence different sets of permutations give nonisomorphic matroids. □

Now we have constructed a family of $(n!)^{st}$ nonisomorphic matroids $\{M_\Sigma\}$ having the same lattice of cyclic flats. We omit the subscript Σ when all st permutations $\sigma_{i,k}$ in Σ are the identity permutations. Our goal in this section is to prove that all matroids in this family have the same Tutte polynomial.

Theorem 5.2.2. *Let n, s, and t be any positive integers. There are at least $(n!)^{st}$ nonisomorphic matroids on $nt(s+t+1) + n + s + \binom{s+2}{2} + \binom{t+1}{2}$ elements with the same lattice of cyclic flats and the same Tutte polynomial.*

In order to prove Theorem 5.2.2, it suffices to show that M and M_Σ have the same Tutte polynomial. The basic idea of the proof is similar to that in the proof of Theorem 5.1.3.

Let R_Σ denote R_{M_Σ} in Definition 3.1.3, i.e., for $X \subseteq E$,

$$R_\Sigma(X) = \{Z \in \mathcal{Z}_\Sigma : r_\Sigma(X) = r_\Sigma(Z) + |X - Z|\}.$$

Recall that $R_\Sigma(X)$ is the set of cyclic flats that determine $r_\Sigma(X)$. We will show that for a given nonspanning dependent subset $X \subseteq E$ in the matroid M_Σ, the sublattice $R_\Sigma(X)$ of $\mathcal{Z}_\Sigma$ is contained in a single saturated chain of $\mathcal{Z}_\Sigma$, and this chain is determined by the cardinalities of $X \cap Z$, where Z ranges over the atoms of $\mathcal{Z}_\Sigma$, i.e., $A_{i,0}$ and H_k. Since these atoms are not altered by Σ, we use this result to establish a rank and cardinality preserving bijection φ between the subsets of the ground set of the two matroids M and M_Σ.

Consider the following $s + t + 1$ sets of subsets of 2^E:

$$S_i = \{X \subseteq E : |X \cap A_{i,0}| \geq |X \cap A_{i',0}| + 2 \text{ for all } i' \neq i\} \text{ for } 0 \leq i \leq s,$$
$$T_k = \{X \subseteq E : X \subseteq H_k, |X| > 2nt + 1\} \text{ for } 1 \leq k \leq t - 1, \text{ and}$$
$$U = \{X \subseteq E : X \notin S_i \text{ and } X \notin T_k \text{ for all } i \text{ and } k\}.$$

We will show that this is indeed a partition of 2^E and we will define the bijection φ separately on the $s + t + 1$ blocks, S_i, T_k, and U. The next lemma shows the relationship between the cardinality of $X \cap A_{i,0}$ and the assertion $A_{i,j} \in R_\Sigma(X)$.

Lemma 5.2.3. *Let X be a subset of E and $A_{i,j} \in R_\Sigma(X)$ for some i and j. Then we have $|X \cap A_{i,0}| \geq |X \cap A_{i',0}|$ for all $i' \neq i$. Furthermore, if X is nonspanning and dependent, then $|X \cap A_{i,0}| \geq |X \cap A_{i',0}| + 2$ for all $i' \neq i$.*

Proof. If $A_{i,j} \in R_\Sigma(X)$, then we have

$$\begin{aligned}
(n+j)t+1 &= r_\Sigma(A_{i,j}) \\
&= r_\Sigma(X) - |X - A_{i,j}| \\
&\leq |X| - |X - A_{i,j}| \qquad (5.3) \\
&= |X \cap A_{i,j}| \\
&\leq |X \cap A_{i,0}| + |A_{i,j} - A_{i,0}| \\
&\leq |X \cap A_{i,0}| + j(t+1).
\end{aligned}$$

Thus,

$$nt - j + 1 \leq |X \cap A_{i,0}|.$$

On the other hand, we have

$$\begin{aligned}
|X \cap A_{i',0}| &\leq |X - A_{i,j}| \\
&= r_\Sigma(X) - r_\Sigma(A_{i,j}) \\
&\leq 2nt + 2 - ((n+j)t+1) \qquad (5.4) \\
&= nt - jt + 1.
\end{aligned}$$

Since $j \geq 0$ and $t \geq 1$, we have $|X \cap A_{i,0}| \geq |X \cap A_{i',0}|$ as desired.

If X is nonspanning and dependent, then the inequalities in (5.4) and (5.3) are strict, so we have $|X \cap A_{i,0}| - |X \cap A_{i',0}| \geq 2$ for all $i' \neq i$. □

The next lemma shows the relationship between the cardinality of $X \cap A_{i,0}$ and the assertion $H_k \in R_\Sigma(X)$.

Lemma 5.2.4. *Let $X \subseteq E$. If $H_k \in R_\Sigma(X)$ for some k with $1 \leq k \leq t-1$, then $|X \cap (\bigcup_{i=0}^{s} A_{i,0})| \leq 1$.*

Proof. Since $X \cap A_{i,0} \subseteq X - H_k$ for $0 \leq i \leq s$ and H_k is a hyperplane of M_Σ, we have

$$\left| X \cap \left(\bigcup_{i=0}^{s} A_{i,0} \right) \right| \leq |X - H_k| = r_\Sigma(X) - r_\Sigma(H) \leq r_\Sigma(E) - r_\Sigma(H) = 1.$$ □

Recall that $C_0, C_1^\Sigma, \ldots, C_s^\Sigma, C'_1, \ldots, C'_{t-1}$ are $s+t$ saturated chains of cyclic flats in $\mathcal{Z}_\Sigma$. For notational convenience, let $C_0^\Sigma = C_0$. Thus, $C_i^\Sigma = \{\emptyset, A_{i,0}, \ldots, A_{i,n}, E\}$ for $0 \le i \le s$. Lemmas 5.2.3 and 5.2.4 have the following corollary, which gives a key property of S_i.

Corollary 5.2.5. *Let $X \subseteq E$. If $X \in S_i$, then $R_\Sigma(X) \subseteq C_i^\Sigma$ for all Σ.*

Proof. Since $X \in S_i$, we have $|X \cap A_{i,0}| \ge |X \cap A_{i',0}| + 2$ for all $i' \ne i$. By the contrapositives of Lemmas 5.2.3 and 5.2.4, we have $A_{i',j} \notin R_\Sigma(X)$ for all $i' \ne i$ and all j, and $H_k \notin R_\Sigma(X)$ for all k. Hence $R_\Sigma(X) \subseteq C_i^\Sigma$. □

The next two lemmas treat properties of T_k and U.

Lemma 5.2.6. *Let $X \subseteq E$. Then $X \in T_k$ if and only if $R_\Sigma(X) = \{H_k\}$ for all (equivalently, some) Σ.*

Proof. Suppose $X \in T_k$. Then

$$r_\Sigma(X) \le r_\Sigma(H_k) = 2nt + 1 < |X|;$$

hence X is dependent and $\mathrm{cyc}(X) \ne \emptyset$. So we have

$$\emptyset \subsetneq 0_{R_\Sigma(X)} \subseteq \mathrm{cl}(X) \subseteq H_k.$$

Since $\emptyset \lessdot H_k$ in the lattice $\mathcal{Z}_\Sigma(M)$, we have $0_{R_\Sigma(X)} = \mathrm{cl}(X) = H_k$. By taking the cyclic parts of each term, we get $0_{R_\Sigma(X)} = 1_{R_\Sigma(X)} = H_k$, i.e., $R(X) = \{H_k\}$.

Suppose $R_\Sigma(X) = \{H_k\}$. Since $\emptyset, E \notin R_\Sigma(X)$, we know that X is dependent and nonspanning. So we have

$$r_\Sigma(H_k) + |X - H_k| = r_\Sigma(X) \le r_\Sigma(H_k).$$

It follows that $X - H_k = \emptyset$ and $r_\Sigma(X) = r_\Sigma(H_k)$. The first implies $X \subseteq H_k$. The second implies

$$|X| > r_\Sigma(X) = r_\Sigma(H_k) = 2nt + 1$$

because X is dependent. Hence $X \in T_k$. □

Lemma 5.2.7. *Let $X \subseteq E$. If $X \in U$, then X is spanning or independent.*

Proof. We prove the contrapositive of the statement. Suppose X is nonspanning and dependent. Then $\emptyset, E \notin R_\Sigma(X)$ by Lemma 3.1.4. Thus, either $H_k \in R_\Sigma(X)$ for some k or $A_{i,j} \in R_\Sigma(X)$ for some i and j (but not both, as $R_\Sigma(X)$ is a lattice). In the first case, we have $R_\Sigma(X) = \{H_k\}$, hence $X \in T_k$ by Lemma 5.2.6. In the second case, we have $|X \cap A_{i,0}| - |X \cap A_{i',0}| \geq 2$ for all $i' \neq i$ by Lemma 5.2.3, hence $X \in S_i$. It follows that $X \notin U$. □

Now, we can verify that S_i, T_k, and U partition 2^E.

Lemma 5.2.8. *In $M_\Sigma(E)$, the $s + t + 1$ subsets S_i, T_k, and U of 2^E partition 2^E.*

Proof. Notice that $U \neq \emptyset$ because $\emptyset \in U$. Clearly, S_i and T_k are nonempty. Thus, it suffices to show that all sets are disjoint. We need to check $S_i \cap T_k = \emptyset$ for all i and k, and $T_k \cap T_{k'} = \emptyset$ for $k \neq k'$. Both cases are immediate from Corollary 5.2.5 and Lemma 5.2.6. □

Using Corollary 5.2.5 and Lemmas 5.2.6 and 5.2.7, we prove Theorem 5.2.2.

Proof of Theorem 5.2.2. By Lemma 5.2.1, all $(n!)^{st}$ nonisomorphic matroids in the family $\{M_\Sigma\}$ have isomorphic lattices of cyclic flats. Let Σ be any $s \times t$ array of permutations in S_n. It suffices to show that M and M_Σ have the same Tutte polynomial. We will establish a bijection $\varphi : 2^E \to 2^E$ which, when viewed as a map from the subsets of $M(E)$ to the subsets of $M_\Sigma(E)$, preserves the cardinality and the rank of each subset, i.e., $|X| = |\varphi(X)|$ and $r(X) = r_\Sigma(\varphi(X))$ for all $X \subseteq E$. Consider the $s + t + 1$ sets of subsets S_i, T_k, and U of 2^E defined above. These sets indeed partition 2^E by Lemma 5.2.8. Notice that this partition does not depend on Σ since the sets $A_{i,0}$ and H_k do not. We define φ by defining it separately on the $s + t + 1$ blocks S_i, T_k, and U.

On S_0, define φ to be the identity map. For $X \in S_0$, we have both $R(X) \subseteq C_0$ and $R_\Sigma(X) \subseteq C_0$ by Corollary 5.2.5. Since C_0 is independent of Σ, $R(X) = R_\Sigma(X)$. Hence $r(X) = r_\Sigma(\varphi(X))$.

On S_i with $1 \leq i \leq s$, let $\varphi(X)$ be obtained from X by replacing each $x_{j,k}$ that is in X by $x_{\sigma_{i,k}(j),k}$. Clearly, φ is a bijection of S_i that preserves cardinality. By Corollary 5.2.5, we have $R(X) \subseteq C_i$ and $R_\Sigma(X) \subseteq C_i^\Sigma$ for $X \in S_i$. Notice that $x_{\sigma_{i,k}(j),k}$ in C_i^Σ plays the same role as $x_{j,k}$ in C_i due to how these chains are obtained via the rows of D and $\Sigma_i(D)$. So we have $r(X) = r_\Sigma(\varphi(X))$. Hence φ is the desired bijection on S_i.

On T_k with $1 \leq k \leq t-1$, define φ to be the identity map. By Lemma 5.2.6, $X \in T_k$ implies $R(X) = R_\Sigma(X) = \{H_k\}$. Since $X \subseteq H_k$, we have $r(X) = r(H_k) = r_\Sigma(H_k) = r_\Sigma(X)$.

On U, define φ to be the identity map. For $X \in U$, X is spanning or independent in both $M(E)$ and $M_\Sigma(E)$ by Lemma 5.2.7. Hence $r(X) = \min\{|X|, 2nt+2\} = r_\Sigma(X)$. □

5.3 Semi-Magic Square Matroids

In this section, another generalization of Giménez's construction is explored with the same aim of producing larger families of matroids with isomorphic lattices of cyclic flats and the same Tutte polynomials. Unlike the generalization in the previous section, we restrict ourselves to the class of primitive cyclic width-2 matroids. Giménez's construction in Section 5.1 gave a family of $n!$ matroids in PCW(2), one matroid for each permutation σ of n elements. Here, the chains of cyclic flats are constructed via an $n \times n$ semi-magic square instead of a permutation of n elements.

An (n,t)-*semi-magic square* is an $n \times n$ square matrix whose entries are nonnegative integers in which the sum of the entries in each row, as well as the sum of those in each column, is t (see [1]). We note that an $(n,1)$-semi-magic square is a permutation matrix. In general, by Birkhoff's theorem (see, e.g.,[13]), (n,t)-semi-magic squares are sums of t permutation matrices.

In this section, we first introduce and discuss the family of (n,t)-semi-magic square matroids defined by (n,t)-semi-magic squares; the number of such nonisomorphic matroids is the number of (n,t)-semi-magic squares. Then we consider a more general construction of matroids called R-single-chain matroids. This construction is a generalization of Giménez's construction. Indeed, it will not be difficult to see that semi-magic square matroids are special case of R-single-chain matroids. These matroids have the property that $R(X)$ is always contained in one of the two saturated chains of cyclic flats whenever X is dependent and nonspanning. We will show that two R-single-chain matroids have the same Tutte polynomial if they have isomorphic lattices of cyclic flats and corresponding cyclic flats have the same rank and cardinality.

For a given (n,t)-semi-magic square $S = (s_{i,j})$ with $t \geq 2$, we construct a matroid M_S. For $1 \leq i \leq n$ and $1 \leq j \leq n$, let $T_{i,j}$ be disjoint sets with $|T_{i,j}| = s_{i,j}$. Let A_0 and B_0 be sets disjoint from each other and from all sets $T_{i,j}$ with $|A_0| = n(t-1)+2$ and $|B_0| = n(t-1)+3$. From A_0, B_0, and $T_{i,j}$, we will construct the rest of the sets that will form $\mathcal{Z}(M_S)$. Let

$$A_i = A_{i-1} \cup \bigcup_{j=1}^{n} T_{i,j} \quad \text{for } 1 \leq i \leq n, \text{ and}$$

$$B_j = B_{j-1} \cup \bigcup_{i=1}^{n} T_{i,j} \quad \text{for } 1 \leq j \leq n.$$

The set E is defined to be the union of A_0, B_0, and all $T_{i,j}$. From this construction, it is clear that $\mathcal{Z}(M_S)$ is the primitive width-2 lattice

$$\mathcal{Z}(M_S) = \{\emptyset : A_0, \ldots, A_n : B_0, \ldots, B_n : E\}.$$

The cardinalities for some of the important sets are

$$\begin{aligned} |A_i - A_{i-1}| = |B_j - B_{j-1}| &= t, \\ |(A_i - A_{i-1}) \cap (B_j - B_{j-1})| &= |T_{i,j}| = s_{i,j}, \text{ and} \\ |E| &= |A_0| + |B_0| + \sum s_{i,j} \\ &= n(t-1) + 2 + n(t-1) + 3 + nt \\ &= 3nt - 2n + 5. \end{aligned}$$

Finally define the ranks of these cyclic flats as follows:

$$\begin{aligned} r(\emptyset) &= 0, \\ r(A_i) = r(B_i) &= (n+i)(t-1) + 1, \text{ and} \\ r(M) &= 2n(t-1) + 2. \end{aligned}$$

This construction defines a matroid M_S; we call it an (n,t)-*semi magic square matroid*.

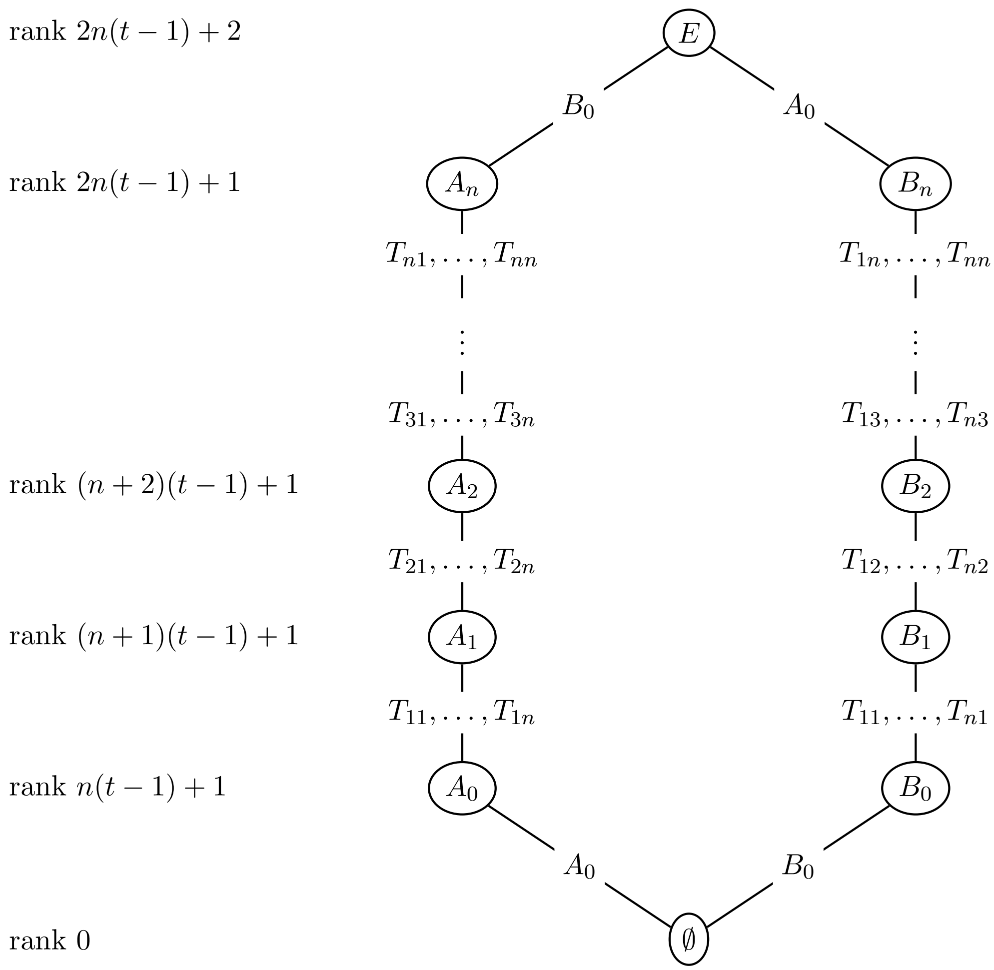

Figure 5.3: The lattice of cyclic flats $\mathcal{Z}(M_S)$ of the (n, t)-semi-magic square matroid M_S

However, we will not prove that M_S is indeed a matroid until Lemma 5.3.2.

By the observations on cardinalities above, the (n, t)-semi-magic square S can be recovered from the (n, t)-semi-magic square matroid M_S. Thus, the number of nonisomorphic (n, t)-semi-magic square matroids is precisely the number of (n, t)-semi-magic squares, which is denoted by $H_n(t)$. For small values of n, we have

$$H_1(t) = 1,$$

$$H_2(t) = t + 1, \text{ and}$$

$$H_3(t) = 3\binom{t+3}{4} + \binom{t+2}{2}.$$

It is very difficult to calculate $H_n(t)$ for $n \geq 4$. The interested reader is referred to [2] (which uses complex analysis) and [8] (which derives asymptotic upper bounds). For fixed t, the number $H_n(t)$ grows superexponentially with respect to n.

Notice that the case $t = 2$ includes the construction by Giménez in Section 5.1. For each $\sigma \in S_n$, let $P_\sigma = (a_{i,j})$ be the $(n, 1)$-semi-magic square (i.e., the permutation matrix) such that

$$a_{i,j} = \begin{cases} 1 & \text{if } i = \sigma(j) \\ 0 & \text{otherwise.} \end{cases}$$

Then the semi-magic square representation of M_σ is $P_\sigma + I$, where I is the $n \times n$ identity matrix. Here, P_σ and I represent the arrangement of sets of elements $\{x_i\}$ and $\{y_i\}$, respectively. Thus, the number of semi-magic square matroids is a lot larger than the number of matroids in Giménez's construction. For example, for $n = 3$, there are $3! = 6$ matroids in the family $\{M_\sigma\}$. However, the number of $(3, 2)$-semi-magic squares is known to be $H_3(2) = 21$; there are 21 nonsomorphic $(3, 2)$-semi-magic square matroids.

We need to show two things about (n, t)-semi-magic square matroids. First, we need to show that they are indeed matroids. Second, we need to show that they have the desired property, i.e., they all have the same Tutte polynomial. We treat both of these statements in a more general context, which we introduce next. As Lemma 5.3.2 shows, (n, t)-semi-magic square matroids are among the matroids addressed in the next lemma.

Lemma 5.3.1. *Let E be a finite set and n and m be positive integers. Let $A_0, \ldots, A_n$ and $B_0, \ldots, B_m$ be subsets of E such that $\mathcal{Z} = \{\emptyset : A_0, \ldots, A_n : B_0, \ldots, B_m : E\}$ is a primitive width-2 lattice under inclusion. Also let r be an integer-valued function on $\mathcal{Z}$ such that*

(Z1) $r(\emptyset) = 0$,

(Z2) $0 < r(Z_2) - r(Z_1) < |Z_2 - Z_1|$ *for all* $Z_1, Z_2 \in \mathcal{Z}$ *with* $Z_1 \subsetneq Z_2$,

(Z3.1) $|A_0 \cap B_0| \le r(A_0) + r(B_0) - r(E)$,

(Z3.2) $|(A_i - A_0) \cap B_0| \le 2r(A_i) - 2r(A_0) - |A_i - A_0|$ *for* $1 \le i \le n$ *and*

$|(B_j - B_0) \cap A_0| \le 2r(B_j) - 2r(B_0) - |B_j - B_0|$ *for* $1 \le j \le m$, *and*

(Z3.3) $|(A_i \cap B_j) - (A_0 \cap B_0)| \le r(A_i) - r(A_0) + r(B_j) - r(B_0)$

for $1 \le i \le n$ *and* $1 \le j \le m$.

Then the lattice $\mathcal{Z}$ and the function r define a matroid M on E for which $\mathcal{Z}(M) = \mathcal{Z}$ and

$r_M|_{\mathcal{Z}(M)} = r$.

Proof. We show that $\mathcal{Z}$ and r satisfy conditions (Z0)–(Z3) in Theorem 3.1.1. By assumption, $\mathcal{Z}$ is a lattice, i.e., (Z0) holds. Also (Z1) and (Z2) are given. We need to show (Z3) for all pairs of cyclic flats of the form (A_i, B_j). The case for the pair (A_0, B_0) is given by (Z3.1). Notice that $A_i \wedge B_j = \emptyset$ and $A_i \vee B_j = E$. By (Z3.1), (Z3.2), and (Z2), we have

$$\begin{aligned}
|A_i \cap B_0| &= |A_0 \cap B_0| + |(A_i - A_0) \cap B_0| \\
&\leq r(A_0) + r(B_0) - r(E) + 2r(A_i) - 2r(A_0) - |A_i - A_0| \\
&= r(A_i) + r(B_0) - r(E) + r(A_i) - r(A_0) - |A_i - A_0| \\
&< r(A_i) + r(B_0) - r(E).
\end{aligned}$$

So (Z3) holds for all the pairs (A_i, B_0) with $1 \leq i \leq n$. Similarly, (Z3) holds for all pairs (A_0, B_j) with $1 \leq j \leq m$. Also by (Z3.1) and (Z3.3), we have

$$\begin{aligned}
|A_i \cap B_j| &= |A_0 \cap B_0| + |(A_i \cap B_j) - (A_0 \cap B_0)| \\
&\leq r(A_0) + r(B_0) - r(E) + r(A_i) - r(A_0) + r(B_j) - r(B_0) \\
&= r(A_i) + r(B_j) - r(E).
\end{aligned}$$

So (Z3) holds for all the pairs (A_i, B_j) with $1 \leq i \leq n$ and $1 \leq j \leq m$. Hence (Z3) holds. Thus, there is a matroid M such that $\mathcal{Z}(M) = \mathcal{Z}$ and $r_M|_{\mathcal{Z}(M)} = r$ by Theorem 3.1.1. □

We call a matroid M of the type described in Lemma 5.3.1 an *R-single-chain matroid* because the sublattice $R(X)$ of $\mathcal{Z}(M)$ stays in one of the two saturated chains whenever X is nonspanning and dependent. The result in Lemma 5.3.3 will explain this terminology. The next lemma shows that (n, t)-semi-magic square matroids are indeed R-single-chain matroids.

Lemma 5.3.2. *Every (n, t)-semi-magic square matroid is an R-single-chain matroid.*

Proof. We need to check the conditions in Lemma 5.3.1. Clearly (Z1), (Z2), and (Z3.1) hold. Condition (Z3.2) reduces to $0 \leq i(t-2)$ and so holds. For (Z3.3), since $A_i \cap B_0 = A_0 \cap B_j = \emptyset$, we have

$$
\begin{aligned}
|(A_i \cap B_j) - (A_0 \cap B_0)| &= |(A_i - A_0) \cap (B_j - B_0)| \\
&\leq \min\{|A_i - A_0|, |B_j - B_0|\} \\
&= t \min\{i, j\} \\
&\leq \frac{t}{2}(i + j) \\
&\leq (t-1)(i+j) \\
&= r(A_i) - r(A_0) + r(B_j) - r(B_0). \qquad \square
\end{aligned}
$$

Suppose M is an R-single-chain matroid. Let C_A and C_B denote its two saturated chains, $C_A = \{\emptyset, A_0, \ldots, A_n, E\}$ and $C_B = \{\emptyset, B_0, \ldots, B_m, E\}$. Now we show that for any nonspanning dependent set X, the set $R(X)$ is contained in one of these two saturated chains.

Lemma 5.3.3. *Let M be an R-single-chain matroid on E with the lattice of cyclic flats $\mathcal{Z}(M) = \{\emptyset : A_0, \ldots, A_n : B_0, \ldots, B_m : E\}$.*

(1) *If $|X \cap A_0| - |X \cap B_0| > r(A_0) - r(B_0)$, then $R(X) \subseteq C_A$.*

(2) *If $|X \cap A_0| - |X \cap B_0| < r(A_0) - r(B_0)$, then $R(X) \subseteq C_B$.*

(3) *If $r(A_0) - r(B_0) - 1 \leq |X \cap A_0| - |X \cap B_0| \leq r(A_0) - r(B_0) + 1$, then X is spanning or independent.*

Proof. Suppose $B_j \in R(X)$. Then

$$
\begin{aligned}
r(B_j) &= r(X) - |X - B_j| \\
&\leq |X| - |X - B_j| \\
&= |X \cap B_j| \\
&\leq |X \cap B_0| + |B_j - B_0|.
\end{aligned}
\tag{5.5}
$$

Hence

$$|X \cap B_0| \geq r(B_j) - |B_j - B_0|.$$

On the other hand, using

$$r(E) \le r(A_0) + r(B_0) - |A_0 \cap B_0| \quad \text{and}$$

$$|A_0 \cap B_j| \le |A_0 \cap B_0| + 2r(B_j) - 2r(B_0) - |B_j - B_0|$$

from (Z3.1) and (Z3.2), respectively, we obtain

$$\begin{aligned} |X \cap A_0| \le |X - B_j| &+ |A_0 \cap B_j| \\ &= r(X) - r(B_j) + |A_0 \cap B_j| \\ &\le r(E) - r(B_j) + |A_0 \cap B_j| \\ &\le r(A_0) + r(B_0) - |A_0 \cap B_0| - r(B_j) \\ &\quad + |A_0 \cap B_0| + 2r(B_j) - 2r(B_0) - |B_j - B_0| \\ &= r(A_0) - r(B_0) + r(B_j) - |B_j - B_0|. \end{aligned} \tag{5.6}$$

Thus, we have

$$|X \cap A_0| - |X \cap B_0| \le r(A_0) - r(B_0).$$

Taking the contrapositive yields (1). A similar argument gives (2). For (3), suppose that X is nonspanning and dependent. Then $\emptyset, E \notin R(X)$. So we have either $A_i \in R(X)$ or $B_j \in R(X)$ for some i or j. If $B_j \in R(X)$, then $|X \cap A_0| - |X \cap B_0| \le r(A_0) - r(B_0) - 2$ because inequalities (5.5) and (5.6) are strict. Similarly, if $A_i \in R(X)$, then

$$|X \cap A_0| - |X \cap B_0| \ge r(A_0) - r(B_0) + 2.$$

By taking the contrapositive, we obtain statement (3). □

Using Lemma 5.3.3, we next give sufficient conditions for two R-single-chain matroids with isomorphic lattices of cyclic flats to have the same Tutte polynomial.

Theorem 5.3.4. *Let n and m be positive integers. Let M and M' be two R-single-chain* *matroids on* E *with rank functions* r *and* r'*, respectively. Let their lattices of cyclic flats be*

$$\mathcal{Z}(M) = \{\emptyset : A_0, \ldots, A_n : B_0, \ldots, B_m : E\}$$

and

$$\mathcal{Z}(M') = \{\emptyset : A'_0, \ldots, A'_n : B'_0, \ldots, B'_m : E\}$$

such that

(1) $r(M) = r'(M')$,

(2) $|A_i| = |A'_i|$ *and* $r(A_i) = r'(A'_i)$,

(3) $|B_j| = |B'_j|$ *and* $r(B_j) = r'(B'_j)$,

(4) $|A_i \cap B_0| = |A'_i \cap B'_0|$, *and*

(5) $|A_0 \cap B_j| = |A'_0 \cap B'_j|$

for all i *and* j *with* $0 \leq i \leq n$ *and* $0 \leq j \leq m$*. Then* M *and* M' *have the same Tutte polynomial.*

Proof. Without loss of generality, we can relabel the elements of M' so that, for all i and j, we have $A_0 = A'_0$, $B_0 = B'_0$, $A_i \cap B_0 = A'_i \cap B'_0$, and $B_j \cap A_0 = B'_j \cap A'_0$. In particular, this implies $A_0 \cup B_0 = A'_0 \cup B'_0$. Also let C'_A and C'_B denote the two saturated chains of M'.

We define a bijection $\varphi : 2^E \to 2^E$ so that for every $X \subseteq E$, we have $|X| = |\varphi(X)|$ and $r(X) = r'(\varphi(X))$. First, partition 2^E as follows:

$$S_A = \{X \subseteq E : |X \cap A_0| - |X \cap B_0| > r(A_0) - r(B_0)\}$$

$$S_B = \{X \subseteq E : |X \cap A_0| - |X \cap B_0| < r(A_0) - r(B_0)\}$$

$$S_C = \{X \subseteq E : |X \cap A_0| - |X \cap B_0| = r(A_0) - r(B_0)\}.$$

We define φ on S_A, S_B, and S_C separately.

Observe that we have $|(A_{i+1} - A_i) - B_0| = |(A'_{i+1} - A'_i) - B'_0|$ for $0 \leq i \leq n-1$ and $|(E - A_n) - B_0| = |(E - A'_n) - B'_0|$. Let $\psi : E \to E$ be any bijection that maps $(A_{i+1} - A_i) - B_0$ to $(A'_{i+1} - A'_i) - B'_0$, and $(E - A_n) - B_0$ to $(E - A'_n) - B'_0$, and is the identity on $A_0 \cup B_0$.

On S_A, let $\varphi : S_A \to S_A$ be induced by ψ, i.e., $\varphi(X) = \{\psi(e) : e \in X\}$. Clearly, φ is a bijection on S_A that preserves cardinality. Recall that $r(X) = r(Z) + |X - Z|$ for all $Z \in R(X)$. Since φ fixes $A_0 \cup B_0$, we have $R_M(X) \subseteq C_A$ and $R_{M'}(\varphi(X)) \subseteq C'_A$ by Lemma 5.3.3. Also $|X - A_i| = |\varphi(X) - A'_i|$ for $0 \leq i \leq n$ because ψ is a bijection between A_i and A'_i for each i. Hence φ preserves rank and cardinality on S_A.

On S_B, we can construct $\varphi : S_B \to S_B$ in a similar fashion. Note that the map $\psi' : E \to E$ used to define φ on S_B differs from ψ.

On S_C, let φ be the identity map. Since X is spanning or independent in both M and M' by Lemma 5.3.3, it follows that φ preserves rank and cardinality. □

Recall that (n,t)-semi-magic square matroids are R-single-chain matroids by Lemma 5.3.2. Thus, we have the following result.

Corollary 5.3.5. *All matroids in the family of (n,t)-semi-magic square matroids have the same Tutte polynomial.*

5.4 Computing the Tutte Polynomial

Let n and t be positive integers with $t \geq 2$. Consider the family of (n,t)-semi-magic square matroids defined in Section 5.3. In this section, we will show that, for a particular choice of S, the (n,t)-semi-magic square matroid M_S is a lattice path matroid. As a result, the Tutte polynomial of (n,t)-semi-magic square matroids is computable in polynomial time in the cardinality of the ground set because the Tutte polynomial of a lattice path matroid is [4, Theorem 6.1].

To define lattice path matroids, we consider lattice paths from $(0,0)$ that use two types of steps: East step $E = (1,0)$ and North step $N = (0,1)$. Thus, each lattice path is a word in the alphabet $\{E, N\}$. Also we will use the notation α^n to denote the concatenation of n letters, or strings of letters, α. Furthermore, for two integers $n < m$, we will use the notation $[n,m]$ for the set of all integers between n and m inclusively, i.e., $\{n, \ldots, m\}$. In particular, we use $[n]$ for $[1,n]$.

Recall transversal matroids in Section 4.1. A lattice path matroid is a special type of transversal matroid that arises from two lattice paths.

Definition 5.4.1. Let $P = p_1 \dots p_{n+r}$ and $Q = q_1 \dots q_{n+r}$ be two lattice paths from $(0, 0)$ to (n, r) with P never going above Q. Let $\{p_{j_1}, \dots, p_{j_r}\}$ and $\{q_{k_1}, \dots, q_{k_r}\}$ be the North steps of P and Q with $j_1 < \dots < j_r$ and $k_1 < \dots < k_r$, respectively. Define $N_i = [k_i, j_i]$ for $1 \le i \le r$. The *lattice path matroid* $M[P, Q]$ is, up to isomorphism, the transversal matroid on the ground set $[n + r]$ that has $(N_1, \dots, N_r)$ as a presentation.

By construction, $M[P, Q]$ has rank r and nullity n. For each subset X of the ground set $[n+r]$ of $M[P, Q]$, define the lattice path $P(X)$ [4, Definition 3.2] as the word $s_1 s_2 \dots s_{n+r}$ in the alphabet $\{E, N\}$ where

$$s_i = \begin{cases} N & \text{if } i \in X \\ E & \text{if } i \notin X. \end{cases}$$

Notice that we can recover the set X from the lattice path $P(X)$ of $n+r$ steps. In particular, a lattice path from $(0, 0)$ to (n, r) is $P(X)$ for some X with $|X| = r$. Let $R[P, Q]$ denote the region bounded by P and Q. Then, for $1 \le i \le r$, we have

$$N_i = \{j : \text{step } j \text{ is the } i\text{-th North step of some } P(X) \text{ that stays in } R[P, Q]\}.$$

Furthermore, if the path $P(X)$ stays in $R[P, Q]$, then X is a basis of $M[P, Q]$ [4, Theorem 3.3].

Theorem 5.4.2. *A subset B of $[n + r]$ is a basis of $M[P, Q]$ if and only if the lattice path $P(B)$ stays in the region $R[P, Q]$.*

Fix two positive integers n and t with $t \ge 2$. We construct a lattice path matroid L and an (n, t)-semi-magic square matroid M_S on the same ground set $E = [3nt - 2n + 5]$. Then we will show that $L = M_S$.

Define the lattice path matroid $L = M[P, Q]$ on $[3nt - 2n + 5]$ by the two boundary

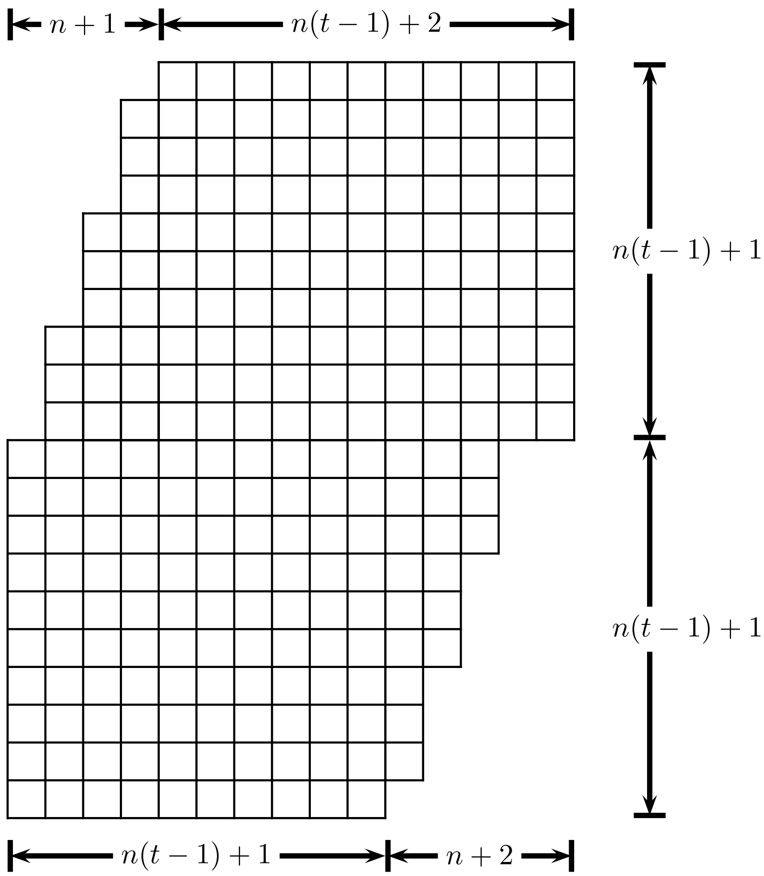

Figure 5.4: $M[P, Q]$ for $n = 3$ and $t = 4$

lattice paths

$$P = E^{n(t-1)+1}N(EN^{t-1})^n E^2 N^{n(t-1)+1} \quad \text{and}$$
$$Q = N^{n(t-1)+1}E(N^{t-1}E)^n N E^{n(t-1)+2}.$$

An example of the lattice diagram is shown in Figure 5.4. We have $r(L) = 2n(t-1)+2$ and $n(L) = nt + 3$ by construction.

Now we define the (n, t)-semi-magic square matroid M_S on the same ground set E. Let $S = (s_{i,j})$ be the (n, t)-semi-magic square with

$$s_{i,j} = \begin{cases} t & \text{if } i + j = n + 1 \\ 0 & \text{otherwise,} \end{cases}$$

i.e.,

$$S = \begin{pmatrix} 0 & 0 & \dots & 0 & t \\ 0 & 0 & \dots & t & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & t & \dots & 0 & 0 \\ t & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Let M_S be the (n,t)-semi-magic square matroid associated with S. From the construction described in Section 5.3, we have

$$\mathcal{Z}(M_S) = \{\emptyset : A_0, \dots, A_n : B_0, \dots, B_n : E\}$$

and

$$|A_0| = n(t-1) + 2,$$

$$|B_0| = n(t-1) + 3, \text{ and}$$

$$|T_{i,j}| = \begin{cases} t & \text{if } i + j = n + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Recall that these sets A_0, B_0, and $T_{i,n+1-i}$ partition the ground set $E = [3nt - 2n + 5]$. We assign integer intervals for each set:

$$A_0 = [n(t-1) + 2],$$

$$B_0 = [n(t-1) + nt + 3, 3nt - 2n + 5], \quad \text{and}$$

$$T_{i,n+1-i} = [n(t-1) + (i-1)t + 3, n(t-1) + it + 2].$$

Then we have

$$A_i = [n(t-1) + it + 2], \quad \text{and}$$

$$B_i = [n(t-1) + (n-i)t + 3, 3nt - 2n + 5]$$

for $1 \le i \le n$. Let r_S denote the rank function of M_S. Then we have

$$r_S(\emptyset) = 0,$$

$$r_S(A_i) = r_S(B_i) = (n+i)(t-1)+1, \text{ for } 1 \le i \le n, \text{ and}$$

$$r(M_S) = 2n(t-1)+2.$$

Our goal in this section is to prove the following theorem.

Theorem 5.4.3. *The (n,t)-semi-magic square matroid M_S and the lattice path matroid L on the common ground set $E = [3nt - 2n + 5]$, as defined above, are equal.*

We will prove this result by two different methods. For the first proof, notice that r_L agrees with r_S on E, A_i, and B_i with $1 \le i \le n$, so the result follows if we show that $\mathcal{Z}(L) = \mathcal{Z}(M_S)$. We will identify the cyclic flats of L from the proper nontrivial connected flats and fundamental flats (which are defined below).

A matroid M is *connected* if, for every pair of distinct elements of $E(M)$, there is a circuit containing both [14, Proposition 4.1.4]. A flat F of a matroid M is *connected* if the restriction $M|F$ is connected. A flat F is *trivial* if F is independent; otherwise F is *nontrivial* [5]. Notice that the only non-cyclic connected flats are the one-point trivial flats. On the other hand, any disconnected cyclic flat is a direct sum of at least two nontrivial connected flats. In the next lemma, we give a sufficient condition so that the proper nontrivial connected flats agree with the proper nontrivial cyclic flats.

Lemma 5.4.4. *Let M be a matroid. If $r(F_1) + r(F_2) \ge r(M)$ for all proper nontrivial connected flats F_1 and F_2 with $F_1 \cap F_2 = \emptyset$, then the proper nontrivial cyclic flats are precisely the proper nontrivial connected flats.*

Proof. Notice that proper nontrivial connected flats are all cyclic. We need to show that the proper nontrivial cyclic flats are all connected. Suppose Z is a proper nontrivial disconnected cyclic flat. Then $M|Z = M|F_1 \oplus M|F_2 \oplus M|Z - (F_1 \cup F_2)$, where F_1 and F_2 are connected flats. So we have $r(Z) \ge r(F_1) + r(F_2) \ge r(M)$, which contradicts the properness of Z. □

We now define the fundamental flats of a matroid [5, Definition 5.1]. We use them as a tool to identify the proper nontrivial cyclic flats of lattice path matroids.

Definition 5.4.5. Let F be a connected flat of a connected matroid M for which $|F| > 1$ and $r(F) < r(M)$. We say that F is a *fundamental flat* of M if for some spanning circuit C of M the intersection $F \cap C$ is a basis of F.

The next two lemmas [5, Lemma 4.3, Theorems 5.3 and 5.7] allow us to identify all proper nontrivial connected flats of a given lattice path matroid from its fundamental flats. A lattice path P has a NE corner at h if step h of P is North and step $h+1$ is East. An EN corner at k is defined similarly [5].

Lemma 5.4.6. *Let M be the connected lattice path matroid $M[P, Q]$ with $r(M) = r$ and $n(M) = n$. Let the EN corners of Q be at $i_1, i_2, \ldots, i_h$, with $i_1 < i_2 < \cdots < i_h$, and the NE corners of P be at $j_1, j_2, \ldots, j_k$, with $j_1 < j_2 < \cdots < j_k$. The fundamental flats of M are $[i_1] \subset [i_2] \subset \cdots \subset [i_h]$ and $[j_k + 1, n + r] \subset \cdots \subset [j_2 + 1, n + r] \subset [j_1 + 1, n + r]$. The rank (resp. nullity) of $[i_s]$ is the number of North (resp. East) steps among the first i_s steps of Q. The rank (resp. nullity) of $[j_t + 1, n + r]$ is the number of North (resp. East) steps among the last $n + r - j_t$ steps of P.*

Lemma 5.4.7. *Let M be the connected lattice path matroid $M[P, Q]$ and, for positive integers h and k, let $F_1 \subset F_2 \subset \cdots \subset F_h$ and $G_1 \subset G_2 \subset \cdots \subset G_k$ be the chains of fundamental flats of M. The proper nontrivial connected flats of M are*
(i) $F_1, F_2, \ldots, F_h, G_1, G_2, \ldots, G_k$, and
(ii) the intersections $F_i \cap G_j$ for which the inequality $n(M) < n(F_i) + n(G_j)$ holds.
A nontrivial connected flat of the form $F_i \cap G_j$ has rank $r(F_i) + r(G_j) - r(M)$.

We now give the first proof of Theorem 5.4.3.

Proof of Theorem 5.4.3. Notice that $L = M[P, Q]$ is connected because P and Q intersect only at $(0, 0)$ and $(n(L), r(L))$ [5, Theorem 3.5]. Observe that the number of EN corners in Q is $n+1$, as is the number of NE corners of P, namely

$$\{n(t-1) + it + 2 : 0 \leq i \leq n\}.$$

Recall that $A_i = [n(t-1) + it + 2]$ and $B_i = [n(t-1) + (n-i)t + 3, 3nt - 2n + 5]$ for

$0 \le i \le n$. Thus, by Lemma 5.4.6, the fundamental flats of L are

$$A_0 \subset \cdots \subset A_n \quad \text{and} \quad B_0 \subset \cdots \subset B_n.$$

We show that these are precisely the proper nontrivial cyclic flats of L. Using Lemma 5.4.6, we have $r_L(A_i) = r_L(B_i) = (n+i)(t-1)+1$, $n_L(A_i) = i+1$, and $n_L(B_i) = i+2$. It follows that

$$\begin{aligned} n_L(A_i) + n_L(B_j) &\le n_L(A_n) + n_L(B_n) \\ &= 2n+3 \\ &\le n(L). \end{aligned}$$

By Lemma 5.4.7, the fundamental flats of L are precisely the proper nontrivial connected flats of L. Also we have

$$\begin{aligned} r_L(A_i) + r_L(B_j) &\ge r_L(A_0) + r_L(B_0) \\ &= 2n(t-1)+2 \\ &= r(L). \end{aligned}$$

By Lemma 5.4.4, the proper nontrivial connected flats of L are precisely the proper nontrivial cyclic flats of L. Hence $\mathcal{Z}(L) = \mathcal{Z}(M_S)$. Since r_L agrees with r_S for all cyclic flats, we have $L = M_S$ by Theorem 3.1.1. □

The alternative method to prove $L = M_S$ is to show that both can be written as the intersection of the same pair of nested matroids. Before proceeding to the proof, we briefly discuss nested matroids and intersections of matroids.

Suppose that two matroids M_1 and M_2 have the same ground set. Consider the collection of sets that are independent in both matroids, i.e., $\mathcal{I}(M_1) \cap \mathcal{I}(M_2)$. If this collection satisfies (I1)–(I3) of the axiom scheme (Definition 2.1.1) and thus defines a matroid, then the resulting matroid is called the *intersection of* M_1 *and* M_2 and is denoted by $M_1 \cap M_2$. Thus, we have $\mathcal{I}(M_1 \cap M_2) = \mathcal{I}(M_1) \cap \mathcal{I}(M_2)$. In general, $\mathcal{I}(M_1) \cap \mathcal{I}(M_2)$ may not satisfy

(I3) so the intersection $M_1 \cap M_2$ may not exist.

A *nested matroid* is a matroid that can be obtained from the empty matroid by iterating the operations of adding isthmuses and taking free extensions [3]. Nested matroids can be characterized in terms of both lattice paths and the lattice of cyclic flats. A nested matroid is a lattice path matroid of the form $M[E^n N^r, Q]$ or $M[P, N^r E^n]$ [5, Definition 4.1, Theorem 5.6]. Also, a matroid M is nested if and only if $\mathcal{Z}(M)$ is a chain [3, Lemma 2.2]. The following corollary identifies the proper nontrivial cyclic flats of nested matroids in terms of lattice paths. It can be viewed as a mild modification of [5, Lemma 4.3].

Corollary 5.4.8. *Let M be the nested matroid $M[E^n N^r, Q]$ with no loops and no isthmuses. Let the EN corners of Q be at $i_1, i_2 \ldots, i_h$ with $i_1 < i_2 < \cdots < i_h$. The proper nontrivial cyclic flats of M are the initial segments $[i_1] \subset [i_2] \subset \cdots \subset [i_h]$.*

Proof. The proper nontrivial connected flats of M are $[i_1] \subset [i_2] \subset \cdots \subset [i_h]$ by Lemmas 5.4.6 and 5.4.7. Then by Lemma 5.4.4, they are precisely the proper nontrivial cyclic flats. □

We now give the second proof of Theorem 5.4.3.

Proof of Theorem 5.4.3. We will show that both M_S and L are intersections of two nested matroids, i.e., $M_S = M_A \cap M_B$ and $L = L_A \cap L_B$ where M_A, M_B, L_A, and L_B are all nested. Then, we will show that corresponding nested matroids are the same, i.e., $M_A = L_A$ and $M_B = L_B$.

First, we construct M_A and M_B on $E = [3nt - 2n + 5]$ by the lattice of cyclic flats. Recall that $\mathcal{Z}(M_S) = \{\emptyset : A_0, \ldots, A_n : B_0, \ldots, B_n : E\}$. Let $\mathcal{Z}(M_A)$ and $\mathcal{Z}(M_B)$ be the two saturated chains of $\mathcal{Z}(M_S)$, i.e.,

$$\mathcal{Z}(M_A) = \{\emptyset, A_0, \ldots, A_n, E\} \quad \text{and} \quad \mathcal{Z}(M_B) = \{\emptyset, B_0, \ldots, B_n, E\}.$$

Let r_A and r_B be the rank functions of M_A and M_B, respectively, such that $r_A(Z) = r_S(Z)$ for $Z \in \mathcal{Z}(M_A)$ and $r_B(Z) = r_S(Z)$ for $Z \in \mathcal{Z}(M_B)$. Clearly, this construction defines nested matroids M_A and M_B [3, Lemma 2.2].

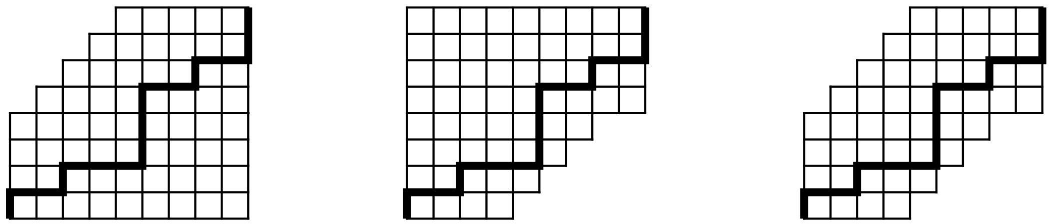

Figure 5.5: The intersection of two lattice path matroids

We now show that $M_S = M_A \cap M_B$. Since $\mathcal{Z}(M_S) = \mathcal{Z}(M_A) \cup \mathcal{Z}(M_B)$, we have

$$\begin{aligned} r_S(X) &= \min_{Z \in \mathcal{Z}(M_S)} \{r_S(Z) + |X - Z|\} \\ &= \min\{\min_{Z \in \mathcal{Z}(M_A)} \{r_A(Z) + |X - Z|\}, \min_{Z \in \mathcal{Z}(M_B)} \{r_B(Z) + |X - Z|\}\} \\ &= \min\{r_A(X), r_B(X)\}. \end{aligned}$$

If $X \in \mathcal{I}(M_S)$, then the inequality $r_S(X) \leq r_A(X) \leq |X|$ implies $r_A(X) = |X|$, hence $X \in \mathcal{I}(M_A)$; similarly, we have $X \in \mathcal{I}(M_B)$. Conversely, if $X \in \mathcal{I}(M_A) \cap \mathcal{I}(M_B)$, then $r_S(X) = \min\{r_A(X), r_B(X)\} = |X|$, hence $X \in \mathcal{I}(M_S)$. Thus, $\mathcal{I}(M_S) = \mathcal{I}(M_A) \cap \mathcal{I}(M_B)$.

Next, we construct two nested matroids L_A and L_B as lattice path matroids using the boundary paths P and Q of $L = M[P, Q]$. Let

$$\begin{aligned} L_A &= M[E^{nt+3} N^{2n(t-1)+2}, Q] \quad \text{and} \\ L_B &= M[P, N^{2n(t-1)+2} E^{nt+3}]. \end{aligned}$$

In order to show that $L = L_A \cap L_B$, it suffices to verify that $\mathcal{B}(L) = \mathcal{B}(L_A) \cap \mathcal{B}(L_B)$. Notice that $r(L) = r(L_A) = r(L_B) = 2n(t-1) + 2$ and $n(L) = n(L_A) = n(L_B) = nt + 3$. Also observe that

$$R[P, Q] = R[E^{nt+3} N^{2n(t-1)+2}, Q] \cap R[P, N^{2n(t-1)+2} E^{nt+3}],$$

i.e., a lattice path is in the region corresponding to L if and only if it is in the regions corresponding to both L_A and L_B (Figure 5.5). We have $\mathcal{B}(L) = \mathcal{B}(L_A) \cap \mathcal{B}(L_B)$ by Theorem 5.4.2.

By Corollary 5.4.8, we know that the set of proper nontrivial cyclic flats of L_A and L_B are $A_0 \subset \cdots \subset A_n$ and $B_0 \subset \cdots \subset B_n$, respectively. Thus, $\mathcal{Z}(L_A) = \mathcal{Z}(M_A)$ and $\mathcal{Z}(L_B) = \mathcal{Z}(M_B)$. Furthermore, if r_{L_A} and r_{L_B} are the rank functions of L_A and L_B, respectively, then we have $r_{L_A}(Z) = r_L(Z)$ and $r_{L_B}(Z) = r_L(Z)$ for all their respective cyclic flats Z. Thus, we have $L_A = M_A$ and $L_B = M_B$. Hence we conclude $L = M_S$. $\square$

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