# Derivation and Investigation of Affine Invariants for Image Processing Based on Wigner Distributions

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A Dissertation submitted to

The Faculty of
The School of Engineering and Applied Science
of The George Washington University
in partial satisfaction of the requirements
for the degree of Doctor of Science

January 31, 2013

Dissertation directed by

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The School of Engineering and Applied Science of The George Washington University certifies that Albert Hoang has passed the Final Examination for the degree of Doctor of Science as of February 23, 2012. This is the final and approved form of the dissertation.

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## Acknowledgments

A lot of thanks and appreciation is to Prof. Berkovich for his many suggestions, comments, observations, and his support, encouragement, and patience. Also much thanks to the Children ‘s Research Institute of Children’s National Medical Center, Washington, DC for the support for many years.

# Abstract

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It is often of interest in computer vision to extract features from images that capture certain image information and are unchanged when the images are seen at different viewpoints. These features can be used in many computer vision applications such as object recognition and image retrieval. Affine Moment Invariants (AMI) are a specific type of features, and are defined as functions of the moments of images in the literature. AMI capture the shape information of objects in images and are unchanged when the images are seen at different affine transformations.

In this dissertation a method is proposed to extract certain features of images that not only capture the shape but also texture information of objects in images and are unchanged under affine transformations. These features are called the Affine Wigner Moment Invariants (AWMI), defined as functions of the moments of the Wigner distributions of images. And they could be considered as a generalization of AMI from capturing only the shape into capturing both the shape and texture information.

Specific contributions of this dissertation consist of three parts. The first part is a method generalizing the method of deriving AMI to deriving AWMI from the so-called algebraic invariants. The second part is a method to find the appropriate algebraic invariants for determining AWMI. The third part shows how to calculate AWMI from real world images, and reports two experiments. These experiments show that the AWMI are indeed affine invariant and better extract texture than AMI, and also show how AWMI can be used in image retrieval.

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# Chapter 1 - Introduction

## 1.1 Dissertation Overview

Object recognition and image retrieval are important applications in the field of computer vision. Many methods in these applications depend on the task of extracting appropriate features such that these features capture the shapes of objects in the images and are invariant when the image are seen at different viewpoints ([1][2][3]).

Affine moment invariants (AMI) are such features calculated from the moments of the images. They capture the information about the shapes of objects in the images and are unchanged when the underlying images are seen at different affine transformations. Since the moments of images are calculated from the spatial coordinates and intensities of the images, so are the affine moment invariants ([1][4][5]).

There has been a great interest in the literature to derive features from images such that the features capture not only the shapes but also the texture of objects in images and are also invariant when the images are seen at different viewpoints ([6][7]).

It has been suggested in the literature that a different representation of images other than the intensities could capture better images' texture information. Such a representation is called a spatial-frequency representation, and Wigner distribution is the most typical and most used spatial-frequency representation. In fact, Wigner distributions have been used in applications to segment and discriminate objects in images based on texture ([8][9][10][11][12]).

While image's intensities give a value for every spatial coordinate on the image, a spatial-frequency representation gives a value for every pair of spatial coordinate and spatial-frequency coordinate. Thus a spatial-frequency representation is expected to capture explicitly more information about images than intensities, including texture information. It is because it describes an image based on both the spatial and spatial-frequency coordinates, while the intensities describe based on spatial coordinates only.

In this dissertation, a method is proposed to derive features based on the Wigner distributions of images such that these features capture the shapes and texture of objects in images and are unchanged or invariant when the images are seen at different affine transformations. Such features are called affine Wigner moment invariants (AWMI).

AWMI could be used in object recognition and image retrieval applications when there is a need to extract features that can capture texture information about objects and are invariant under affine transformations.

The content of this dissertation consists of some overview and introduction materials and then a presentation of the proposed method, and is organized into 6 chapters.

This chapter 1 gives an overview about the dissertation topic, overviews about some background topics, and then an overview about the contributions of this dissertation.

Chapter 2 gives some mathematical reviews of a number of topics that serve as a foundation for the dissertation. The topics reviewed include spatial-frequency representations, theory of algebraic forms, and affine invariants in pattern recognition.

Chapter 3, 4, and 5 make up the contributions of this dissertation.

Chapter 3 basically generalizes the approach of deriving AMI. In ([5][4]) a relationship between AMI and algebraic invariants was derived for intensities. In this chapter a relationship between AWMI and algebraic invariants was derived for Winger distributions.

In chapter 4, a method to derive some algebraic invariants that are needed to determine AWMI is proposed. A total of 13 AWMI have been derived using this method. Nine of these 13 AWMI involve frequency variables, and can be computed and compared only when the knowledge of affine transforms between images are known. Four other AWMI involve only spatial variables and thus can be computed and compared for any images.

Chapter 5 deals with the issues of calculating the AWMI, and reports two experiments. One experiment is to test if the AWMI derived in chapter 4 are indeed affine invariant, and if these AWMI indeed capture texture information better than AMI. Another experiment is to show how AWMI could be used in image retrieval tasks, and compare their performance with the performance of the AMI and the SIFT method.

Chapter 6 provides some conclusions about the dissertation.

## 1.2 Background

### 1.2.1 Affine Invariants

In object recognition or image retrieval in computer vision, a task of central importance is to determine if an image is part of another image even when the two images are obtained under different viewpoints. Here two images of the same object but obtained from two different viewpoints can be mathematically interpreted as one image is an affine transformed image of the other ([1],[2], [13]).

A typical approach to solve this task of determining if an image is part of another image consists of three steps. In the first step, determine the points or regions of interest in the two images such that these points or regions can also be re-generated if the underlying images are affine transformed. Note that a region of interest in an image could be any region determined from the underlying algorithm, or just the contour or boundary of an object in the image determined from some segmentation algorithm.

In the second step, extract certain features of the neighborhoods around the points of interest or features of the regions of interest determined from the first step. These features should be unchanged, or invariant, when the underlying images are affine transformed.

In the third step, match the features of the neighborhoods of points of interest or regions of interest of one image to those of the other image. If a match is found, this means one image is found to be part of the other object ([14], [15], [13]).

In each of the above three steps described above, there are many possible solutions. For step 1, there are algorithms that determine the points and regions of interest that can be re-generated when the underlying images are under certain transforms such as scaling, rotations, or more generally affine transforms ([16]). Examples of implementations for step 3 can be found in references ([14] [15]).

For step 2 of extracting certain features in the neighborhoods of certain interest points or regions in an image, there are three possible approaches that can be found in the literature.

The first approach is based on extracting features in the neighborhoods of interest points, and can be called the neighborhood-based approach ([17],[13]). The second approach is to extract features from the contour or boundary of the regions of interest, and can be called the contour-based approach ([18], [19]). The third approach is to extract features from the regions of pixels making up the regions of interest, and can be called the region-based approach ([20],[21]). In this dissertation, a method to extract features of regions invariant to affine transformations is proposed and belongs to the third approach.

In the neighborhood-based approach, an image or a region in the image is represented by a collection of points of interests, where these points are assumed to be found from certain algorithms. In a neighborhood of each point, a number of statistics such as orientations of gradients can be determined and invariant to certain transformations. The collection of statistics for each interest point can be considered as a feature vector representing the neighborhood around the point.

In the contour-based approach, the boundary of a region or an object in an image is assumed to have been calculated and represented as a curve. The Fourier coefficients, called Fourier descriptors, of the Fourier transform of the curve representing the boundary can be determined. These Fourier descriptors are then normalized in certain ways to obtain a feature vector invariant to the affine transformations of the region ([24][25][26][27]).

In the region-based approach, an object is represented by the *moments* or *integral transforms* calculated from the region making up the object. A function defined on these moments or integral transform is then determined such that the function is unchanged when the image is transformed according some transformation. Such a function is also called an *invariant* in the literature. Thus the word *invariant* is used both in *invariant feature vector* and as a function whose value is unchanged when the image is transformed in certain ways. For an object, many invariants, considered as different functions, can be grouped together to make up an invariant feature vector representing the object ([1][22][23][24]).

Using the concepts in the theory of algebraic forms, invariants defined on the moments of objects can be found with respect to the translation, scaling, and rotation transformations ([1][5][4]). There are many other different types of moments that can be defined on objects, and some offer certain computational advantages such as orthogonal moments, which include Zernike moments. Invariants defined on Zernike moments could be found for translations, scaling, and rotations. For an example of integral invariants, Fourier-Mellin transform of an object could be considered as an integral transform. And invariants defined on this transform are called integral invariants, and could also be determined for translations, scaling, and rotations ([25][22]).

Invariants defined on the central moments could also be found for affine transformations that include translations, scaling, and rotations as special cases. These invariants are called *affine moment invariants* in the literature. Various methods have been used to determine these invariants, such as methods from the theory of algebraic forms, or the graph method. Affine moment invariants have been used in many object recognition and image retrieval applications ([26][27][1]).

### 1.2.2 Derivation of Affine Moment Invariants from Algebraic Invariants

Derivation of invariants defined on images when the images are under some transformations is an important concern in pattern recognition. Hu was the first to derive invariants of images from algebraic invariants defined and derived within the theory of algebraic forms ([5]). Hu's paper, unfortunately, contained some derivation errors, but the general approach was still valid and the errors were later corrected by Reiss and Mamistvalov ([4][28]).

The theory of algebraic forms was very active in the late half of 19$^{th}$ and first half of 20$^{th}$ century. It mainly concerned about finding functions defined on the coefficients of homogeneous polynomials such that these functions are unchanged when the variables of the polynomials are changed under linear transformations. These functions are called *algebraic invariants* or just *invariants* ([29][30][31][32][33]). The theory of algebraic forms has given rises to many modern branches of mathematics, and nowadays algebraic invariants could be reformulated within the context modern algebra ([34][35][36][37][38]).

Hu's approach was to express the moment generating function of a two dimensional image as an infinite sum of homogenous polynomials whose variables are contragredient to the image's two spatial coordinates and whose coefficients are the image's moments. Thus when an image is under an affine transformation, then an algebraic invariant for a polynomial becomes basically an image invariant defined on the moments. That is in a nutshell how an affine moment invariant is derived from an algebraic invariant ([5][4][28]).

### 1.2.3 Spatial - Frequency Representation

In this dissertation, for simplification of explanations, the terms *spatial coordinates* as related to image analysis and *time coordinates* as related to signal processing are used interchangeably. There is no loss of generality when either of the terms is used since the underlying mathematics is the same.

In signal processing, or similarly in image analysis, the typical representation of a signal is the intensity function expressed as a function of the spatial coordinates. To assess the effect of a frequency on the signal, the Fourier transform of the signal, which is considered as a function of the spatial frequency coordinates, is often examined.

In this intensity representation there is no way to assess the signal at any combination of spatial and spatial frequency coordinates simultaneously. In real life, however, many signals have spatial -dependent frequency components that make the intensity representation inadequate for analysis ([8][9][39][40]). Extensive research has been carried out to devise consistent ways to represent a signal or an image as a function of both spatial and spatial frequency coordinates. Such a function of both spatial and spatial frequency coordinates is called a *spatial-frequency representation* ([41][42])

The main approach in spatial-frequency representation is to define a function in both the spatial and frequency coordinates that acts similarly like a density function in the field of Statistics. Such a function is called a *distribution*, and this approach can be called the *distribution approach* ([9][41]). Another approach in spatial-frequency representation is *linear decomposition* that decomposes any intensity function into a sum like a Fourier series and considers the coefficients in the sum as a spatial – frequency representation ([41][43]).

In the distribution approach, two classes of distributions have been proposed in the literature. One class of distributions is called the *Cohen's class* and the other less frequently mentioned is the class of the so-called *affine distributions* ([41][42]). The two distributions proposed earliest and used in signal processing are the so-called s*pectrograms* and the so-called *Wigner distributions*, with the Wigner having a lot more desirable properties ([9][41][44]). Despite those properties the Wigner distributions still have some weaknesses that have prompted extensive research into fixing these weaknesses but still maintaining other properties. The Cohen's class is a class of distributions of which the spectrograms and the Wigner distributions are members. And it contains many other distributions that are modifications of the Wigner distributions ([9][43]).

### 1.2.4 Analysis of Texture of Images

The *texture* of an object in an image is usually understood as repetitive patterns on the object's surface ([45][46][2]). The texture of an object can be grouped into homogeneous regions whereby the repetitive patterns within each region are similar but across different regions are different. There are three main problems texture analysis of images often has to deal with. The first problem is called *texture classification* that means classifying homogeneous regions on object's surfaces into known classes. The second problem is *texture segmentation* or *texture clustering* that means grouping pixels into homogeneous groups such that patterns in each group are the same and across groups are different ([45][46][47][48]). The third problem is *image retrieval based on texture* ([49][50]).

There are generally four different groups of methods of texture analysis, which are statistical methods, geometrical or structural methods, model-based methods, and signal processing or spectral methods ([45][6]).The spectral method group includes any approach that uses frequency information in the analysis of texture. In image analysis spatial-frequency representation is often used for texture analysis and thus could be grouped into the group of spectral methods. A spatial - frequency representation frequently used texture analysis tasks is the Wigner distribution ([12][47][51]).

Recall that a typical approach in object recognition and image retrieval is first to extract relevant features and then apply some appropriate methods to work on those features. The same approach is often used in texture classification and segmentation, whereby relevant texture features are first needed to be extracted then some appropriate methods are used on these texture features.

Texture analysis problems in some applications may require the results of the analysis unchanged when objects in images under consideration are viewed or transformed in certain ways. Similar to the object recognition and image retrieval tasks mentioned earlier, in texture analysis great deal of efforts have been made in finding those functions defined on the texture feature vector such that the function is unchanged when the underlying object is changed. Such a function is called a *texture invariant,* a term that stresses that fact that the function is defined on the texture feature vectors, not the general feature vector ([6][52][53][7]).

As mentioned earlier, affine moment invariants are functions defined on the moments of objects that are unchanged when the objects are transformed by the affine transformation. Here the moments are calculated based on the intensity and spatial coordinates. And the affine moment invariants can be used for object recognition and image retrieval tasks based on objects' shapes.

A Wigner distribution, or generally any spatial - frequency representation, is a function defined on the spatial and spatial frequency coordinates, the moments of the distribution can be defined and calculated ([54][55]). To the knowledge of the author of this dissertation, there has not been any published research in exploring ways to derive invariants based the moments of Wigner distributions with respect to affine transformations.

As Wigner distributions can be considered as more general than the intensity functions, image analysis tasks carried out using affine moment invariants could also be carried out with invariants defined on Wigner moments. Moreover, since Wigner distributions are often used for texture analysis, it is expected that invariants defined on Wigner moments could also be used for texture analysis. This is a potential advantage of affine Wigner moment invariants over affine moment invariants, since affine moment invariants are mainly used for recognition tasks based on objects' shapes, not on objects' texture ([21][56][20]).

### 1.2.5 Digital Holography

Image processing as a field generally focuses on analyses of images. A real world scene typically can be captured on some medium through the technology of photography. The medium contains the so-called intensities representing the scene, and analyses are carried out on the intensities or some alternative representation such as spatial-frequency representation.

Besides photography, another optical technique called holography can be used to capture a real world scene and display a copy of the scene later. In holography, a scene by some optical devices can be captured and recorded on some medium such as some type of papers. The medium recording the scene is called a hologram, and can be processed by some other optical devices to construct a copy of the scene later. What makes holography a much better technique than photography is that a real world scene can be reconstructed in the three dimensional real world to produce a look-alike copy of the scene. In photography, the scene can only be seen on a two dimensional image. The reason why holography is better is that holograms not only record intensities like photography but also the phase information of the scene ([57][58]).

The technique of holography has continuously been improved in order to become more practical. One particular application is called holographic data storage that has a potential storage capability larger than the current storage technology ([59][60]). Other potential applications of holography are possible explanations of the working of human memory and the working of universe ([61]). There is scarce research of algorithms applied on holograms in the computer vision literature. The reason is holography is yet to mature to become a reliable technology so people can take pictures of real world scenes.

While computer vision tasks such as image retrieval and object recognitions are important in applications involving images, these tasks probably are still valid for applications involving holograms. Whether images or holograms, it is probably still necessary to devise algorithms for object recognition and image retrieval. For these tasks, deriving features that extract the information in images or holograms and are used for later analysis is an important approach. There is a difference in deriving features for images and holograms, however. An image is either represented as intensities or a spatial-frequency representation, and features are derived based on either of these representations. A hologram, however, has been shown to be a Fresnel transform or more generally a linear canonical transform of a real world scene( [58] [62][63]). In this situation, features if to be derived must be derived based on these transforms, not on intensities or spatial-frequency representation like in images.

## 1.3 Dissertation Contribution

In this dissertation, a method is proposed to derive features from images such that these features capture the shape and texture information from the images and are unchanged when the images are seen at different viewpoints. A viewpoint basically can be represented as an affine transformation. These features are called AWMI.

The method proposed in this dissertation extends the method proposed in the literature that derived the features called AMI. These AMI capture the shape information of objects in images and are invariant with respect to affine transformations.

Potential applications of AWMIs could extend the applications of AMI. AMI have been used in object recognition and image retrieval applications that use shape information of objects and need to be affine invariant. The AWMI derived in this dissertation could be used in those applications that use both shape and texture information of objects and need to be affine invariant.

The method proposed in this dissertation will be presented in chapter 3, 4, and 5.

Chapter 3 is a generalization of the approach deriving AMI as proposed in the literature ([5][4][28]). Essentially the approach in the literature was to derive a relationship between AMI and algebraic invariants. And from the known algebraic invariants already derived within the theory of algebraic forms, AMI can be determined from the derived relationship.

The approach of deriving a relationship between AMI and algebraic invariants will be extended in chapter 3 to derive a relationship between AWMI and certain types of algebraic invariants. The tasks of the extension is to generalize the concepts and results for AMI defined on moments of intensities to concepts and results for AWMI defined on moments of Wigner distributions.

In the relationship between AMI and algebraic invariants, a number of algebraic invariants are known in the theory of algebraic forms and thus a number of AMI can be determined from the relationship.

But in the relationship between AWMI and algebraic invariants as derived in chapter 3, the algebraic invariants required to calculate AWMI will be too complicated if determined from the theory of algebraic forms. These supposedly complicated algebraic invariants will give rise to complicated AWMI.

In chapter 4 of this dissertation, a method is proposed to determine the algebraic invariants required for the calculations of AWMI. These algebraic invariants are supposedly simpler than those calculated straight from the theory of algebraic forms. From the algebraic invariants derived in this method, a total of 13 AWMI have been derived.

The derivation of AWMI from chapter 3 and 4 is based on continuous Wigner distributions. For real world practice, the AWMI need to be calculated from real world images that would give rise to discrete Wigner distributions.

Moreover, the claims that the derived AWMI are affine invariant and capture texture information need to be verified with real world images.

Chapter 5 shows how discrete Wigner distributions can be calculated and therefore how AWMI can be calculated from real world images. A number of assumptions have been made in this chapter for the calculations possible.

Also in chapter 5 a simulated experiment has been carried out in Matlab to verify if AWMI are indeed invariant and to compare the performance of AWMI and four AMI as reported in the literature in extracting texture information.

The results of the experiment show that the 13 derived AWMI are indeed invariant with respect to affine transformations, and extract texture information from images better the four chosen AMI do.

# Chapter 2 - Literature Reviews

## 2.1 Introduction

The previous chapter provides a broad overview of this dissertation and related research background. This chapter provides a mathematical review of some specific topics from the literature that could serve as a foundation for the method proposed in this dissertation. Here, the proposed method is to derive some affine invariants from Wigner distributions.

Three topics reviewed in this chapter include spatial frequency representation, theory of algebraic forms, and affine invariants in pattern recognition.

In section 2.2, basically the mathematical definition of spatial frequency representation will be given, in which Wigner distribution is a special case.

In section 2.3, a part of theory of algebraic forms directly related to the method constructed in this dissertation will be reviewed.

In section 2.4, some definitions of invariants used in pattern recognition will be reviewed. Especially, the method of derivation of affine moment invariants from the literature will be described. This method will be generalized to the method proposed in this dissertation.

## 2.2 Spatial Frequency Representations

The materials reviewed in this section can be found in ([43] [40][41] [9] [8] [10][64]).

### 2.2.1 Power Spectrum Density

Time-frequency analysis arises whenever there is a need to assess the signal, or similarly an image, at different time and frequency. Consider a signal $s(t)$ and its Fourier transform $S(\omega)$ in signal processing, $s(t)$ is used to assess the signal at $t$, and $S(\omega)$ at $\omega$. There is no way to assess the signal at any spatial - frequency pair $(t, \omega)$ simultaneously.

A signal in general by the Fourier transform can be considered as a sum of many component signals. In real life many signals have component signals at different frequencies existing for only a short time. Thus the sum of these component signals make the underlying signal having different values at different time-and frequency pair $(t, \omega)$'s. Thus if there is a way to assess these signals at every $(t, \omega)$, it's much better than the capability of assessing the strength with only $s(t)$ and $S(\omega)$ at each separate $t$ or $\omega$.

A *spatial - frequency representation* of a signal $s(t)$ is a function $P(t, w)$ that satisfies certain consistency and other conditions so that $|P(t, \omega)|^2$ can be interpreted as the strength or energy of the signal at $(t, \omega)$. There are a number of methods to spatial-frequency representations, such as signal decomposition, energy distribution with Cohen's class and affine distributions.

Among all the conditions a time-frequency representation $P(t,\omega)$ should satisfy, the following are two key conditions:

$$\frac{1}{2\pi}\int P(t,\omega)d\omega = |s(t)|^2 \tag{2.1}$$

$$\int P(t,\omega)dt = |S(\omega)|^2 \tag{2.2}$$

Conditions (2.1), (2.2) are called the *spatial marginal condition* and *frequency marginal condition*. These properties are basically the same marginal properties required by density functions defined in Statistics.

The Cohen's class of distributions consisting of spatial-frequency representations satisfying the above conditions and other conditions can be constructed as follows:

Given a signal $s(t)$ and its Fourier transform $S(\omega)$. The autocorrelation giving the similarity between signal $s(t)$ and signal $s(t-\tau)$ is defined to be function $R(\tau)$:

$$R(\tau) = \int s(t)s^*(t-\tau)\,dt = \int s\left(t+\tfrac{\tau}{2}\right)s^*(t-\tfrac{\tau}{2})\,dt \tag{2.3}$$

Where $s^*(u)$ is the complex conjugate of $s(u)$ for any $u$. By the Wiener-Khinchin theorem, $|S(\omega)|^2$ is the Fourier transform of $R(\tau)$:

$$|S(\omega)|^2 = \int R(\tau)exp\{-j\omega\tau\}d\tau \tag{2.4}$$

If a function $R(t,\tau)$ can be defined so that:

$$\int R(t,\tau)dt = R(\tau) \tag{2.5}$$

Then the following function $P(t,\omega)$ defined as the Fourier transform of $R(t,\tau)$:

$$P(t,\omega) = \int R(t,\tau)\ exp\{-j\omega\tau\}d\tau \tag{2.6}$$

is called the *power spectrum density* and is considered as a spatial-frequency representation.

Thus $P(t,\omega)$ is the coordinate of $R(t,\tau)$ with respect to the basis functions $\{exp\{-j\omega\tau\}: \tau \text{ in } \boldsymbol{R}\}$. Different choices of $R(t,\tau)$ gives rises to different spatial-frequency representations $P(t,\omega)$'s, as long as $P(t,\omega)$ satisfies the required conditions.

### 2.2.2 Spectrograms

Often, to examine the strength of a signal $s(t)$ at some specific $(t,\omega)$, the Fourier transform of a small section of the signal is considered. Specifically, use a window function $h(\tau)$ and take the Fourier transform of the modified signal $s(t)h^*(t-\tau)$ to obtain a function called the short time Fourier transforms $S(t,\omega)$ as:

$$S(t,\omega) = \int s(t)h^*(t-\tau)exp\{-j\omega\tau\}d\tau \tag{2.7}$$

The function $P(t,\omega)$ defined as

$$P(t,\omega) = |S(t,\omega)|^2 \tag{2.8}$$

is called a the *spectrogram.* Within the context of the Cohen's class, it can be shown that $P(t,\omega)$ as defined in (2.8) is a spatial – frequency representation for $s(t)$.

### 2.2.3 Wigner –Ville and Wigner Distributions

Another way to obtain a spatial-frequency representation is define function $R(t,\tau)$ as the integrand of $R(t)$defined in (2.3):

$$R(t,\tau) = s\left(t+\frac{\tau}{2}\right)s^*(t-\frac{\tau}{2}) \tag{2.9}$$

Then the power spectrum density defined in (2.6) is called the *Wigner-Ville distribution* of signal $s(t)$, and denoted as $W(t,\omega)$:

$$W(t,\omega) = \int s\left(t+\frac{\tau}{2}\right)s^*\left(t-\frac{\tau}{2}\right)exp\{-j\omega\tau\}d\tau \tag{2.10}$$

The Wigner –Ville distribution was developed by Wigner in 1932 in the context of quantum mechanics. It was then brought into signal processing analysis by Ville 15 years later.

If the signal under consideration $s(t)$ is a real signal, then $s\left(t+\frac{\tau}{2}\right) = s^*\left(t-\frac{\tau}{2}\right)$ is a real number. In this case the Wigner-Ville distribution is called the *Wigner distribution*.

It can be shown $W(t,\omega)$ as defined in (2.10) is a real-valued function.

$W(t,\omega)$ is also unchanged under translation. This means if $s(t)$ and $s'(t^{'})$ are two functions and there is a constant $c$ such that $s(t) = s'(t^{'})$ for $t = t^{'} + c$. Then for any $\omega$:

$$W(t,\omega) = W'(t^{'},\omega) \tag{2.11}$$

Where $W(t,\omega)$, $W'(t^{'},\omega)$ are the Wigner-Ville distributions of $s(t)$ and $s'(t^{'})$ respectively. Another property of $W(t,\omega)$ is that the energy contained in $W(t,\omega)$ is equal to that in $s(t)$

$$\frac{1}{2\pi}\iint W(t,\omega)d\omega dt = \frac{1}{2\pi}\int |S(\omega)|^2 d\omega = \int |s(t)|^2 dt \tag{2.12}$$

The Wigner-Ville distribution is often a preferred representation, because it better characterizes the underlying signal than the spectrogram and many others. For examples, the so-called the instantaneous frequency and group delay can be calculated from the Wigner-Ville distribution, but can't be derived from spectrogram or the scalogram, which is the square of the Wavelet transform.

One weakness of the Wigner-Ville distribution often cited is the presence of a so-called *cross-term* or *interference term* in the calculation of the Wigner-Ville distribution of a sum of two signals. Consider signal $s(t)$ that is the sum of two signals $s_1(t)$ and $s_2(t)$, $s(t) = s_1(t) + s_2(t)$. Let $W(t,\omega)$, $W_1(t,\omega)$, and $W_2(t,\omega)$ be the Wigner-Ville distributions of $s(t)$, $s_1(t)$, and $s_2(t)$ respectively. Then it can be shown that:

$$W(t,\omega) = W_1(t,\omega) + W_1(t,\omega) + U(t,\omega) \tag{2.13}$$

Where $U(t,\omega)$ is a real-valued number called the *cross-term* or *interference term*. Also $W_1(t,\omega)$ and $W_1(t,\omega)$ are called *auto terms*. The cross term often is large and sometimes dominates the auto terms and hides the contributions of the auto terms to $W(t,\omega)$. Many representations have been proposed to eliminate or suppress the effect of the cross-term.

### 2.2.4 Cohen's Class of distributions

Since the appearance of the Wigner-Ville distribution, may other representations have been proposed in literature. Cohen has suggested a unified approach that encompasses many distributions to within one family called the *Cohen's class* ([8][9][43]). Spectrograms and Wigner-Ville distributions can be seen as member of the Cohen's class.

### 2.2.5 Two Dimensional Wigner-Ville Distributions

The Wigner-Ville distribution as defined in (2.10) and more generally the Cohen's class can be generalized to higher dimensions ([62][65]).

In the case of two dimensional signal $s(t_1,t_2)$, the Wigner-Ville distribution for $s(t_1,t_2)$ is defined to be :

$$W(t_1,t_2,\omega_1,\omega_2)=\iint s\left(t_1+\tfrac{s_1}{2},\ t_2+\tfrac{s_2}{2}\right)s^*\left(t_1-\tfrac{s_1}{2},\ t_2-\tfrac{s_2}{2}\right)\times exp\{-2j(s_1\omega_1+s_2\omega_2)\}\,ds_1ds_2 \tag{2.14}$$

Where $t_1, t_2$ are the spatial coordinates, and $\omega_1, \omega_2$ are the frequency coordinates.

For three dimensional or higher dimensional signal, the corresponding Wigner-Ville distribution can be defined similarly.

## 2.3 Theory of Algebraic Forms

The main references for the materials reviewed in this section include ([29][30]).

The theory of algebraic forms concerns with the following situation.

Consider a homogeneous multivariate polynomial $P(\boldsymbol{x}; \boldsymbol{a})$ with $\boldsymbol{x}$ being the vector of all variables and $\boldsymbol{a}$ the vector of all coefficients. Here homogeneous means the sum of all the exponents of all variables in a term is the same. Suppose $\boldsymbol{x}$ is a linear transform of vector $\boldsymbol{x'}$ through a matrix $\mathrm{M}$, with $\boldsymbol{x} = \mathrm{M}\boldsymbol{x'}$**.** Substitute $\boldsymbol{x} = \mathrm{M}\boldsymbol{x}'$ into $P(\boldsymbol{x}; \boldsymbol{a})$ to obtain another homogeneous polynomial $P(\boldsymbol{x}'; \boldsymbol{a}')$, expressed in vector of variables $\boldsymbol{x}'$ and vector $\boldsymbol{a}'$ of all coefficients.

The problem the theory of algebraic forms basically deals with is how to find functions $I(\boldsymbol{a})$ defined on the vector coefficients such that $I(\boldsymbol{a}) = (\det M)^{\omega} \boldsymbol{I}(\boldsymbol{a}^{'})$**.** Here, $\omega$ is an integer. Such a function $I(\boldsymbol{a})$ is called an *invariant*, or an *algebraic invariant* in order to distinguish it from many other types of invariants encountered in this dissertation.

The theory of algebraic forms was studied extensively in the late 19th and early 20th century. It has given rise to many branches of modern mathematics such as abstract algebra, differential geometry, manifolds. In the context of this dissertation, it helps to determine the moment invariants as described in the literature ([5][4][28]), and it helps to determine the Wigner-Ville moment invariants proposed in this dissertation.

### 2.3.1 Algebraic Forms

**Multilinear Forms**

Given $p$ sets of variables $\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{t}$ where each set has $n$ variables. Each set can be written as a vector such as $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^T$, …, $\boldsymbol{t} = (t_1, t_2, \ldots, t_n)^T$. The sets $\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{t}$ can be all different or some or all of them are identical.

A *multilinear form* $f(\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{t})$ in $p$ sets of variables $\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{t}$ is an expression of the following format:

$$f(\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{t}) = \sum_{i_1, i_2, \ldots, i_p \in \{1,2,\ldots,n\}} a_{i_1 i_2 \ldots i_p} x_{i_1} y_{i_2} \ldots t_{i_p} \tag{2.15}$$

Where $a_{i_1 i_2 \ldots i_p}$'s are called the *coefficients* and are scalar numbers.

A multilinear form $f(\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{t})$ in $p$ sets of variables can be called a *$p$-linear form*. In the case $p = 2, 3$, the associated multilinear forms are called *bilinear* and *trilinear forms* respectively.

In general if $j_1, j_2, \dots, j_p$ are a permutation of $i_1, i_2, \dots, i_p$, then $a_{i_1 i_2 \dots i_p}$ and $a_{j_1 j_2 \dots j_p}$ are different.

**Multivariate Forms**

In the case all the $p$ sets of variables are identical $\boldsymbol{x} = \boldsymbol{y} = \cdots = \boldsymbol{t} = (x_1, x_2, \dots, x_n)^T$ and if $j_1, j_2, \dots, j_p$ are a permutation of $i_1, i_2, \dots, i_p$, then $a_{i_1 i_2 \dots i_p} = a_{j_1 j_2 \dots j_p}$. Then the resulting multilinear form is called a *multivariate* form. Such a multivariate form is also called an $n$ *-ary order* $p$ *multivariate form* $f(\boldsymbol{x})$.

By re-arranging (2.15), an $n$ *-ary order* $p$ *multivariate form* $f(\boldsymbol{x})$ can be expressed as:

$$f(\boldsymbol{x}) = \sum_{\substack{i_1+i_2+\cdots+i_n=p \\ i_1,i_2,\dots,i_n \in \{0,2,\dots,n\}}} \binom{p}{i_1 i_2 \dots i_n} a_{i_1 i_2 \dots i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n} \tag{2.16}$$

where

$$\binom{p}{i_1 i_2 \dots i_p} = \frac{p!}{i_1!\, i_2! \dots i_p!} \tag{2.17}$$

is called the *binomial coefficient*.

In general, an $n$-ary form of order $p$ as expressed in (2.16) has $\binom{n+p-1}{p}$ terms.

When $n = 2, 3, 4$, the $n$-ary form is called *binary, ternary, and quaternary* form respectively. When an $n$-ary form has order $p = 1, 2, 3, 4$, it is said to have *linear, quadratic, cubic*, or *quartic* order respectively.

A multilinear or multivariate form will be referred to generically as an *algebraic form*.

### 2.3.2 Cogredience and Contragredience

Let $T$ be a linear transformation from $\boldsymbol{R}^n$ to $\boldsymbol{R}^n$, represented by an $n \times n$ non-singular matrix $M$ with respect to certain basis. For every $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \boldsymbol{R}^n$, let $\mathbf{x} = \mathrm{T}(\mathbf{x}')$**,** then $\mathbf{x} = \mathrm{M}\,\mathbf{x}'$**.** Loosely speaking, matrix $M$ can be said to be a linear transformation.

Let $\mathbf{x}, \mathbf{y} \in \boldsymbol{R}^n$, there are situations that $\mathbf{x}$ and $\mathbf{y}$ are under some linear transformations that are related. Let $\mathbf{x}$ be under a linear transformation represented by an $n \times n$ non-singular matrix $M$, i.e. $\mathbf{x} = \mathrm{M}\,\mathbf{x}'$.

If $\mathbf{y}$ is under the same linear transformation, i.e. $\mathbf{y} = \mathrm{M}\,\mathbf{y}'$, then $\mathbf{x}, \mathbf{y}$ are said to be *cogredient.* If $\mathbf{y}$ is under the linear transformation $\mathbf{y} = \mathrm{A}\,\mathbf{y}'$, where $\mathrm{A} = (\mathrm{M}^{\mathrm{T}})^{-1}$, then $\mathbf{x}$**,** $\mathbf{y}$ are said to be *contragredient.* Note that $\mathbf{y} = \mathrm{A}\,\mathbf{y}'$ is equivalent to $\mathbf{y}' = \mathrm{M}^{\mathrm{T}}\mathbf{y}$**.**

For any $\mathbf{x}, \mathbf{y} \in \boldsymbol{R}^n$, the *inner product* of $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ and $\mathbf{y} = (\mathrm{y}_1, \mathrm{y}_2, \dots, \mathrm{y}_\mathrm{n})^{\mathrm{T}}$ is defined to be: $\langle \mathbf{x}, \mathbf{y} \rangle = \sum \mathrm{x}_i y_i$. In matrix notations $\langle \mathbf{x}, \mathbf{y} \rangle = \boldsymbol{x}^T \boldsymbol{y} = \boldsymbol{y}^T \boldsymbol{x}$**.**

A necessary and sufficient condition for $\mathbf{x}, \mathbf{y}$ to be contragredient is $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}', \mathbf{y}' \rangle$.

### 2.3.3 Symbolic Notations

To facilitate the manipulations and derivations of results for multilinear and multivariate forms, a consistent way of using symbols to represent the forms was introduced.

Consider any $p$-linear form $f(\boldsymbol{x}, \boldsymbol{y}, \dots, \boldsymbol{t})$. Let $\boldsymbol{a} = (a_1, a_2, \dots, a_n)^T$, $\boldsymbol{b} = (b_1, b_2, \dots, b_n)^T$, …, $\boldsymbol{h} = (h_1, h_2, \dots, h_n)^T$ be $p$ vectors of symbols such that

$$f(\boldsymbol{x}, \boldsymbol{y}, \dots, \boldsymbol{t}) = \left(\sum_{i=1}^{n} a_i x_i\right)\left(\sum_{i=1}^{n} b_i y_i\right) \dots \left(\sum_{i=1}^{n} h_i t_i\right) \tag{2.18}$$

$$= \sum_{i,j,\dots,k=1}^{n} a_i\, b_j \dots h_k x_i y_j \dots t_k \tag{2.19}$$

Where vectors $\boldsymbol{a}$ is associated with $\boldsymbol{x}$, $\boldsymbol{b}$ is associated with $\boldsymbol{y}$, …, $\boldsymbol{h}$ is associated with $\boldsymbol{t}$.

By equating (2.19) with (2.15), the product of symbols such as $a_i b_j \dots h_k$ can be used to denote the actual coefficient $a_{ij\dots k}$. This means the product of symbols $a_i b_j \dots h_k$ represents the actual coefficient $a_{ij\dots k}$.

Thus in any expression involving symbols, the symbols can be manipulated as numbers. But as long as when the symbols are multiplied together, there must be exactly $p$ symbols multiplied together in order to represent a valid actual coefficient.

To distinguish between cogredient and contragredient sets of variables in a multilinear form, some new notations are now introduced.

Let $f(\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{u}, \boldsymbol{v})$ be a multilinear form in $p$ sets of variables $\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{u}, \boldsymbol{v}$.

Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^T$, … be the sets that are cogredient with $\boldsymbol{x}$.

Let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)^T$, $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)^T$,… be the sets that are contragredient with $\boldsymbol{x}$.

Let the vectors of symbols for the cogredient $\boldsymbol{x}, \boldsymbol{y},$ … be expressed in the regular letters $\boldsymbol{a}, \boldsymbol{b}$, ….

Let the vectors of symbols for the contragredient $\boldsymbol{u}, \boldsymbol{v},$ … be expressed in the Greek letters $\boldsymbol{\alpha}, \boldsymbol{\beta}$, ….

Denote by $a_x = \langle \boldsymbol{a}, \boldsymbol{x} \rangle$, $b_y = \langle \boldsymbol{b}, \boldsymbol{y} \rangle$, …, $u_\alpha = \langle \boldsymbol{\alpha}, \boldsymbol{u} \rangle$, $v_\beta = \langle \boldsymbol{\beta}, \boldsymbol{v} \rangle$. Using matrix notations, $a_x = \boldsymbol{a}^T \boldsymbol{x}$, $b_x = \boldsymbol{b}^T \boldsymbol{y}$, …, $u_\alpha = \boldsymbol{\alpha}^T \boldsymbol{u}$, $\boldsymbol{v}_\beta = \boldsymbol{\beta}^T \boldsymbol{v}$.

Then the multilinear form $f(\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{u}, \boldsymbol{v})$ in (2.18) can be expressed as

$$f(\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{u}, \boldsymbol{v}) = a_x b_y \ldots u_\alpha v_\beta \tag{2.20}$$

Note that when some of the sets of variables are identical, the right hand side of the above can be simplified. For example, if $\boldsymbol{x} = \boldsymbol{y}$, then $\boldsymbol{a} = \boldsymbol{b}$ and $a_x b_y$ in the right hand side of (2.20) becomes $a_x^2$.

Now, also a multivariate form $f(\boldsymbol{x})$ as a special case of (2.18) with all $p$ sets of variables being identical can be expressed as:

$$f(\boldsymbol{x}) = a_x^p \tag{2.21}$$

### 2.3.4 Symbolic Notations under Congredience and Contragredience

**Transformations of Multilinear Forms**

Consider the multilinear form $f(\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{u}, \boldsymbol{v})$ as in (2.20), where $\boldsymbol{x}, \boldsymbol{y}, \ldots$ are cogredient, and $\boldsymbol{u}, \boldsymbol{v}, \ldots$ are contragredient with $\boldsymbol{x}$**.** Here $\boldsymbol{x} = \mathrm{M}\boldsymbol{x}'$, and $\boldsymbol{u} = \mathrm{A}\,\boldsymbol{u}'$, where $\mathrm{A} = (\mathrm{M}^{\mathrm{T}})^{-1}$ and M is a non-singular squared matrix.

Note that $f(\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{u}, \boldsymbol{v})$ can be expressed as the matrix notations as:

$f(\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{u}, \boldsymbol{v}) = a^T \boldsymbol{x}\, b^T \boldsymbol{y} \ldots \alpha^T \boldsymbol{u}\, \beta^T \boldsymbol{v}$**.**

By substituting $\boldsymbol{x} = \mathrm{M}\boldsymbol{x}'$, $\boldsymbol{y} = \mathrm{M}\,\boldsymbol{y}'$**, …,** $\boldsymbol{u} = \mathrm{A}\boldsymbol{u}'$**,** $\boldsymbol{v} = \mathrm{A}\boldsymbol{v}'$ in $f(\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{u}, \boldsymbol{v})$, form$f(\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{u}, \boldsymbol{v})$ becomes $f(\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{u}, \boldsymbol{v}) = a'^T \boldsymbol{x}'\, b'^T \boldsymbol{y}' \ldots \alpha'^T \boldsymbol{u}'\, \beta'^T \boldsymbol{v}'$**.** Here $a' = \mathrm{M}^{\mathrm{T}} a$, $b' = \mathrm{M}^{\mathrm{T}} b$, …, $\alpha' = \mathrm{A}^{\mathrm{T}} \alpha$, $\beta' = \mathrm{A}^{\mathrm{T}} \beta$.

Thus $f(\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{u}, \boldsymbol{v})$ expressed using symbols $a, b, \ldots, \alpha, \beta$ transforms into a new form $f'(\boldsymbol{x}', \boldsymbol{y}', \ldots, \boldsymbol{u}', \boldsymbol{v}') = a'^T \boldsymbol{x}'\, b'^T \boldsymbol{y}' \ldots \alpha'^T \boldsymbol{u}'\, \beta'^T \boldsymbol{v}'$ using symbols $a', b', \ldots, \alpha', \beta'$.

Under $= \mathrm{M}\boldsymbol{x}'$, $a_x$ in $f(\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{u}, \boldsymbol{v})$ transforms into $a'_{x'}$, this implies $a_x = a'_{x'}$ or $\langle \boldsymbol{a}, \boldsymbol{x} \rangle = \langle \boldsymbol{a}', \boldsymbol{x}' \rangle$. Thus $\boldsymbol{a}$ is contragredient with $\boldsymbol{x}$**.**

Since $\boldsymbol{a}$ is contragredient with $\boldsymbol{x}$**,** but $\boldsymbol{x}$ is contragredient with $\boldsymbol{u}$, therefore $\boldsymbol{a}$ is cogredient with $\boldsymbol{u}$**.** Similarly $b_y = b'_{y'}$, so $\boldsymbol{b}$ is contragredient with $\boldsymbol{x}$**,** **y** and cogredient with $\boldsymbol{u}, \boldsymbol{v}$**.**

Under $\boldsymbol{u} = A\boldsymbol{u}'$, $u_\alpha$ in $f(\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{u}, \boldsymbol{v})$ transforms into $u'_{\alpha'}$, this implies $u_\alpha = u'_{\alpha'}$. Thus $\boldsymbol{\alpha}$ is contragredient with $\boldsymbol{u}$.

Since $\boldsymbol{\alpha}$ is contragredient with $\boldsymbol{u}$, but $\boldsymbol{u}$ is contragredient with $\boldsymbol{x}$, therefore $\boldsymbol{\alpha}$ is cogredient with $\boldsymbol{x}$. Similarly $v_\beta = v'_{\beta'}$, so $\boldsymbol{\beta}$ is contragredient with $\boldsymbol{u}, \boldsymbol{v}$ and cogredient with $\boldsymbol{x}, \boldsymbol{y}$.

**Equivalent Vectors of Symbols**

In many calculations related to forms, it is often necessary to work with the power of actual coefficients, such as $a^k_{i_1 i_2 \ldots i_p}$ where $a_{i_1 i_2 \cdots i_p}$ is an actual coefficient of a form. Since a vector of symbols $\boldsymbol{a}$ can only be used to represent a product of different $a_{i_1 i_2 \cdots i_p}$'s. There is a need to use symbols to work with $a^k_{i_1 i_2 \ldots i_p}$ unambiguously.

This need is solved by using $k$ vectors of symbols to represent the same form. In this case, those $k$ vectors of symbols are said to be *equivalent*.

**Transformations of Vectors of Symbols**

Now, let $\mathrm{M}$ be an $n \times n$ matrix $\mathrm{M} = \begin{bmatrix} \xi_1 \ \eta_1 \ \ldots \ \omega_1 \\ \xi_2 \ \eta_2 \ \ldots \ \omega_2 \\ \ldots \\ \xi_n \ \eta_n \ \ldots \ \omega_n \end{bmatrix}$ representing a linear transformation from $\boldsymbol{R}^n$ to $\boldsymbol{R}^n$. In the theory of algebraic forms, the determinant of $\mathrm{M}$ is denoted as $(\xi\eta \ldots \omega)$. Also $\det M$ will be used to denote the determinant of $\mathrm{M}$. Thus both notations $(\xi\eta \ldots \omega)$ and $\det M$ will be used to denote determinant of $\mathrm{M}$.

The following are some important results that will be used very often.

Now let $a_x^p$, $b_x^q$, …, $m_x^r$ be $n$-ary forms which have orders $p, q, \dots, r$. Then under $\boldsymbol{x} = M\,\boldsymbol{x}'$, $a_x^p$ transforms to $a'^p_{x'}$ and $b_x^q$ to $b'^q_{x'}$ , …, and $m_x^r$ to $m'^r_{x'}$ .

And the symbols $\boldsymbol{a}', \boldsymbol{b}', \dots, \boldsymbol{m}'$ are determined from $\boldsymbol{a}' = M^T\boldsymbol{a}$ **,** $\boldsymbol{b}' = M^T\boldsymbol{b}$ **, …,** $\boldsymbol{m}' = M^T\boldsymbol{m}$**.** This fact can be written in a matrix notation as:

$$[\boldsymbol{a}'\,\boldsymbol{b}'\,\dots\boldsymbol{m}'] = M^T[\boldsymbol{a}\;\;\boldsymbol{b}\;\;\dots\boldsymbol{m}] \tag{2.22}$$

Where the left hand side is the matrix formed by the column vectors $\boldsymbol{a}', \boldsymbol{b}', \dots \boldsymbol{m}'$**.** Similarly for the right hand side.

Then by taking determinants of the above, the following result is obtained:

$$(a'\,b'\,\dots m') = (\xi\;\eta\;\dots\;\omega)(a\;b\;\dots m) \tag{2.23}$$

Where $(a\;b\;\dots m)$ and $(a'\,b'\,\dots m')$, and $(\xi\;\eta\;\dots\;\omega)$ are the determinants of matrices $[\boldsymbol{a}\;\;\boldsymbol{b}\;\;\dots\boldsymbol{m}]$, $[\boldsymbol{a}'\,\boldsymbol{b}'\,\dots\boldsymbol{m}']$, and $M$ respectively.

An application of the above is when $a_x^2$, $b_x^2$ represent the same binary quadratic form with $\boldsymbol{a}$ and $\boldsymbol{b}$ being two equivalent vectors of symbols. Then $(a'\,b') = (\xi\;\eta)(a\;b)$**,** meaning the determinant of the matrix formed by $\boldsymbol{a}'$ and $\boldsymbol{b}'$ is a product of the determinant of matrix $M$ and the determinant of the matrix formed by $\boldsymbol{a}$ and $\boldsymbol{b}$**.**

Now, consider any multilinear form $f(\boldsymbol{x}, \boldsymbol{y}, \dots, \boldsymbol{u}, \boldsymbol{v}) = a_x b_y \dots u_\alpha v_\beta$, with each set of $\boldsymbol{x}, \boldsymbol{y}, \dots, \boldsymbol{u}, \boldsymbol{v}$ has only two variables**.** And consider the transformation $\boldsymbol{x} = M\boldsymbol{x}'$, where $M = \begin{bmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{bmatrix}$**,** and $\boldsymbol{y}$**,…** are cogredient with $\boldsymbol{x}$, and $\boldsymbol{u}, \boldsymbol{v}$**, …** are contragredient with $\boldsymbol{x}$. Then if $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)^T$**,** $\boldsymbol{\beta} = (\beta_1, \beta_2)^T$ are two vectors of symbols associated with two sets such as $\boldsymbol{u}, \boldsymbol{v}$, which are contragredient with $\boldsymbol{x}$**,** then

$$(\alpha'\beta') = \frac{1}{(\xi\eta)}(\alpha\beta) \tag{2.24}$$

Where $(\alpha\beta)$ is the determinant of $\begin{bmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{bmatrix}$, and $(\alpha'\beta')$ is the determinant of

$\begin{bmatrix}\alpha'_1 & \beta'_1\\ \alpha'_2 & \beta'_2\end{bmatrix}$, and where $u_\alpha$ transforms into $u'_{\alpha'}$ with $u_\alpha = u'_{\alpha'}$ and $v_\beta$ transforms into $v'_{\beta'}$ with $v_\beta = v'_{\beta'}$.

### 2.3.5 Algebraic Invariants

**Definition of Alegebraic Invariants**

Let $\Gamma = \{f_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \dots, \boldsymbol{u}, \boldsymbol{v}; \boldsymbol{a}_i) : i = 1,2, \dots, n\}$ be a set of forms where $f_i(\boldsymbol{x}, \boldsymbol{y}, \dots \boldsymbol{u}, \boldsymbol{v}; \boldsymbol{a}_i)$ is either multivariate or multilinear forms. Here $\boldsymbol{y}, \boldsymbol{z}, \dots$ are cogredient with $\boldsymbol{x}$, and $\boldsymbol{u}, \boldsymbol{v}$ are contragredient with $\boldsymbol{x}$, and $\boldsymbol{a}_i$ is the vector of coefficients.

Let $I(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)$ be a real-valued function defined on $\boldsymbol{a}_i$'s, the vectors of coefficients of $f_i(\boldsymbol{x}, , \boldsymbol{y}, \dots \boldsymbol{u}, \boldsymbol{v}; \boldsymbol{a}_i)$'s.

Let $M$ be a non-singular matrix representing a linear transformation $\boldsymbol{x} = M\boldsymbol{x}'$. Let $\boldsymbol{a}'_i$ be the vector of coefficients of the form obtained from $f_i(\boldsymbol{x}, , \boldsymbol{y}, \dots \boldsymbol{u}, \boldsymbol{v}; \boldsymbol{a}_i)$ under $\boldsymbol{x} = M\boldsymbol{x}', \dots, \boldsymbol{u} = A\boldsymbol{u}', \dots$ where $A = (M^T)^{-1}$

Function $I(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)$ is said to be *an invariant* if it satisfies the following condition:

$$I(\boldsymbol{a}'_1, \boldsymbol{a}'_2, \dots, \boldsymbol{a}'_n) = (\det M)^\omega I(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n) \tag{2.25}$$

Where $\omega$ is an integer and could be negative, zero, or positive, $det\ M$ denotes the determinant of $M$.

When $\omega$ is non-zero, $I(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)$ is called a *relative invariant* of weight $\omega$. When $\omega$ is zero, $I(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n)$ is called an *absolute invariant*.

The invariants as defined here sometimes are called *algebraic invariants* in this dissertation to distinguish it from many other types of invariants defined in the field of pattern recognition.

The following is a theorem that gives the general format showing how the invariants of multivariate forms look like.

**Fundamental Theorem for Invariants of Multivariate Forms**

Consider a set of multivariate forms whose symbols are $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \dots, \boldsymbol{m}$. Every invariant for the set can be expressed as a sum of terms, each term is a constant times a product of the type $(\boldsymbol{ab} \dots \boldsymbol{m})$, which is the determinant of the matrix created by vector of symbols $\boldsymbol{a}, \boldsymbol{b}, \dots, \boldsymbol{m}$.

**Example**: Consider a set consisting of just one binary quadratic form $f = a_{20}x_1^2 + 2a_{11}x_1x_2 + a_{02}x_2^2$.

Let $\boldsymbol{a}, \boldsymbol{b}$ be two vectors of equivalent symbols representing the form. Then it can be checked that function $I$ defined as $I = (\boldsymbol{ab})^2$, as specified by the above theorem, is an invariant of $f$. It can be shown that $I$ is of weight 2, and can be expressed in the actual coefficient as $I = 2(a_{20}a_{02} - a_{11}^2)$ and is of order 2.

The following is a theorem that gives the general format showing how the invariants of multilinear forms look like.

**Fundamental Theorem for Invariants of Multilinear Forms**

Consider a set of multilinear forms whose symbols are $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots, \boldsymbol{m}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots\ldots \boldsymbol{\mu}$ . Every invariant for the set is a polynomial, where each term is a constant times a product of the three types: $\boldsymbol{a}_{\boldsymbol{\alpha}}$, $(\boldsymbol{ab} \ldots \boldsymbol{m})$, $(\boldsymbol{\alpha\beta}\ \boldsymbol{\gamma} \ldots\ \boldsymbol{\mu})$.

Here $\boldsymbol{a}_{\boldsymbol{\alpha}} = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$, $(\boldsymbol{ab} \ldots \boldsymbol{m})$ is the determinant of a matrix formed by $\boldsymbol{a}, \boldsymbol{b} \ldots \boldsymbol{m}$, and $(\boldsymbol{\alpha\beta}\ \boldsymbol{\gamma} \ldots\ \boldsymbol{\mu})$ is the determinant of a matrix formed by $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots, \boldsymbol{\mu}$.

**Example**: Consider a set consisting of just one binary bilinear form $f = a_{11}x_1u_1 + a_{12}x_1u_2 + a_{21}x_2u_1 + a_{22}x_2u_2$, where $\boldsymbol{x} = (x_1, x_2)^T$ and $\boldsymbol{u} = (u_1, u_2)^T$ are contragredient.

Let $\boldsymbol{a}, \boldsymbol{\alpha}$ be the symbols representing the form. Then it can be checked that function $I$ defined as $I = \boldsymbol{a}_{\boldsymbol{\alpha}}$, as specified by the above theorem, is an invariant of $f$. It can be shown that $I$ can be expressed in terms of actual coefficients as $I = a_{11} + a_{22}$, and is an absolute invariant.

Classical reference to the theory of algebraic forms and invariants can be found in ([29][30][31][32]) and modern reference formalized in advanced abstract algebras can be found in ([36][35][34][37][38][66]).

## 2.4 Affine Invariants in Pattern Recognition

This section gives a brief review of invariants derived under different transformations, gives definitions of a number of moments, and sketch how affine moment invariants can be derived from algebraic invariants. Most of the materials reviewed can be found in ([3][1][5][4][17]).

### 2.4.1 Affine Transformations

An *affine transformation*, restricted to $\boldsymbol{R}^2$, is a function mapping point $\boldsymbol{x}' = (x'_1, x'_2)^{\mathrm{T}}$ to point $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}}$**,** according to

$$\boldsymbol{x} = \mathrm{A}\boldsymbol{x}' + \boldsymbol{b} \tag{2.26}$$

Where $\mathrm{A}$ is a $2 \times 2$ real-valued non-singular matrix, and $\boldsymbol{b} = (b_1, b_2)^{\mathrm{T}}$ is a 2-dim real-valued vector.

When $\mathrm{A} = \boldsymbol{0}$**,** and $\boldsymbol{b} = (b, b)^{\mathrm{T}}$**,** the affine transformation is called a *translation* that translates every point by a distance $b$. When $\mathrm{A} = \alpha\boldsymbol{I}$**,** where $\boldsymbol{I}$ is the $2 \times 2$ identity matrix and $\alpha > 0$, and $\boldsymbol{b} = \boldsymbol{0}$**,** the affine transformation is called a *scaling* that scales every point by a factor $\alpha$. When $\boldsymbol{b} = \boldsymbol{0}$**,** the resulting affine transformation is called a *linear* transformation. When $\boldsymbol{b} = \boldsymbol{0}$ and $\mathrm{A} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$, the affine transformation is called a *rotation* that rotates every point by an angle $\varphi$ counter clockwise.

A 2-dimensional image $f(\boldsymbol{x})$ is said to be an *affine transform* of another 2-dim image $f'(\boldsymbol{x}')$ if for any points $\boldsymbol{x}$ and $\boldsymbol{x}'$ related by $\boldsymbol{x} = \mathrm{A}\boldsymbol{x}' + \boldsymbol{b}$ then $f(\boldsymbol{x}) = f'(\boldsymbol{x}')$, for some real-valued non-singular matrix $\mathrm{A}$, and a vector $\boldsymbol{b}$.

### 2.4.2 Approximate Invariants

In object recognition and image retrieval applications, it is required to match an image to another image to see if the first image is part of the second image. As mentioned in chapter 1, a typical approach to this matching task consists of three steps. First, detect all the points or regions in each image such that these points and regions can be detected again under certain transformations. The points or regions are called points or regions of interest. Second, extract features for those detected points or regions such that these features are invariant with respect to certain transformations. Third, match those points or features in the two images together using the features determined from the second step. If a match is found then conclude the first image is part of the second image, otherwise conclude the first image is not part of the second.

There are a number of algorithms in the literature to determine either or all of the above three steps ([16], [67], [1]). In many of these algorithms, the features derived in the second step are not exactly invariant but rather approximately invariant with respect to certain transformations. Such an algorithm is the popular SIFT algorithm to solve both the first and second steps, and is reviewed here to serve as a background for this dissertation ([17],[13]).

The SIFT algorithm is an empirical procedure to determine features of any image such that these features are approximately invariant with respect to translation, scaling, rotation, and additionally invariant to certain degree of noise and illumination. The algorithm aims to detect a number of points of interest for any image, and then determines a vector of features from a neighborhood around each point of interest. The following is an outline of the algorithm.

Let $f(x, y)$ be an input image. Let $G(x, y, \sigma)$ be a Gaussian function $G(x, y, \sigma) = \frac{1}{2\pi\sigma^2} exp\{-(x^2 + y^2)/2\pi\sigma^2\}$. For each $k = 1, 2, \ldots, 5$, let $L_k(x, y)$ be a blurred image of $f(x, y)$ obtained from convolving $f(x, y)$ with $G(x, y, k\sigma)$:

$$L_k(x, y) = G(x, y, k\sigma) * f(x, y) \tag{2.27}$$

The difference of Gaussian image $D_k(x, y)$ is defined to be:

$$D_k(x, y) = L_{k+1}(x, y) - L_k(x, y) \tag{2.28}$$

In the computer vision literature, the difference of Gaussian $D_k(x, y)$ can be used to determine the edges and corners from the input image. In this algorithm, it is used to determine the corners of image $f(x, y)$ that are defined as the local extrema of $D_k(x, y)$'s. The corners are called keypoints in this algorithm. The main idea of the SIFT algorithm can be roughly sketched into four steps as follows:

- In the first step, calculate the difference of Gaussian images $D_k(x, y)$, and determine the local extrema of images $D_k(x, y)$, $k = 2, \ldots 4$. This step is carried in a loop.

  Loop 4 times

  - First, Calculate $L_k(x, y)$ for $k = 1, \ldots 5$, then calculate $D_k(x, y)$, $k = 1, \ldots 4$

- Second, for every pixel at location $(x_0, y_0)$ in $D_k(x, y)$ , $k = 2, \dots 4$, determine a neighborhood centered at location $(x_0, y_0)$ in each of $D_{k-1}(x, y)$, $D_k(x, y)$, $D_{k+1}(x, y)$. If the intensity of the pixel is the largest or smallest in all three neighborhoods then it is a local extremum.
- Third, let $f(x, y)$ be the image obtained from down sampling $L_5(x, y)$ by a factor of 2.
- Fourth, repeat the loop

- In the second step, eliminate those local extrema that have low contrast or are on edges. The remaining local extrema are considered as the keypoints.
- In the third step, determine a vector aggregating all gradients in a neighborhood of each keypoint that is on certain $D_k(x, y)$
  - Consider a neighborhood at every keypoint, determine the gradient and orientation for each point in the neighborhood.
  - Divide the interval (0,360) of degrees into 36 equal-length subintervals
  - For each point, locate the subinterval the orientation of the point's gradient falls into, calculate the sum of the magnitudes of all the gradients whose orientations fall into the subinterval.
  - Consider the subinterval whose associated total magnitude is largest. Consider the vector whose orientation is the middle of the subinterval and whose magnitude is the associated total magnitude. This vector located at the keypoint is an aggregate of all gradients of points in the neighborhood.

- In the fourth step, determine features from a neighborhood of each keypoint. This step basically repeats the third step to determine the features.
    - Consider a 16x16 neighborhood of pixels centered at any keypoint.
    - Divide the above 16x16 neighborhood into 16 4x4 squares.
    - Divide the interval (0,360) of degrees into 8 equal-length subintervals
    - For each point in each 4x4 square, find the subinterval the orientation of the gradient of the point falls into, sum the magnitude of the gradient to the total magnitude associated with the subinterval.
    - Collect 8 total magnitudes associated with 8 subintervals and consider them as 8 features determined from one 4x4 square.
    - Repeat the immediately above two steps for all 16 4x4 squares to collect altogether 16x8 = 128 features representing the 16x16 neighborhood.

It has been argued in that the keypoints determined from the above algorithm can be obtained again when the underlying image is under any scaling transformations. The translation and orientation don't affect the determination of the keypoints. For the 128 features determined for each keypoint, normalization of total magnitudes will make them scale-invariant. The features are orientation-invariant because the 8 subintervals mentioned in the fourth step don't change under orientation transformations. Translation don't affect the determination these 128 features. So, in general, the features determined from the SIFT algorithm are scaling, orientation, and translation invariant.

Thus the SIFT algorithm determines the keypoints and the features associated with them so that these features are approximately invariant to scaling, orientation, and translation. An algorithm called ASIFT has been proposed in the literature to generalize the SIFT algorithm to derive features that are approximately affine invariant ([13]).

### 2.4.3 Vector Spaces of Invariants

An *invariant* $I$ of an image $f(x_1, x_2)$ with respect to a transformation $T$ is defined to be a function $I(f)$, such that if $f(x_1, x_2)$ is transformed into $g(y_1, y_2)$ then $I(f) = I(g)$.

Note that the addition of two invariants is an invariant, and scalar multiplication of a real number and an invariant is also an invariant. Thus the space generated from a set of invariants becomes a vector space, and accepts a basis in the case of finite dimensions.

There is a large class of invariants that are defined in terms of the moments of an image or of some function derived from the image.

**Moments of Images**

Given an image $f(x_1, x_2)$. Let be $\{\psi_{pq}(x_1, x_2): p, q = 0,1,2, \dots\}$ be a set of polynomials defined on $\boldsymbol{R}^2$. A *general moment* of *order* $k = p + q$ is defined to be:

$$M_{pq} = \iint \psi_{pq}(x_1, x_2)\, f(x_1, x_2) dx_1 dx_2 \tag{2.29}$$

In Statistics, $f(x_1, x_2)$ under consideration is a density function, and the general moments provide information about $f(x_1, x_2)$. And sometimes only a few low order moments provide enough useful information. Using the similar interpretation, general moments provide information about image $f(x_1, x_2)$, and sometimes just a few low order moments are enough to represent the whole image $f(x_1, x_2)$.

The polynomials $\psi_{pq}(x_1, x_2)$ are called the *kernels* of the general moments.

If $\psi_{pq}(x_1, x_2) = x_1^p x_2^q$, then the general moment $M_{pq}$ is called the *geometric moments* of the image, and denoted as $m_{pq}$. The 0-order moment is thus denoted as $m_{00}$, and there are two first order moments and denoted as $m_{10}$ and $m_{01}$.

For any image, there is a point that is considered as the center of the image. The point $\boldsymbol{x}_c = (x_{1c}, x_{2c})^{\mathrm{T}}$**,** with $x_{1c} = \frac{m_{10}}{m_{00}}$ and $x_{2c} = \frac{m_{01}}{m_{00}}$, is considered as the image's center and called the *centroid* of the image.

In the definition of the general moments, if let $\psi_{pq}(x_1, x_2) = (x_1 - x_{1c})^p (x_2 - x_{2c})^q$, then the resulting general moment $M_{pq}$ is called the *central moment* and denoted as $\mu_{pq}$. Given an image, if the coordinate system is translated into the centroid of the image, then the coordinate of centroid with respect to the new coordinate system is **0**. And under the new coordinate system, the central moments and the geometric moments are identical.

**Complex Moments**

In the definition of general moments, if take $\psi_{pq}(x_1, x_2) = (x_1 + ix_2)^p (x_1 - ix_2)^q$, then the resulting moments are called the *complex moments* and denoted by $c_{pq}$. Note that $c_{pq}$ is a complex number.

The complex moment of all orders can be shown to be rotation invariant.

It has been shown that the space of all rotation invariants created from moments of any kind up to any order $k$ has a basis. This basis constitutes of some functions of the complex moments up to any order $k$. In other words, the set of all complex moments up to any order $k$ can be used to calculate a basis for the space of all invariants with respect to the rotations.

**Orthogonal Moments**

The moments when given can be used to reconstruct the underlying image, similar to the situation that given a Fourier transform the original function can be reconstructed.

The reconstruction is, however, simpler and more reliable in practice when the kernels used to define the moments are *orthogonal* and/or *normal*. Here, orthogonal means the inner product of any two different kernels is 0, and normal means the inner product of a kernel with itself is 1.

In the case of geometric moments, the kernels are $\psi_{pq}(x_1, x_2) = x_1^p x_2^q$ are not normal. There have been a number of choices of orthogonal kernels in the literature leading up to the so-called *Zernike moments*, *Legendre moments*, *Fourier-Mellin moments*, etc.

While these moments defined from some specific orthogonal kernels, derivations of invariants under certain transformations such as rotations are complicated.

### 2.4.4 Affine Moment Invariants

Recall that *an invariant* with respect to some transformation $T$ is a function $I$ defined on the space of all images such that if an image $f'$ is transformed into image $f$ under $T$ then $I(f) = I(f')$.

In general, any image $f$ can be represented by the set $\boldsymbol{\mu}^f$ of all its central moments. This means from $f$, central moment $\mu_{pq}^f$ of $f$ can be calculated for any non-negative integers $p, q$, and from all these $\mu_{pq}^f$ image $f$ can be uniquely reconstructed.

Now, a *moment invariant* with respect to a transformation $T$ is defined to be a function $I$ defined on the space $\{\boldsymbol{\mu}^f : f \text{ an image}\}$, such that if image $f$ is obtained from $f'$ under $T$ then $I(\boldsymbol{\mu}^f) = I(\boldsymbol{\mu}^{f'})$. Thus a moment invariant is an invariant defined on the moments of the image.

An *affine moment invariant* is defined to be a moment invariant under the affine transformation of images.

From Hu's landmark paper ([5]) and other work that follow ([4][28]), a number of affine moment invariants have been derived. The derivation process is as follows.

Suppose image $f(\boldsymbol{x})$ is a linear transformation of image $f'(\boldsymbol{x}')$**,** i.e. for any $\boldsymbol{x}$ and $\boldsymbol{x}'$ such that $\boldsymbol{x} = \mathrm{A}\boldsymbol{x}'$ then $f(\boldsymbol{x}) = f'(\boldsymbol{x}')$**.**

The moment generating functions $M(\boldsymbol{s})$ for $f(\boldsymbol{x})$ can be written as:

$$M(\boldsymbol{s}) = \sum_{\boldsymbol{p}} \frac{1}{p!} P_p(\boldsymbol{s}; \boldsymbol{\mu}_p) \tag{2.30}$$

Where $\boldsymbol{s} = (s_1, s_2)^{\mathrm{T}}$, and $P_p(\boldsymbol{s}; \boldsymbol{\mu}_p)$ is a binary form of order $p$ with coefficient $\boldsymbol{\mu}_p$, which is a collection of moments of image $f(\boldsymbol{x})$**.** Similarly, the moment generating functions $M'(\boldsymbol{s}')$ for $f'(\boldsymbol{x}')$ can be written as:

$$M'(\boldsymbol{s}') = \sum_{\boldsymbol{p}} \frac{1}{p!} P_p(\boldsymbol{s}'; \boldsymbol{\mu}'_p) \tag{2.31}$$

Where $\boldsymbol{s}' = (s'_1, s'_2)^{\mathrm{T}}$, and $P_p(\boldsymbol{s}'; \boldsymbol{\mu}'_p)$ is binary form of order $p$ with coefficient $\boldsymbol{\mu}'_p$, which is a collection of moments of image $f'(\boldsymbol{x}')$**.**

Now, transform $\boldsymbol{s}'$ to $\boldsymbol{s}$ under $\boldsymbol{s} = \mathrm{M}\boldsymbol{s}'$ with $\mathrm{M} = (\mathrm{A}^{\mathrm{T}})^{-1}$, i.e. $\boldsymbol{x}$ and $\boldsymbol{s}$ are contragredient, then

It can be shown that $P_p(\boldsymbol{s}; \boldsymbol{\mu}_p)$ transforms to $P_p\left(\boldsymbol{s}'; \frac{\mathbf{1}}{|J|}\boldsymbol{\mu}'_p\right)$, where $J = \det M$ and $|J|$ is the absolute value of J. This means:

$$P_p(\boldsymbol{s};\boldsymbol{\mu}_p) = P_p\left(\boldsymbol{s}';\frac{1}{|J|}\boldsymbol{\mu}'_p\right) \tag{2.32}$$

Now, in the theory of algebraic forms, a number of algebraic invariants for binary form of small order $p$ have been found. Using these algebraic invariants, moment invariants for the linear transformations can be constructed as follows:

Let $K(\boldsymbol{\mu})$ be such an algebraic invariant with weight $w$ and degree $k$, also called order $k$. Then function $I(\boldsymbol{\mu})$ defined as:

$$I(\boldsymbol{\mu}) = \begin{cases} \dfrac{K(\boldsymbol{\mu})}{\mu^{w+k}} & if\ w\ is\ even \\ \dfrac{|K(\boldsymbol{\mu})|}{|\mu|^{w+k}} & if\ w\ is\ odd \end{cases} \tag{2.33}$$

is a moment invariant under $\boldsymbol{x} = A\boldsymbol{x}'$**.** Here $\mu = \mu_{00}$ is the geometric of order 0 of image $f(\boldsymbol{x})$.

Now if an image $f(\boldsymbol{x})$ is obtained from image $f'(\boldsymbol{x}')$ under an affine transformation $\boldsymbol{x} = \boldsymbol{A}\boldsymbol{x}^{'} + \boldsymbol{b}$**.** Then image $f(\boldsymbol{x})$ can be considered as the translation of image $g(\boldsymbol{y})$ under $\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{b}$, while image $g(\boldsymbol{y})$ is the linear transformation of image $f'(\boldsymbol{x}')$ under $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}^{'}$**.**

For each of images $'(\boldsymbol{x}')$ , $g(\boldsymbol{y})$ , $f(\boldsymbol{x})$ , translate the original coordinate system to the centroid of the image and then calculate the geometric moments. Then the moments of $g(\boldsymbol{y})$ and $f(\boldsymbol{x})$ are identical, and function $I(\boldsymbol{\mu})$ is the same for $f'(\boldsymbol{x}')$ and $g(\boldsymbol{y})$. Thus $I(\boldsymbol{\mu})$ is the same for $f'(\boldsymbol{x}')$ and $f(\boldsymbol{x})$. In other words, $I(\boldsymbol{\mu})$ as obtained in (2.33) is an affine moment invariant for image $f(\boldsymbol{x})$.

The four affine moment invariants $\psi_1.\ \psi_2$, $\psi_3$, and $\psi_4$ as derived in ([4]) are:

$$\psi_1 = \frac{1}{\mu^4}[\mu_{20}\mu_{02} - \mu_{11}^2] \tag{2.34}$$

$$\psi_2 = \frac{1}{\mu^{10}}[(\mu_{03}\mu_{30} - \mu_{21}\mu_{12})^2 - 4(\mu_{30}\mu_{12} - \mu_{21}^2)(\mu_{03}\mu_{21} - \mu_{12}^2)] \tag{2.35}$$

$$\psi_3 = \frac{1}{\mu^7}[\mu_{20}(\mu_{21}\mu_{03} - \mu_{12}^2) - \mu_{11}(\mu_{30}\mu_{03} - \mu_{12}\mu_{21}) + \mu_{02}(\mu_{30}\mu_{12} - \mu_{21}^2)] \tag{2.36}$$

$$\begin{aligned}\psi_4 = \frac{1}{\mu^{11}}[&\mu_{30}^2\mu_{02}^3 - 6\mu_{30}\mu_{21}\mu_{11}\mu_{02}^2 + 6\mu_{30}\mu_{12}\mu_{02}(2\mu_{11}^2 - \mu_{20}\mu_{02}) \\ &+ \mu_{30}\mu_{03}(6\mu_{20}\mu_{11}\mu_{02} - 8\mu_{11}^3) + 9\mu_{21}^2\mu_{20}\mu_{02}^2 \\ &- 18\mu_{21}\mu_{12}\mu_{20}\mu_{11}\mu_{12} + 6\mu_{21}\mu_{03}\mu_{20}(2\mu_{11}^2 - \mu_{20}\mu_{02}) \\ &+ 9\mu_{11}^2\mu_{02}\mu_{20}^2 - 6\mu_{12}\mu_{03}\mu_{11}\mu_{20}^2 + \mu_{03}^2\mu_{02}^3]\end{aligned} \tag{2.37}$$

Where $\mu$ denotes $\mu_{00}$, and $\mu_{pq}$ denotes the moments of the image with respect to the coordinate system that is centered at the image's centroid.

### 2.4.5 A Summary of Steps to Derive Affine Moment Invariants

The following is a summary of the steps to derive affine moment invariants (AMI) for two-dimensional images.

1. Given an image $f(\boldsymbol{x})$ that is an affine transformation of image $f'(\boldsymbol{x}')$. This means $f(\boldsymbol{x}) = f'(\boldsymbol{x}')$ whenever $\boldsymbol{x} = A\boldsymbol{x}' + \boldsymbol{b}$, where A is $2 \times 2$ non singular matrix and $\boldsymbol{b}$ is a vector in $\boldsymbol{R^2}$.
2. Calculate the centroids of image $f(\boldsymbol{x})$ and $f'(\boldsymbol{x}')$.
3. Decompose the moment generating functions of $f(\boldsymbol{x})$ and $f'(\boldsymbol{x}')$ into sums of binary forms as in (2.30) and (2.31). Here a binary form $P_p(\boldsymbol{s}; \boldsymbol{\mu}_p)$ of order $p$ is

$$P_p(\boldsymbol{s}; \boldsymbol{\mu}_p) = \sum_{k_1+k_2=p} \binom{p}{k_1 \; k_2} \mu_{k_1 k_2} s_1^{k_1} s_2^{k_2}$$

   And $\boldsymbol{s} = (s_1, s_2)^T$ is a vector in $\boldsymbol{R^2}$, and $\mu_{k_1 k_2}$ are moments of $f(\boldsymbol{x})$ with respect to the coordinate system having the origin at the centroid of $f(\boldsymbol{x})$.
4. Substitute $\boldsymbol{s} = \mathrm{M}\boldsymbol{s}'$ into $P_p(\boldsymbol{s}; \boldsymbol{\mu}_p)$ to obtain (2.32), where $M = (A^T)^{-1}$ and $J = |\det M|$.
5. From the theory of algebraic forms, obtain a known algebraic invariant $K(\boldsymbol{\mu})$ for $\{P_2(\boldsymbol{s}; \boldsymbol{\mu}_2), P_3(\boldsymbol{s}; \boldsymbol{\mu}_3)\}$.
6. Use (2.33) to obtain an AMI $I(\boldsymbol{\mu})$. Here $I(\boldsymbol{\mu}) = I(\boldsymbol{\mu'})$, with $\boldsymbol{\mu}$ and $\boldsymbol{\mu'}$ being vectors of moments of $f(\boldsymbol{x})$, $f(\boldsymbol{x}')$ with respect to the coordinate systems centered at the centroids.

The above steps of derivation can be illustrated as in the following figure.

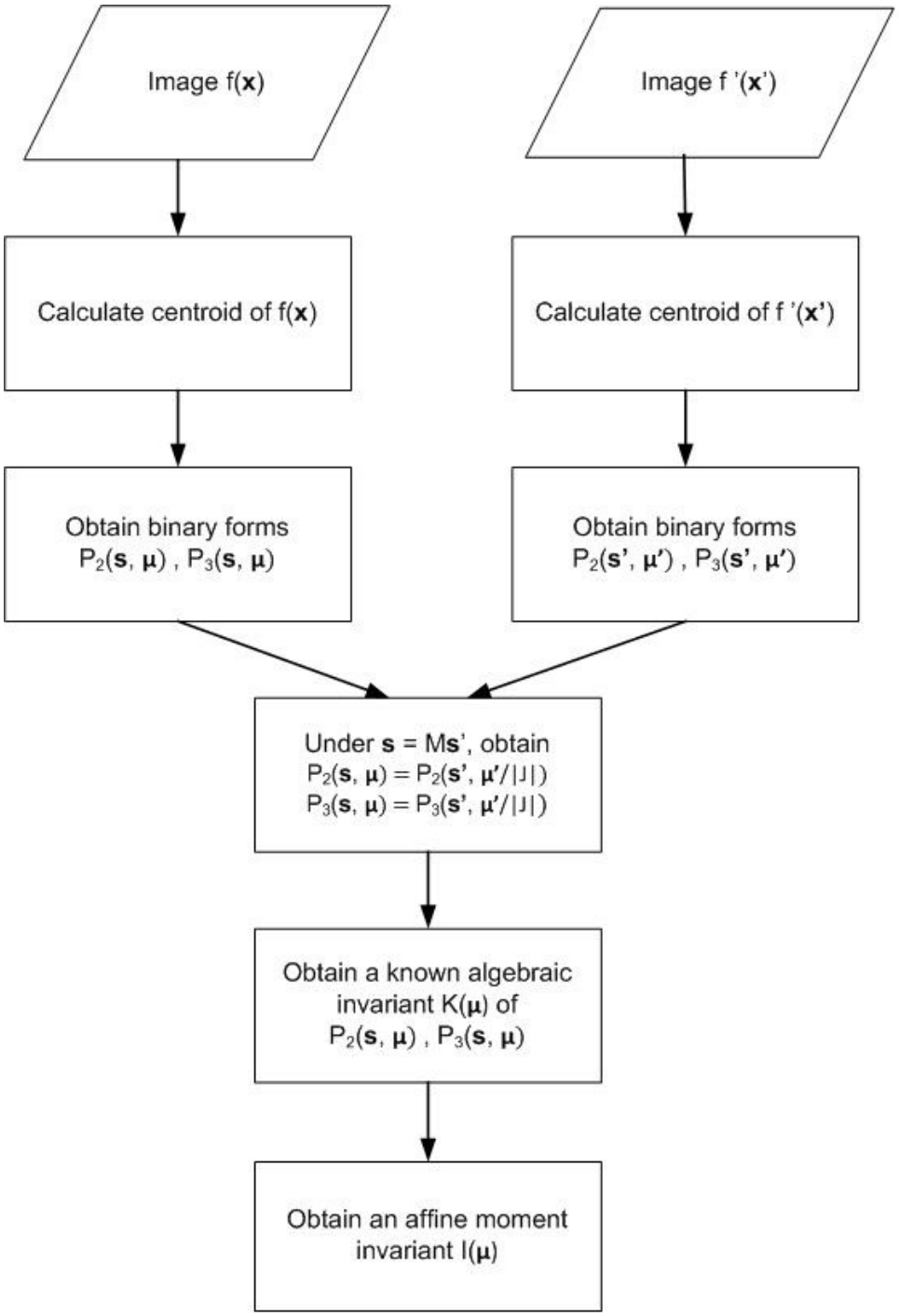


Figure 2.1: Steps in deriving affine moment invariants

# Chapter 3 – Wigner Distributions under Linear Transformations

## 3.1 Introduction

Affine moment invariants are functions defined on the moments of an underlying image such that when the image is changed under any affine transformation these functions are unchanged. In other words, the value of such a function calculated based on the moments of the original images and the value calculated based on the image obtained from any affine transformation are the same ([1][5][4])

Since the moments of an image are a summarization of the image, an affine moment invariant as a function of the moments can be also considered as a rough summarization of the image. Thus an affine moment invariant of an image not only captures certain information about the image but also is unchanged when the image is affine transformed. Affine moment invariants have been used in a number of object recognition applications ([27][26][68][69]).

The moments of an image are essentially calculated from the image's spatial coordinates and intensities, thus an affine moment invariant captures the information about the image through the spatial coordinates and intensities.

For many types of images such as images having texture, the intensities of such an image often are considered as not explicitly containing all information, especially information about texture. Specifically, the intensities do not provide explicit information about the image at every combination of spatial coordinate and spatial-frequency coordinate. Thus an affine moment invariant calculated based on intensities could be considered as not containing enough information for images having texture.

A spatial-frequency representation of an image is a representation of the image giving a value for every combination of spatial coordinate and spatial frequency coordinate. Essentially it is defined and calculated from the intensities of the underlying image. The representation is explicit because it gives a value for every pair of spatial and spatial frequency coordinates, while the intensity gives a value for every spatial coordinate only ([43][64][41][9]).

A spatial-frequency representation of an image could be considered as capturing both the information about the shapes and texture of objects in the image. Wigner distribution is the most popular and most often used spatial-frequency representation ([10][43][44]).It has been used to segment and classifications of objects in textured images ([47][70][12][71]). This implies the information about shapes and textures of the objects is contained in the Wigner distribution.

This dissertation is about deriving invariants from the moments of Wigner distribution of any image such that these invariants are unchanged when the underlying image is affine transformed. Since Wigner distributions are expected to contain the shape and texture information, so are these invariants. Derivations of invariants that capture the texture information is a topic of great interest ([6][53][52][7][72]), invariants derived from Wigner distributions and unchanged with respect to affine transformations as proposed in this dissertation could be considered as a contribution to this literature.

This chapter sets out to derive a relationship between invariants to be defined on the moments of Wigner distribution and some types of algebraic invariants as defined in the theory of algebraic forms. The invariants to be defined on moments of Wigner distributions will be invariant with respect to affine transformation and will be called *affine Wigner moment invariants.*

The approach taken follows the approach relating affine moment invariants to algebraic invariants of binary forms ([5][4][28]). In the derivation process, a number of definitions and properties related to Wigner distributions are made and shown.

In the next section 3.2, some definitions related to Wigner distributions are given, along with a few properties. In section 3.3, a decomposition of the moment generating function of any Wigner distribution into a sum of multivariate forms is derived, as in (3.15). The coefficients of these forms consist of the moments of the Wigner distribution. Then it is shown how two multivariate forms of the same order are related when an image is under a linear transformation, as in (3.17). The goal of this chapter to derive a relationship between affine Wigner moment invariants and some algebraic invariants is accomplished in section 3.4. The relationship is established through propositions 3.7 and 3.8 in this chapter.

By the end of this chapter a relationship between affine Wigner moment invariants and some algebraic invariants is determined. It basically says that if an algebraic invariant for the quaternary forms having contragredient variables can be found, then an affine Wigner moment invariant can be determined. The next chapter 4 presents a method to derive some algebraic invariants of these quaternary forms of order 2 and 3, and thus determines some affine Wigner moment invariants.

## 3.2 Wigner Distributions

Let $f(\boldsymbol{x})$ be a continuous function defined on $\boldsymbol{R}^2$ representing an image. A space-spatial frequency representation of $f(\boldsymbol{x})$ is a function of both the space and spatial frequency coordinates**.** Wigner Ville distribution, or its specific case called Wigner distribution, is such a representation and is the most typical and most used.

**Definition**

For any $\boldsymbol{x} = (x_1, x_2)^T$ and $\boldsymbol{w} = (w_1, w_2)^T$ in $\boldsymbol{R}^2$, the *Wigner-Ville distribution* for $f(\boldsymbol{x})$ is defined to be the following function:

$$W(\boldsymbol{x}, \boldsymbol{w}) = \int g_{\boldsymbol{x}}\left(\frac{\tau_1}{2}, \frac{\tau_2}{2}\right) exp\{-j(\tau_1 w_1 + \tau_2 w_2)\} d\tau_1 d\tau_2 \tag{3.1}$$

where

$$g_{\boldsymbol{x}}(s_1, s_2) = f(x_1 + s_1, x_2 + s_2) f^*(x_1 - s_1, x_2 - s_2) \tag{3.2}$$

In the case considered in this dissertation $f(\boldsymbol{x})$ is a real-valued function, thus $f^*(\boldsymbol{x}) = f(\boldsymbol{x})$ for any $\boldsymbol{x}$**.** The Wigner-Ville as defined above is called the *Wigner distribution*.

When considered as a function of $\boldsymbol{w}$ at some fixed $\boldsymbol{x}$**,** $W(\boldsymbol{x}, \boldsymbol{w})$ is denoted by $W_{\boldsymbol{x}}(\boldsymbol{w})$. By change of variables $u_1 = \frac{\tau_1}{2}$, and $u_2 = \frac{\tau_2}{2}$, then

$$W_{\boldsymbol{x}}(\boldsymbol{w}) = 4 \int g_{\boldsymbol{x}}(u_1, u_2) exp\{-j(u_1 2w_1 + u_2 2w_2)\} du_1 du_2 \tag{3.3}$$

Let $G_{\boldsymbol{x}}(\boldsymbol{w})$ be the Fourier transform of $g_{\boldsymbol{x}}(u_1, u_2)$, then

$$W_{\boldsymbol{x}}(\boldsymbol{w}) = 4\, G_x(2\boldsymbol{w}) \tag{3.4}$$

Thus besides the proportional constants, Wigner distribution $W_{\boldsymbol{x}}(\boldsymbol{w})$ is basically the Fourier transform of function $g_{\boldsymbol{x}}(u_1, u_2)$ defined around $\boldsymbol{x}$**,** some what similar to the short time Fourier transform.

**Definition**

For any non-negative integers $p,\ q, r, s,$ a *geometric moment* of order $k = p + q + r + s$ for $W(\boldsymbol{x}, \boldsymbol{w})$ is defined to be:

$$\mu_{pqrs} = \int x_1^p x_2^q w_1^r w_2^s W(\boldsymbol{x}, \boldsymbol{w}) d\boldsymbol{x} d\boldsymbol{w} \tag{3.5}$$

The set of all $\mu_{pqrs}$'s could be considered as a summarization of distribution $W(\boldsymbol{x}, \boldsymbol{w})$**.** This interpretation is a generalization of the interpretation that the moments of an image are a summarization of the image. Let

$$x_{1c} = \frac{\mu_{1000}}{\mu_{0000}} \qquad and \qquad x_{2c} = \frac{\mu_{0100}}{\mu_{0000}} \tag{3.6}$$

Point $\boldsymbol{x}_c = (x_{1c}, x_{2c})^T$ is defined to be the *centroid* of $(\boldsymbol{x}, \boldsymbol{w})$ .

The rest of this section presents some properties of Wigner distributions.

**Proposition 3.1**

If $f(\boldsymbol{x})$ and $f'(\boldsymbol{x}')$ are two functions defined in $\boldsymbol{R}^2$ and there is a constant vector $\boldsymbol{d}$ such that $f(\boldsymbol{x}) = f'(\boldsymbol{x}')$ for $\boldsymbol{x} = \boldsymbol{x}' + \boldsymbol{d}$. Then $W(\boldsymbol{x}, \boldsymbol{w})$ is unchanged under translation, i.e.

$$W(\boldsymbol{x}, \boldsymbol{w}) = W'(\boldsymbol{x}', \boldsymbol{w}) \tag{3.7}$$

Where $W(\boldsymbol{x}, \boldsymbol{w})$, $W'(\boldsymbol{x}', \boldsymbol{w})$ are the Wigner-Ville distributions of $f(\boldsymbol{x})$ and $f'(\boldsymbol{x}')$ respectively.

Proof) Consider the definition of $W(x_1, x_2, w_1, w_2)$, from the relationship between $f(\boldsymbol{x})$ and $f'(\boldsymbol{x}')$:

$$f\left(x_1 + \frac{s_1}{2},\ x_2 + \frac{s_2}{2}\right) = f'\left(x_1 + \frac{s_1}{2} - d_1\ ,\ x_2 + \frac{s_2}{2} - d_2\right)$$

Substituting $f'\left(x_1 + \frac{s_1}{2} - d_1\ ,\ x_2 + \frac{s_2}{2} - d_2\right)$ for $f\left(x_1 + \frac{s_1}{2},\ x_2 + \frac{s_2}{2}\right)$ in the definition of $W(x_1, x_2, w_1, w_2)$ in (3.1) and (3.2) to obtain:

$$W(x_1, x_2, w_1, w_2) = W'(x_1 - d_1, x_2 - d_2, w_1, w_2)$$

By substituting $x'_1 = x_1 - d_1$ and $x'_2 = x_2 - d_2$, obtained from the relationship $\boldsymbol{x} = \boldsymbol{x}' + \boldsymbol{d}$, into the above, the proposition follows. QED.

Other properties for Wigner –Ville distributions for continuous function $f(\boldsymbol{x})$, $\boldsymbol{x}$ in $\boldsymbol{R}^2$, are:

$$\int W(x_1, x_2, w_1, w_2) dx_1 dx_2 = |F(w_1, w_2)|^2 \tag{3.8}$$

$$\frac{1}{(2\pi)^2} \int W(x_1, x_2, w_1, w_2) dw_1 dw_2 = |f(x_1, x_2)|^2 \tag{3.9}$$

Where $F(w_1, w_2)$is the Fourier transform of $f(x_1, x_2)$.

The last property implies that there is a one-one correspondence between moment $m_{pq\,00}$ of $W(x_1, x_2, w_1, w_2)$ and $m_{pq}$ of $|f(x_1, x_2)|^2$. Thus information about non-negative $f(x_1, x_2)$ or equivalently $|f(x_1, x_2)|^2$ captured in $m_{pq}$'s is also captured in $m_{pqrs}$'s.

## 3.3 Moment Generating Functions under Linear Transformations

This section will show that when an image $f'(\boldsymbol{x}')$ is transformed to another image $f(\boldsymbol{x})$ under a linear transformation, how their Wigner-Ville distributions and moment generating functions are related.

**Definition**

For any Wigner distribution $W(\boldsymbol{x}, \boldsymbol{w})$**,** its moment generating function $\mathcal{M}(\boldsymbol{s}, \boldsymbol{u})$ is defined to be: For $\boldsymbol{s}, \boldsymbol{u}$ in $\boldsymbol{R}^2$:

$$\mathcal{M}(\boldsymbol{s}, \boldsymbol{u}) = \int exp(\boldsymbol{s}^T \boldsymbol{x} + \boldsymbol{u}^T \boldsymbol{w}) W(\boldsymbol{x}, \boldsymbol{w}) d\boldsymbol{x} d\boldsymbol{w} \tag{3.10}$$

The following notational set up is used in the remainder of this section. Image $f(\boldsymbol{x})$ is a linear transform of $f'(\boldsymbol{x}')$, i.e. if $\boldsymbol{x} = A\boldsymbol{x}'$ with $\boldsymbol{x}'$ in $\boldsymbol{R}^2$ then $f(\boldsymbol{x}) = f'(\boldsymbol{x}')$, where $A = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ is a non-singular $2 \times 2$ matrix.

The Fourier transforms, the Wigner-Ville distributions, and the moment generating functions of $f(\boldsymbol{x})$ and $f'(\boldsymbol{x}')$ are denoted by $F(\boldsymbol{w})$ and $F'(\boldsymbol{w}')$ respectively, by $W(\boldsymbol{x}, \boldsymbol{w})$ and $W'(\boldsymbol{x}', \boldsymbol{w}')$ respectively, and by $\mathcal{M} = \mathcal{M}(\boldsymbol{s}, \boldsymbol{u})$ and $\mathcal{M}' = \mathcal{M}'(\boldsymbol{s}', \boldsymbol{u}')$ respectively.

Points $\boldsymbol{x}$ and $\boldsymbol{w}$ are assumed to be contragredient, i.e. if $\boldsymbol{x} = A\boldsymbol{x}'$ then $\boldsymbol{w}' = A^T \boldsymbol{w}$**.**

And $\begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{u} \end{pmatrix}$ and $\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{w} \end{pmatrix}$ are assumed contragredient, this implies $\boldsymbol{s} = M\boldsymbol{s}'$ and $\boldsymbol{u} = (M^T)^{-1}\boldsymbol{u}'$, where $M = (A^T)^{-1}$, thus $(M^T)^{-1} = A$. And let $J = det\, M$.

It can be verified that the Fourier transforms $F(\boldsymbol{w})$ and $F'(\boldsymbol{w}')$ are proportional with $F(\boldsymbol{w}) = |detA|F'(\boldsymbol{w}')$. The following property of Wigner-Ville distribution is a generalization of this fact.

**Proposition 3.2**

The Wigner-Ville distributions $W(\boldsymbol{x}, \boldsymbol{w})$ and $W'(\boldsymbol{x}', \boldsymbol{w}')$ are proportional. Specifically:

$$W(\boldsymbol{x}, \boldsymbol{w}) = |det\, A|\, W'(\boldsymbol{x}', \boldsymbol{w}') \tag{3.11}$$

Proof)

Let $W = W(\boldsymbol{x}, \boldsymbol{w})$ be the Wigner-Ville distribution of $f(\boldsymbol{x})$. By definition:

$$W = \int f\left(\boldsymbol{x} + \frac{1}{2}\boldsymbol{\tau}\right) f^*\left(\boldsymbol{x} - \frac{1}{2}\boldsymbol{\tau}\right) exp\{-j(\boldsymbol{\tau}^{\boldsymbol{T}}\boldsymbol{w})\}\, d\boldsymbol{\tau}$$

Let $W' = W'(\boldsymbol{x}', \boldsymbol{w}')$ be the Wigner-Ville distribution of $'(\boldsymbol{x}')$ . By definition:

$$W' = \int f'\left(\boldsymbol{x}' + \frac{1}{2}\boldsymbol{\tau}'\right) f'^*\left(\boldsymbol{x}' - \frac{1}{2}\boldsymbol{\tau}'\right) exp\{-j(\boldsymbol{\tau}'^{\boldsymbol{T}}\boldsymbol{w}')\}\, d\boldsymbol{\tau}'$$

The relationship between $W$ and $W'$ can be obtained through the change of variables $\boldsymbol{\tau} = A\boldsymbol{\tau}'$ **,** where $\boldsymbol{\tau} = (\tau_1, \tau_2)^T$ and $\boldsymbol{\tau}' = (\tau'_1, \tau'_2)^T$.

Under the transformation $= A\boldsymbol{\tau}'$ **,** and since $f(\boldsymbol{x}) = f'(\boldsymbol{x}')$ for $\boldsymbol{x} = A\boldsymbol{x}'$ ,

$$f\left(\boldsymbol{x} + \frac{1}{2}\boldsymbol{\tau}\right) = f\left(A\boldsymbol{x}' + \frac{1}{2}A\boldsymbol{\tau}'\right)$$

$$= f'\left(\boldsymbol{x}' + \tfrac{1}{2}\boldsymbol{\tau}'\right)$$

Similarly,

$$f\left(\boldsymbol{x}+\frac{1}{2}\boldsymbol{\tau}\right)= f'\left(\boldsymbol{x}'+\frac{1}{2}\boldsymbol{\tau}'\right)$$

Also since $\boldsymbol{w}$ is contragredient with $\boldsymbol{x}$, but $\boldsymbol{x}$ is cogredient with $\boldsymbol{\tau}$, then $\boldsymbol{w}$ is contragredient with $\boldsymbol{\tau}$. This implies $\boldsymbol{\tau}^{T}\boldsymbol{w} = \boldsymbol{\tau}'^{T}\boldsymbol{w}'$.

Thus $W = |det\, A|\ W'$. QED.

Denote by $\mu \equiv \mu_{0000}$ and $\mu' \equiv \mu'_{0000}$, from the above proposition it is clear that $\mu = |det\, A|\ \mu'$. Thus

$$\mu = |det\, A|\ \mu' \qquad \text{or equivalently} \qquad \mu' = |det\, M|\ \mu \tag{3.12}$$

**Proposition 3.3**

Let the centroids of $W(\boldsymbol{x},\boldsymbol{w})$ and $W'(\boldsymbol{x}',\boldsymbol{w}')$ be $\boldsymbol{x}_c$ and $\boldsymbol{x}'_{c'}$ respectively, then

$$\boldsymbol{x}_c = A\boldsymbol{x}'_{c'} \tag{3.13}$$

Proof)

The moment $\mu_{1000}$ of $f(\boldsymbol{x})$ is:

$$\mu_{1000} = \int x_1 W(\boldsymbol{x},\boldsymbol{w})d\boldsymbol{w}d\boldsymbol{x}$$

$$= (2\pi)^2 \int x_1 |f(\boldsymbol{x})|^2 d\boldsymbol{x}$$

Where in the last expression was obtained by using (3.9). By changing variable $\boldsymbol{x} = A\boldsymbol{x}'$ and using $f(\boldsymbol{x}) = f'(\boldsymbol{x}')$ in the last expression, $\mu_{1000}$ becomes

$$\mu_{1000} = (2\pi)^2 |\det A| \left[\alpha \int x'_1 |f'(\boldsymbol{x}')|^2 d\boldsymbol{x}' + \beta \int x'_2 |f'(\boldsymbol{x}')|^2 d\boldsymbol{x}'\right]$$

$$= |det\ A| \left[\alpha \int x'_1 W'(\boldsymbol{x}', \boldsymbol{w}') d\boldsymbol{w}' d\boldsymbol{x}' + \beta \int x'_2 W(\boldsymbol{x}', \boldsymbol{w}') d\boldsymbol{w}' d\boldsymbol{x}'\right]$$

$$= |detA| [\alpha \mu'_{1000} + \beta \mu'_{0100}]$$

Where the second to the last expression was obtained using (3.9) again, and the last expression was obtained using the definitions of $\mu'_{1000}$ and $\mu'_{0100}$. Similarly,

$$\mu_{0100} = |detA| [\gamma \mu'_{1000} + \delta \mu'_{0100}]$$

Thus

$$\begin{pmatrix} \mu_{1000} \\ \mu_{0100} \end{pmatrix} = |\det A|\ A \begin{pmatrix} \mu'_{1000} \\ \mu'_{0100} \end{pmatrix}$$

Now applying $\boldsymbol{x}_c = \frac{1}{\mu_{0000}} \begin{pmatrix} \mu_{1000} \\ \mu_{0100} \end{pmatrix}$, $\boldsymbol{x}'_{c'} = \frac{1}{\mu'_{0000}} \begin{pmatrix} \mu'_{1000} \\ \mu'_{0100} \end{pmatrix}$, and (3.12) in the above expression, expression (3.13) can be obtained. QED

**Proposition 3.4**

The moment generating functions $M(\boldsymbol{s}, \boldsymbol{u})$ and $M'(\boldsymbol{s}', \boldsymbol{u}')$ are proportional. Specifically:

$$\mathcal{M}(\boldsymbol{s}, \boldsymbol{u}) = |det\ A|\ \mathcal{M}'(\boldsymbol{s}', \boldsymbol{u}') \tag{3.14}$$

Proof)

The relationship between the moment generating functions $\mathcal{M}$ and $\mathcal{M}'$ can be obtained through the change of variables $\boldsymbol{s} = \mathrm{M}\boldsymbol{s}'$, and $\boldsymbol{u} = A\boldsymbol{u}'$.

Since $\boldsymbol{x}$ and $\boldsymbol{w}$ are contragredient, $W = |det\, A|\, W'$ by (3.11)

Since $\begin{pmatrix}\boldsymbol{s}\\ \boldsymbol{u}\end{pmatrix}$ and $\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{w}\end{pmatrix}$ are contragredient, and contragredience is equivalence equality of inner products as reviewed in section 2.2.1:

$$\boldsymbol{s}^T\boldsymbol{x} + \boldsymbol{u}^T\boldsymbol{w} = \boldsymbol{s}'^T\boldsymbol{x}' + \boldsymbol{u}'^T\boldsymbol{w}'$$

And since $det\begin{bmatrix}M & 0\\ 0 & A\end{bmatrix} = 1$, because $M = (A^T)^{-1}$, $\mathcal{M}$ can be expressed as:

$$\mathcal{M} = |det\, A| \iint exp\left(\boldsymbol{s}'^T\boldsymbol{x}' + \boldsymbol{u}'^T\boldsymbol{w}'\right) W'(\boldsymbol{x}', \boldsymbol{w}')d\boldsymbol{s}'d\boldsymbol{w}'$$

Thus $\mathcal{M} = |det\, A|\, \mathcal{M}'$. QED

**Proposition 3.5**

The moment generating function $\mathcal{M}(\boldsymbol{s}, \boldsymbol{u})$ of $W(\boldsymbol{x}, \boldsymbol{w})$ can be decomposed into:

$$\mathcal{M}(\boldsymbol{s}, \boldsymbol{u}) = \sum_{p=0}^{\infty} \frac{1}{p!} P_p(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}) \tag{3.15}$$

Where $P_p(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$'s are quaternary forms of order $p$, $\boldsymbol{\mu} = \left(\mu_{k_1k_2k_3k_4}\right)$ is a vector of moments of $W(\boldsymbol{x}, \boldsymbol{w})$.

Proof)

The moment generating $\mathcal{M}$ of $W(\boldsymbol{x}, \boldsymbol{w})$ is:

$$\mathcal{M} = \int exp(\boldsymbol{s}^T \boldsymbol{x} + \boldsymbol{u}^T \boldsymbol{w}) W(\boldsymbol{x}, \boldsymbol{w}) d\boldsymbol{x} d\boldsymbol{w}$$

Where the exponential term can be expressed as:

$$exp(\boldsymbol{s}^T \boldsymbol{x} + \boldsymbol{u}^T \boldsymbol{w}) = \sum_{p=0}^{\infty} \frac{1}{p!} (\boldsymbol{s}^T \boldsymbol{x} + \boldsymbol{u}^T \boldsymbol{w})^p$$

$$= \sum_{p=0}^{\infty} \frac{1}{p!} (s_1 x_1 + s_2 x_2 + u_1 w_1 + w_2 w_2)^p$$

$$= \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{k_1+k_2+k_3+k_4=p} \binom{p}{k_1 k_2 k_3\ k_4} s_1^{k_1} s_2^{k_2} u_1^{k_3} u_2^{k_4}\ x_1^{k_1} x_2^{k_2} w_1^{k_3} w_2^{k_4}$$

Thus

$$\mathcal{M} = \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{k_1+k_2+k_3+k_4=p} \binom{p}{k_1 k_2 k_3\ k_4} s_1^{k_1} s_2^{k_2} u_1^{k_3} u_2^{k_4} \iint x_1^{k_1} x_2^{k_1} w_1^{k_3} w_2^{k_4}\, W(\boldsymbol{x}, \boldsymbol{w}) d\boldsymbol{x} d\boldsymbol{w}$$

$$= \sum_{p=0}^{\infty} \frac{1}{p!} \sum_{k_1+k_2+k_3+k_4=p} \binom{p}{k_1 k_2 k_3\ k_4} \mu_{k_1 k_2 k_3 k_4} s_1^{k_1} s_2^{k_2} u_1^{k_3} u_2^{k_4}$$

$$= \sum_{p=0}^{\infty} \frac{1}{p!} P_p(\boldsymbol{u}, \boldsymbol{v}; \boldsymbol{\mu})$$

QED.

From the above proposition, the moment generating function $\mathcal{M}'(\boldsymbol{s}', \boldsymbol{u}')$ of $W'(\boldsymbol{x}', \boldsymbol{w}')$ can also be decomposed into:

$$\mathcal{M}(\boldsymbol{s}', \boldsymbol{u}') = \sum_{p=0}^{\infty} \frac{1}{p!} P_p(\boldsymbol{s}', \boldsymbol{u}'; \boldsymbol{\mu}') \tag{3.16}$$

Where $P_p(\boldsymbol{s}', \boldsymbol{u}'; \boldsymbol{\mu}')$'s are quaternary forms of order $p$, $\boldsymbol{\mu}' = \left(\mu'_{k_1 k_2 k_3 k_4}\right)$ is a vector of moments of $W'(\boldsymbol{x}', \boldsymbol{w}')$.

**Proposition 3.6**

Under the transformations $\boldsymbol{s} = M\boldsymbol{s}'$ and $\boldsymbol{u} = (M^T)^{-1}\boldsymbol{u}'$, all the quaternary forms $P_p(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ are transformed into $P_p(\boldsymbol{s}', \boldsymbol{u}'; \frac{\mathbf{1}}{|J|}\boldsymbol{\mu}')$**,** i.e.

$$P_p(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}) = P_p(\boldsymbol{s}', \boldsymbol{u}'; \tfrac{\mathbf{1}}{|J|}\boldsymbol{\mu}') \tag{3.17}$$

for all positive integer $p$.

Proof)

From the definition of moment generating function (3.10), $\mathcal{M}'$ can be expressed as a power series:

$$\mathcal{M}' = \sum_{p=0}^{\infty} \sum_{k_1+k_2+k_3+k_4=p} \mu'_{k_1 k_2 k_3 k_4} \frac{u_1'^{k_1}}{k_1!} \frac{u_2'^{k_2}}{k_2!} \frac{v_1'^{k_3}}{k_3!} \frac{v_2'^{k_4}}{k_4!} \tag{3.17.1}$$

Now, under the transformation $\boldsymbol{s} = M\boldsymbol{s}'$ and $\boldsymbol{u} = (M^T)^{-1}\boldsymbol{u}'$, $P_p(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ becomes $P_p(\boldsymbol{s}', \boldsymbol{u}'; \boldsymbol{a}')$**.** Thus from (3.15):

$$\mathcal{M} = \sum_{p=0}^{\infty} \frac{1}{p!} P_p(\boldsymbol{u}', \boldsymbol{v}'; \boldsymbol{a}')$$

But by (3.14), $\mathcal{M}' = |J|\mathcal{M}$ because $|J| = |\,det\,M| = |\,det\,A|^{-1}$ , thus

$$\begin{aligned}\mathcal{M}^{'} &= |J|\sum_{p=0}^{\infty}\frac{1}{p!}P_p(\boldsymbol{u}^{'},\boldsymbol{v}^{'};\boldsymbol{a}^{'}) \\ &= |J|\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{k_1+k_2+k_3+k_4=p}\binom{p}{k_1k_2k_3\ k_4}a^{'}{}_{k_1k_2k_3k_4}u_1^{'k_1}u^{'k_1}_{\ 2}v^{'k_3}_{\ 1}v_2^{'k_4} \\ &= \sum_{p=0}^{\infty}\sum_{k_1+k_2+k_3+k_4=p}|J|a^{'}{}_{k_1k_2k_3k_4}\frac{u_1^{'k_1}}{k_1!}\frac{u_2^{'k_2}}{k_2!}\frac{v_1^{'k_3}}{k_3!}\frac{v_2^{'k_4}}{k_4!}\end{aligned}$$

The right hand side of last expression is another power series expansion for $\mathcal{M}^{'}$ besides (3.17.1). Since the power series expansion for $\mathcal{M}^{'}$ must be unique, it must be the case that:

$$a^{'}{}_{k_1k_2k_3k_4} = \frac{1}{|J|}\mu^{'}{}_{k_1k_2k_3k_4}$$

QED

## 3.4 Determination of Affine Wigner Moment Invariants

### 3.4.1 Affine Wigner Moment Invariants

The general problem of determining invariants of images can be stated as this. When an image is under certain transformation $T$, it is of interest to determine a feature vector that not only represents the image but also is unchanged under the transformation.

The concepts of invariants, moment invariants, and affine moment invariants have been reviewed in chapter 2, and can be restated and modified as follows:

**Definitions**

A*n invariant* for some transformation $T$ is a function $I$ defined on the space of all images such that if an image $f'$ is transformed into image $f$ under $T$ then $I(f) = I(f')$.

Denote by $S$ and $S_w$ the space of vectors of moments of all images, and the space of vectors of Wigner moments of the Wigner distributions of all images, respectively. Specifically

$$S = \{\boldsymbol{\mu}^f : \boldsymbol{\mu}^f \text{ vector of moments of any image } f\}$$

$$S_w = \{\boldsymbol{\mu}^f : \boldsymbol{\mu}^f \text{ vector of moments of the Wigner distribution of any image } f\}$$

A *moment invariant* for a transformation $T$ is a function $I$ defined on either $S$ or $S_w$ such that if image $f$ is obtained from $f'$ under $T$ then $I(\boldsymbol{\mu}^f) = I(\boldsymbol{\mu}^{f'})$.

Thus a moment invariant is an invariant defined on either the moments of an image or the moments of the Wigner distribution of an image.

An *affine moment invariant* (AMI) is a moment invariant defined on $S$ and for all affine transformations.

The goal of this dissertation is to propose a new type of invariants based on the moments of the Wigner distributions of images. These new invariants are expected to capture certain texture information and are invariant under any affine transformation. The following is the definition of this new type of invariants.

**Definition**

An *affine Wigner moment invariant* (AWMI) is a moment invariant defined on $S_w$ and for all affine transformations of the underlying image.

### 3.4.2 Affine Wigner Moment Invariants Determined from Algebraic Invariants

According to ([5][4]), an AMI is basically a moment invariant defined on $S$ where the moments are the coefficients of certain number of binary forms. This moment invariant is actually determined from an algebraic invariant of the binary forms.

The derivation of AWMI proposed in this dissertation will follow the same steps of deriving an AMI from an algebraic invariant. This means an AWMI will be shown to be a moment invariant defined from a set of algebraic forms, and this moment invariant will be determined from an algebraic invariant of the set of the forms.

The following definition specifies what sets of algebraic forms can be used to determine an AWMI.

**Definition**

Let $\Gamma$ be a set of $K$ multilinear forms $\Gamma = \{\gamma_k(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{p_k}; \boldsymbol{a}_k) : k = 1,2, \dots, K\}$ satisfying the following conditions:

- Each $\gamma_k(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{p_k}; \boldsymbol{a}_k)$ is a $p_k$-linear form defined on $p_k$ sets of vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{p_k}$ with coefficient vector $\boldsymbol{a}_k$.

- For the derivation of AMI, for each $i = 1, \dots, p_k$ each $\boldsymbol{x}_i = \boldsymbol{s} \in R^2$. And for any image $f(\boldsymbol{x})$, each form $\gamma_k(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{p_k}; \boldsymbol{a}_k)$ is mapped to form $\gamma_k(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{p_k}; \boldsymbol{\mu}_k)$**,** where $\boldsymbol{\mu}_k$ is a vector of moments of $f(\boldsymbol{x})$, $k = 1,2, \dots, K$

- For the derivation of AWMI, for each $i = 1, \dots, p_k$ each $\boldsymbol{x}_i = \boldsymbol{s} \in R^2$or $\boldsymbol{x}_i = \boldsymbol{u} \in R^2$, where $\boldsymbol{s}$ and $\boldsymbol{u}$ are contragredient. And for any image $f(\boldsymbol{x})$, each form $\gamma_k(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{p_k}; \boldsymbol{a}_k)$ is mapped to form $\gamma_k(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{p_k}; \boldsymbol{\mu}_k)$**,** where $\boldsymbol{\mu}_k$ is a vector of moments of the Wigner distribution of $f(\boldsymbol{x})$, $k = 1,2, \dots, K$

- If image $f(\boldsymbol{x})$ is a linear transform of image $f'(\boldsymbol{x}')$, i.e. $f(\boldsymbol{x}) = f'(\boldsymbol{x}')$ whenever $\boldsymbol{x} = A\boldsymbol{x}'$, then each $p_k$-linear form $\gamma_k(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{p_k}; \boldsymbol{\mu}_k)$ will transform to another $p_k$-linear form $\gamma_k\left(\boldsymbol{x'}_1, \boldsymbol{x'}_2, \dots, \boldsymbol{x'}_{p_k}; \frac{\mathbf{1}}{|J|}\boldsymbol{\mu'}_k\right)$**,** where $\boldsymbol{s}$ is contragredient with $\boldsymbol{x}$**.**

  Here $J = \det M$, and $M = (A^T)^{-1}$. And $\boldsymbol{\mu'}_k$ is a vector of moments for either $f'(\boldsymbol{x}')$ or its Wigner distribution, depending if deriving AMI or AWMI.

The set $\Gamma$ is said to be *compatible with linear transformations*.

The following defines moment invariants from a set $\Gamma$ of forms compatible with linear transformations.

**Definition**

Let $\Gamma$ be a set of forms compatible with linear transformations.

Let $I(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_K)$ be a homogeneous polynomial function defined on the vectors of coefficients $\boldsymbol{\mu}_k$'s of the form $\gamma_k(\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_{p_k}; \boldsymbol{\mu}_k)$**'**s such that

$$I(\boldsymbol{\mu'}_1, \boldsymbol{\mu'}_2, \dots, \boldsymbol{\mu'}_p) = I(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_p) \tag{3.18}$$

Then such as polynomial function $I(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_p)$ is called *a moment invariant defined from the set* $\Gamma$.

It is obvious from the definition of $\Gamma$ that function $I(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_K)$ as defined as above is a moment invariant for any linear transformation.

An illustration of the above definition for the case of AMI is as follows.

Let $\Gamma$ be any subset of $\{P_p(\boldsymbol{s}; \boldsymbol{a}); p = 1,2, \dots\}$, where for any image, $P_p(\boldsymbol{s}; \boldsymbol{\mu}_p)$ is as obtained from (2.30). Here $P_p(\boldsymbol{s}; \boldsymbol{\mu}_p)$ is the binary form of order $p$ and $\boldsymbol{\mu}_p$ being a vector of moments of the image.

Note that any binary form $P_p(\boldsymbol{s}; \boldsymbol{\mu}_1)$ of order $p$ is also a p-linear form $f(\boldsymbol{s}, \dots, \boldsymbol{s}; \boldsymbol{\mu}_1)$, where the $p$ sets of variables are identically equal to $\boldsymbol{s}$ in the form.

From the results obtained in the derivation of AMI, $\Gamma$ is compatible with linear transformations.

Now take $\Gamma$ to be any subset set of $\{P_2(\boldsymbol{s};\boldsymbol{a_1}), P_3(\boldsymbol{s};\boldsymbol{a_2})\}$, a number of algebraic invariants for $\Gamma$ are known from the theory of algebraic forms. From any of these algebraic invariants, a moment invariant $I$ defined from $\Gamma$ can be determined. Any such moment invariant $I$ can be shown to be an AMI.

As said earlier, the derivation of AWMI presented in this chapter follows the derivation of AMI from algebraic invariants, just like the illustration given above. The definition moment invariant $I(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_p)$ from a set $\Gamma$ of multilinear forms , not a set of just multivariate forms, is necessary for the more general case of AWMI. This can be seen more clearly in the next chapter.

The following proposition shows how to obtain a moment invariant from an algebraic invariant of a set of multilinear forms compatible with linear transformations.

**Proposition 3.7**

Let $K$ be an algebraic invariant of a set $\Gamma$ of forms compatible with linear transformations. Let the index and order of $K$ be $\omega$ and $k$ respectively. Then function $I(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_p)$ defined as

$$I(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_p) = \begin{cases} \frac{1}{\mu^{\omega+k}} K(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_p) & \text{if } \omega \text{ even} \\ \frac{1}{|\mu|^{\omega+k}} \left| K(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_p) \right| & \text{if } \omega \text{ odd} \end{cases} \tag{3.19}$$

is a moment invariant defined from the set $\Gamma$.

Proof)

From the definition of algebraic invariant, $K$ is an algebraic invariant of $\Gamma$ with index $\omega$ and order $k$ then:

$$K\left(\frac{\boldsymbol{\mu}'_1}{|J|}, \frac{\boldsymbol{\mu}'_2}{|J|}, \dots, \frac{\boldsymbol{\mu}'_K}{|J|}\right) = (det\ M)^{\omega} K(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_K)$$

But $K$ is a homogeneous polynomial of order $k$, then

$$K(\boldsymbol{\mu}'_1, \boldsymbol{\mu}'_2, \dots, \boldsymbol{\mu}'_K) = |J|^k (det\ M)^{\omega} K(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_K)$$

Dividing both sides of the above by $\mu'^{\omega+k}$, and using the fact that $\mu' = |J|\ \mu$ as in (3.12), with $J = det\ M$, the above expression becomes:

$$\frac{1}{\mu'^{\omega+k}} K(\boldsymbol{\mu}'_1, \boldsymbol{\mu}'_2, \dots, \boldsymbol{\mu}'_K) = \left(\frac{J}{|J|}\right)^{\omega} \frac{1}{\mu^{\omega+k}} K(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_K)$$

Thus $I(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_K)$ defined as in (3.19) is a moment invariant defined from $\Gamma$.

QED

In the case of AMI, a moment invariant defined from a set of multilinear forms compatible with linear transformations is an AMI. Similarly, a moment invariant for a set of forms compatible with linear transformations is also an AWMI, as shown in the following.

**Proposition 3.8**

Let $\Gamma$ be a set of multilinear forms compatible with linear transformations.

Let $f(\boldsymbol{x})$ be an image obtained from another image $f'(\boldsymbol{x}')$ through any affine transformation represented by a $2 \times 2$ non-singular matrix $A$ and vector $\boldsymbol{b}$. This means $f(\boldsymbol{x}) = f'(\boldsymbol{x}')$, whenever $\boldsymbol{x} = A\boldsymbol{x}' + \boldsymbol{b}$.

Let $I(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_K)$ be a moment invariant defined from $\Gamma$. Here $\boldsymbol{\mu}_k$ is a vector of moments of the Wigner distribution of the underlying image. For any image $f(\boldsymbol{x})$. Denote by $\boldsymbol{\mu}^f$ the vector $(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_K)$.

Let $\boldsymbol{\mu}^{f'}$ and $\boldsymbol{\mu}^f$ be a vector of all moments for image $f'(\boldsymbol{x}')$ and image $f(\boldsymbol{x})$ respectively.

Here $\boldsymbol{\mu}^{f'}$ and $\boldsymbol{\mu}^f$ are determined from the coordinate systems centered at the centroids of the Wigner distribution of $f'(\boldsymbol{x}')$ and of $f(\boldsymbol{x})$ respectively.

Then $I$ is an affine Wigner moment invariant. i.e. $I(\boldsymbol{\mu}^{f'}) = I(\boldsymbol{\mu}^f)$ .

Proof) Let $\boldsymbol{x}_c$ and $\boldsymbol{x'}_c$ be the centroids of the Wigner distributions of $f(\boldsymbol{x})$ and $f'(\boldsymbol{x'})$ respectively.

Consider the changes of coordination: $\boldsymbol{X} = \boldsymbol{x} - \boldsymbol{x}_c$, and $\boldsymbol{X'} = \boldsymbol{x'} - \boldsymbol{x'}_c$, and define two new images $f(\boldsymbol{X})$ and $f'(\boldsymbol{X'})$ as $f(\boldsymbol{X}) = f(\boldsymbol{x})$ and $f'(\boldsymbol{X}') = f'(\boldsymbol{x}')$.

If $\boldsymbol{x} = A\boldsymbol{x}' + \boldsymbol{b}$ when $f(\boldsymbol{x}) = f'(\boldsymbol{x})$, then $\boldsymbol{X} + \boldsymbol{x_c} = A\boldsymbol{X}' + A\boldsymbol{x'_c} + \boldsymbol{b}$, with $f(\boldsymbol{X}) = f'(\boldsymbol{X}')$. Here $\boldsymbol{x_c} = (\mu_{1000}, \mu_{0100})^{\mathrm{T}}/\mu_{0000}$ **and** $\boldsymbol{x'_c} = (\mu'_{1000}, \mu'_{0100})^{\mathrm{T}}/\mu'_{0000}$.

Now it can be shown that $\boldsymbol{x_c} = A\boldsymbol{x'_c} + \boldsymbol{b}$. Indeed, let $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Follow the proof of proposition 3.3, the following can be obtained:

$$\mu_{0000} = |det\ A|\ \mu'_{0000}$$

$$\mu_{1000} = |\det A|(\alpha\mu'_{1000} + \beta\mu'_{1000} + b_1\mu'_{0000})$$

$$\mu_{0100} = |\det A|(\gamma\mu'_{1000} + \delta\mu'_{1000} + b_2\mu'_{0000})$$

Thus $\boldsymbol{x_c} = \boldsymbol{x'_c} + \boldsymbol{b}$, so that $\boldsymbol{X} = A\boldsymbol{X}'$ and $f(\boldsymbol{X}) = f'(\boldsymbol{X}')$.

Now if $I(\boldsymbol{\mu}^f)$ is moment invariant for $f(\boldsymbol{X})$, with $\boldsymbol{\mu}^f$ being the moments of the Wigner distribution of $f(\boldsymbol{X})$, then $I(\boldsymbol{\mu}^f)$ is unchanged for any affine transformation $f(\boldsymbol{x}) = f'(\boldsymbol{x}')$, whenever $\boldsymbol{x} = A\boldsymbol{x}' + \boldsymbol{b}$. QED

In the case of deriving AMI, the above proposition 3.8 and its proof can be slightly modified to show that a moment invariant for a set of forms compatible to linear transformations is also an AMI.

## 3.5 An Alternative Derivation of Transformations of Forms

In section 3.3, proposition 3.6, it has been shown how quaternary forms $P_p(\boldsymbol{s}, \boldsymbol{u}; \mu)$ of order $p$ transforms to another quaternary form via the use of moment generating functions.

While using moment generating can show every quaternary or binary form transforms into another quaternary or binary form, it doesn't show how such a transformation works out. In this sub section, a more constructive derivation of how an appropriate binary or quaternary form actually transforms into another form is presented. This derivation would provide a better insight of how moments can be used to derive invariants based on the theory of algebraic forms.

The usual set up between variables as in section 3.3 is assumed. Specifically, $f(\boldsymbol{x})$ is a linear transform of $f'(\boldsymbol{x}')$, i.e. if $\boldsymbol{x} = A\boldsymbol{x}'$ then $f(\boldsymbol{x}) = f'(\boldsymbol{x}')$, where $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ is a non-singular $2 \times 2$ matrix.

Also while spatial variable $\boldsymbol{x}$ follows the transform $= A\boldsymbol{x}'$ **,** frequency variable $\boldsymbol{w}$ follows the transform $\boldsymbol{w} = M\boldsymbol{w}'$**,** where $M = (A^T)^{-1}$. In this case $\boldsymbol{x}$ and $\boldsymbol{w}$ are said to be contragredient. Here matrix $M$ will be denoted as $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$.

And let $\boldsymbol{s}$ and $\boldsymbol{u}$ be two 2x1 vectors, with $\boldsymbol{s}$ following the transform $\boldsymbol{s} = M\boldsymbol{s}'$, and $\boldsymbol{u}$ following the transform $\boldsymbol{u} = A\boldsymbol{u}'$. Thus $\boldsymbol{s}$ and $\boldsymbol{u}$ are also contragredient.

Let $\mu_{pq}$ and $\mu'_{pq}$ be the moments of $f(\boldsymbol{x})$ and $f'(\boldsymbol{x}')$, and let $\mu_{pqrs}$ and $\mu'_{pqrs}$ be the moments of the Wigner distributions $W(\boldsymbol{x}, \boldsymbol{w})$ of $f(\boldsymbol{x})$ and $W'(\boldsymbol{x}', \boldsymbol{w}')$ of $f'(\boldsymbol{x}')$.

In this sub section, to simplify derivations, all the functions and variables are assumed to be discrete.

The following proposition shows how $\mu_{pq}$ of order $p+q$ is related to all $\mu'_{pq}$ 's also of order $p+q$:

**Proposition 3.9**

$$\mu_{pq} = \sum_{\substack{k_1=0,..,p \\ k_2=0,..,q}} \binom{p}{k_1}\binom{q}{k_2} a_1^{p-k_1} a_2^{k_1} a_3^{q-k_2} a_4^{k_2} \mu'_{p+q-(k_1+k_2),k_1+k_2} \tag{3.20}$$

Proof) By definition:

$$\mu_{pq} = \sum_{x_1,x_2} x_1^p x_2^q f(\boldsymbol{x})$$

From $\boldsymbol{x} = A\boldsymbol{x}'$, substitute in $x_1 = a_1 x'_1 + a_2 x'_2$ and $x_2 = a_3 x'_1 + a_4 x'_2$ into the above and expand the expression using the binomial theorem to obtain:

$$\mu_{pq} = \sum_{\substack{k_1=0,..,p \\ k_2=0,..,q}} \binom{p}{k_1}\binom{q}{k_2} a_1^{p-k_1} a_2^{k_1} a_3^{q-k_2} a_4^{k_2} \sum_{x_1,x_2} x'^{(p+q-k_1-k_2)}_1 x'^{k_1+k_2}_2 f(\boldsymbol{x})$$

Using $f(\boldsymbol{x}) = f'(\boldsymbol{x}')$ for $\boldsymbol{x} = A\boldsymbol{x}'$ in the above, (3.20) is obtained. QED

From (3.20), $\mu_{10}$ and $\mu_{01}$ can be written together in the following matrix format:

$$\begin{pmatrix}\mu_{10}\\ \mu_{01}\end{pmatrix} = \begin{pmatrix}a_1 & a_2\\ a_3 & a_4\end{pmatrix}\begin{pmatrix}\mu'_{10}\\ \mu'_{01}\end{pmatrix} \tag{3.21}$$

Similarly, $\mu_{20}$, $\mu_{11}$ and $\mu_{02}$ can be written together in the following matrix format:

$$\begin{pmatrix}\mu_{20}\\ \mu_{11}\\ \mu_{02}\end{pmatrix} = \begin{pmatrix}a_1^2 & 2a_1a_2 & a_2^2\\ a_1a_3 & a_1a_4 + a_2a_3 & a_2a_4\\ a_3^2 & 2a_3a_4 & a_4^2\end{pmatrix}\begin{pmatrix}\mu'_{20}\\ \mu'_{11}\\ \mu'_{02}\end{pmatrix} \tag{3.22}$$

The following proposition show how a binary form of order 2 involving moments transforms into another binary form of order 2.

**Proposition 3.10**

The following binary form of order 2:

$$\boldsymbol{P_2}(\boldsymbol{s};\boldsymbol{\mu}) = \mu_{20}s_1^2 + 2\mu_{11}s_1s_2 + \mu_{02}s_2^2 \tag{3.23}$$

transforms into

$$\boldsymbol{P_2}(\boldsymbol{s}';\boldsymbol{\mu}') = \boldsymbol{\mu'_{20}s'^2_1} + \boldsymbol{2\mu'_{11}s'_1s'_2} + \boldsymbol{\mu'_{02}s'^2_2} \tag{3.24}$$

Proof) By direct substitution $\boldsymbol{s} = M\boldsymbol{s}'$ into $P_2(\boldsymbol{s};\boldsymbol{\mu})$ to obtain:

$$P_2(\boldsymbol{s}';\boldsymbol{\mu}) = a'_{20}s'^2_1 + 2a'_{11}s'_1s'_2 + a'_{02}s'^2_2$$

Where $$\begin{pmatrix} a'_{20} \\ a'_{11} \\ a'_{02} \end{pmatrix} = \begin{pmatrix} m_1^2 & 2m_1m_3 & m_3^2 \\ m_1m_2 & m_1m_4 + m_2m_3 & m_3m_4 \\ m_2^2 & 2m_2m_4 & m_4^2 \end{pmatrix} \begin{pmatrix} \mu_{20} \\ \mu_{11} \\ \mu_{02} \end{pmatrix}$$

Denote the above equation as $\boldsymbol{a}' = \boldsymbol{M}\boldsymbol{\mu}$, and denote (3.22) as $\boldsymbol{\mu} = \boldsymbol{A}\boldsymbol{\mu}'$. Then $\boldsymbol{a}' = \boldsymbol{M}\boldsymbol{A}\boldsymbol{\mu}'$.

Using $= (A^T)^{-1}$ , it can be verified that $\boldsymbol{M}\boldsymbol{A} = \boldsymbol{I}$, the identity matrix.

Thus $\boldsymbol{a}' = \boldsymbol{\mu}'$**, and** then $P_2(\boldsymbol{s}';\ \boldsymbol{\mu})$ is as in (3.24) is obtained. QED.

Similar derivations can be done to determine how binary forms of higher order transform to.

The following proposition parallel proposition 3.9 and shows how $\mu_{pqrs}$ of order $p+q+r+s$ is related to a number of $\mu'_{pqrs}$ 's, also of order $p+q+r+s$ :

**Proposition 3.11**

$$\mu_{pqrs} = \sum_{\substack{k_1=0,..,p \\ k_2=0,..,q \\ l_1=0,..,r \\ l_2=0,..,s}} \binom{p}{k_1}\binom{q}{k_2}\binom{r}{l_1}\binom{s}{l_2}\ a_1^{p-k_1}a_2^{k_1}a_3^{q-k_2}a_4^{k_2}m_1^{r-l_1}m_2^{l_1}m_3^{s-l_2}m_4^{l_2} \times \mu'_{p+q-(k_1+k_2),k_1+k_2,r+s-(l_1+l_2),l_1+l_2} \tag{3.25}$$

Proof) By definition:

$$\mu_{pqrs} = \sum_{x_1,x_2,w_1,w_2} x_1^p x_2^q w_1^r w_2^s\ W(\boldsymbol{x},\boldsymbol{w})$$

Define

$$\mu_{rs}(\boldsymbol{x}) = \sum_{w_1,w_2} w_1^r w_2^s \, W(\boldsymbol{x}, \boldsymbol{w})$$

Using $W(\boldsymbol{x}, \boldsymbol{w}) = W^{'}(\boldsymbol{x}^{'}, \boldsymbol{w}')$ for $\boldsymbol{x} = A\boldsymbol{x}'$ and $\boldsymbol{w} = M\boldsymbol{w}'$ for discrete variables, and using proposition (3.10) into the expression for $\mu_{rs}(\boldsymbol{x})$ to obtain:

$$\mu_{rs}(\boldsymbol{x}) = \sum_{\substack{l_1=0,..,r \\ l_2=0,..,s}} m_1^r m_2^{r-l_1} m_3^s m_4^{s-l_2} \mu^{'}_{r+s-(l_1+l_2),l_1+l_2}(\boldsymbol{x}^{'})$$

Here $\mu^{'}_{r+s-(l_1+l_2),l_1+l_2}(\boldsymbol{x}^{'})$ is defined in the same way as $\mu_{rs}(\boldsymbol{x})$ is defined. Thus

$$\mu_{pqrs} = \sum_{\substack{l_1=0,..,r \\ l_2=0,..,s}} m_1^r m_2^{r-l_1} m_3^s m_4^{s-l_2} \sum_{x_1,x_2} x_1^p x_2^q \, \mu^{'}_{r+s-(l_1+l_2),l_1+l_2}(\boldsymbol{x}^{'})$$

Substitute $x_1 = a_1 x'_1 + a_2 x'_2$ and $x_2 = a_3 x'_1 + a_4 x'_2$ into the above, and use the binomial theorem to expand $x_1^p x_2^q$, expression (3.25) can be obtained. QED

From (3.25), $\mu_{2000}$, $\mu_{1100}$ and $\mu_{0200}$ can be written together in the following matrix format:

$$\begin{pmatrix} \mu_{2000} \\ \mu_{1100} \\ \mu_{0200} \end{pmatrix} = \begin{pmatrix} a_1^2 & 2a_1a_2 & a_2^2 \\ a_1a_3 & a_1a_4 + a_2a_3 & a_2a_4 \\ a_3^2 & 2a_3a_4 & a_4^2 \end{pmatrix} \begin{pmatrix} \mu'_{2000} \\ \mu'_{1100} \\ \mu'_{0200} \end{pmatrix} \qquad (3.26)$$

From (3.25), $\mu_{0020}$, $\mu_{0011}$ and $\mu_{0002}$ can be written together in the following matrix format:

$$\begin{pmatrix}\mu_{0020}\\ \mu_{0011}\\ \mu_{0002}\end{pmatrix}=\begin{pmatrix}m_1^2 & 2m_1m_2 & m_2^2\\ m_1m_3 & m_1m_4+m_2m_3 & m_2m_4\\ m_3^2 & 2m_3m_4 & m_4^2\end{pmatrix}\begin{pmatrix}\mu'_{0020}\\ \mu'_{0011}\\ \mu'_{0002}\end{pmatrix} \tag{3.27}$$

From (3.25), $\mu_{1010}$, $\mu_{1001}$, $\mu_{0110}$, and $\mu_{0101}$ can be written together in the following matrix format:

$$\begin{pmatrix}\mu_{1010}\\ \mu_{1001}\\ \mu_{0110}\\ \mu_{0101}\end{pmatrix}=\begin{pmatrix}a_1m_1 & a_1m_2 & a_2m_1 & a_2m_2\\ a_1m_3 & a_1m_4 & a_2m_3 & a_2m_4\\ a_3m_1 & a_3m_2 & a_4m_1 & a_4m_2\\ a_3m_3 & a_3m_4 & a_4m_3 & a_4m_4\end{pmatrix}\begin{pmatrix}\mu'_{1010}\\ \mu'_{1001}\\ \mu'_{0110}\\ \mu'_{0101}\end{pmatrix} \tag{3.28}$$

The following proposition shows how some multivariate and multilinear forms transform to other multivariate and multilinear forms.

**Proposition 3.12**

i) The following binary form involving the moments of spatial variables:

$$f_0(\boldsymbol{s};\boldsymbol{\mu})=\mu_{2000}s_1^2+2\mu_{1100}s_1s_2+\mu_{0200}s_2^2 \tag{3.29}$$

transforms to another binary form also involving the moments of spatial variables:

$$f_0(\boldsymbol{s}';\boldsymbol{\mu}')=\mu'_{2000}s'^2_1+2\mu'_{1100}s'_1s'_2+\mu'_{0200}s'^2_2 \tag{3.30}$$

ii) The following binary form involving the moments of spatial and frequency variables:

$$f_1(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})=\mu_{1010}s_1u_1+\mu_{1001}s_1u_2+\mu_{0110}s_2u_1+\mu_{0101}s_2u_2 \tag{3.31}$$

transforms to another binary form also involving the moments of spatial and frequency variables:

$$f_1(\boldsymbol{s}', \boldsymbol{u}'; \boldsymbol{\mu}') = \mu'_{1010} s'_1 u'_1 + \mu'_{1001} s'_1 u'_2 + \mu'_{0110} s'_2 u'_1 + \mu'_{0101} s'_2 u'_2 \quad (3.32)$$

iii) The following binary form involving the moments of frequency variables:

$$f_2(\boldsymbol{u}; \boldsymbol{\mu}) = \mu_{0020} u_1^2 + 2\mu_{0011} u_1 u_2 + \mu_{0002} u_2^2 \quad (3.33)$$

transforms to another binary form also involving the moments of frequency variables:

$$f_2(\boldsymbol{u}'; \boldsymbol{\mu}') = \mu'_{0020} u'^2_1 + 2\mu'_{0011} u'_1 u'_2 + \mu'_{0002} u'^2_2 \quad (3.34)$$

vi) The following quaternary form of order 2:

$$P_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}) = f_0(\boldsymbol{s}; \boldsymbol{\mu}) + 2f_1(\boldsymbol{s}', \boldsymbol{u}'; \boldsymbol{\mu}') + f_2(\boldsymbol{u}; \boldsymbol{\mu}) \quad (3.35)$$

transforms to another quaternary form of order 2:

$$P_2(\boldsymbol{s}', \boldsymbol{u}'; \boldsymbol{\mu}') = f_0(\boldsymbol{s}'; \boldsymbol{\mu}') + 2f_1(\boldsymbol{s}', \boldsymbol{u}'; \boldsymbol{\mu}') + f_2(\boldsymbol{u}'; \boldsymbol{\mu}') \quad (3.36)$$

Proof) For i), similar to proposition (3.10), $f_0(\boldsymbol{s}; \boldsymbol{\mu})$ as in (3.29) transforms to (3.30).

For ii), consider $f_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ as defined (3.31). Substitute $\boldsymbol{s} = M\boldsymbol{s}'$, $\boldsymbol{u} = A\boldsymbol{u}'$ to $f_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ to transform it to:

$$f_1(\boldsymbol{s}', \boldsymbol{u}'; \boldsymbol{\mu}') = c_{11} s'_1 u'_1 + c_{12} s'_1 u'_2 + c_{21} s'_2 u'_1 + c_{22} s'_2 u'_2$$

Where

$$\begin{pmatrix} c_{11} \\ c_{12} \\ c_{21} \\ c_{22} \end{pmatrix} = \begin{pmatrix} a_3m_1 & a_1m_1 & a_3m_3 & a_1m_3 \\ a_4m_1 & a_2m_1 & a_4m_3 & a_2m_3 \\ a_3m_2 & a_1m_2 & a_3m_4 & a_1m_4 \\ a_4m_2 & a_2m_2 & a_4m_4 & a_2m_4 \end{pmatrix} \begin{pmatrix} \mu_{1001} \\ \mu_{1010} \\ \mu_{0101} \\ \mu_{0110} \end{pmatrix}$$

Denote the above equation as $\boldsymbol{c} = \boldsymbol{M\mu}$, and denote (3.28) as $\boldsymbol{\mu} = \boldsymbol{A\mu'}$. Then $\boldsymbol{c} = \boldsymbol{MA\mu'}$**.** Using $= (A^T)^{-1}$ , it can be verified that $\boldsymbol{MA} = \boldsymbol{I}$, the identity matrix.

Thus $\boldsymbol{c} = \boldsymbol{\mu'}$**, and** then Thus $f_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ in (3.31) transforms to $f_1(\boldsymbol{s'}, \boldsymbol{u'}; \boldsymbol{\mu'})$ in (3.32). QED.

For iii), proposition (3.10) can also be applied to $f_2(\boldsymbol{u}; \boldsymbol{\mu})$ as defined in (3.33) with $\boldsymbol{u} = A\boldsymbol{u'}$, and $\boldsymbol{w} = M\boldsymbol{w'}$**.** It can be verified that $f_2(\boldsymbol{u}; \boldsymbol{\mu})$ transforms to (3.34).

For iv), consider $f(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ defined as the sum of $f_0$, $2f_1$, and $f_2$ as in (3.35) . Using $\boldsymbol{s} = A\boldsymbol{s'}$, $\boldsymbol{u} = A\boldsymbol{u'}$, (3.26)-(3.28), and from the results of i), ii), and iii), it is clear that $f(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ transforms to (3.36). QED.

In the above proposition, quaternary form $P_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ as in (3.35) is the form of order 2 in the decomposition of moment generating function in (3.15). In proposition (3.6), in the case of discrete variables, each form $P_p(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ is shown to transform to $P_p(\boldsymbol{s'}, \boldsymbol{u'}; \boldsymbol{\mu'})$ using the moment generating function.

In this sub-section, $P_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ is shown to transform to $P_2(\boldsymbol{s'}, \boldsymbol{u'}; \boldsymbol{\mu'})$ by the transformations of its sub-forms $f_0(\boldsymbol{s}; \boldsymbol{\mu})$, $f_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$, $f_2(\boldsymbol{u}; \boldsymbol{\mu})$. And the transformations of the sub-forms are obtained from the actual substitutions of variables $\boldsymbol{s} = A\boldsymbol{s'}$, $\boldsymbol{u} = A\boldsymbol{u'}$, and substitutions of moments (3.26)-(3.28).

Similar derivations can be derived for higher order quaternary forms.

The decomposition of quaternary forms into sub-forms as described in this sub-section will be used in the next chapter to derive invariants for the quaternary forms.

# Chapter 4 – A Method to Derive Affine Wigner Moment Invariants

## 4.1 Introduction

The goal of this chapter is to determine a number of AWMI based on the results developed in the last chapter. Specifically, a number of AWMI will be determined from the algebraic invariants of the quaternary forms $P_p(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$'s, obtained from the decomposition (3.15) of the underlying moment generating function.

In the literature, often a few moment invariants are used to represent an image ([4][1]), thus in this chapter only 13 AWMI are derived for two quaternary forms $P_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ and $P_3(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ of order 2 and 3 respectively.

Now, if an algebraic invariant for $P_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ and $P_3(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ could be found, then from propositions 3.7 and 3.8 in the last chapter, an AWMI can be determined. But the problem now is how to find the algebraic invariants for $P_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ and $P_3(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$, where $\boldsymbol{s}$ and $\boldsymbol{u}$ are contragredient.

Textbooks of theory of algebraic forms often give explicit formulas for algebraic invariants for binary quadratic and binary cubic forms. These algebraic invariants are then used in the literature to determine moment invariants ([4][28]). While these textbooks don't give explicit formulas for quaternary forms, they do give the general formats to determine algebraic invariants for any multivariate form of any order and for any multilinear form having any number of sets of variables ([29][30]).

The problem now faced is to find algebraic invariants of $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ and $P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ with $\boldsymbol{s}$ and $\boldsymbol{u}$ being contragredient. Here $P_p(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$'s in general are quaternary forms with $\boldsymbol{s}$ and $\boldsymbol{u}$ considered together as vector $(\boldsymbol{s},\boldsymbol{u})^T$ of four variables. Note that $P_p(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})'s$ could be considered as multilinear forms as well, but the information about the contragredience between $\boldsymbol{s}$ and $\boldsymbol{u}$ are lost.

A brute-force approach is to use the general formats for algebraic invariants for general multivariate or multilinear forms to derive algebraic invariants for the quaternary forms $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ and $P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$. This approach theoretically is possible but the final expressions for the desired algebraic invariants could become too overly complicated. For a derivation of invariants of multivariate quaternary forms expressed in canonical forms, please refer to ([73][66][74][75]). And looking ahead to generalization to 3-dimensional images, finding algebraic invariants instead of quaternary but 6-ary forms with this brute-force approach would become too cumbersome.

In this chapter a different approach is proposed to find algebraic invariants of quaternary forms $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ and $P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$, taking into account the fact that $\boldsymbol{s}$ and $\boldsymbol{u}$ are contragredient. This approach is simpler than the brute-force approach because it tries to find algebraic invariants of only parts of $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ and $P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$. And also, it could potentially be used for generalizing from two-dimensional to three-dimensional images.

The idea of the approach proposed in this chapter is to decompose $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ and $P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ into a sum of multivariate and multilinear forms that are simpler. Then using the fundamental theorems in the theory of algebraic forms, algebraic invariants for these simpler forms can be found. These algebraic invariants will be used to determine AWMI from proposition 3.7 and 3.8.

Section 4.2 of this chapter shows how to decompose $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ and $P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ into simpler forms. Section 4.3 shows how to use proposition 3.7 and 3.8 to obtain AWMI from the algebraic invariants for the simpler forms. Section 4.4 shows how to obtain 7 AWMI for $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ and $P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ separately. Section 4.5 shows how to obtain 6 AWMI for $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ and $P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ simultaneously. Altogether there 9 AWMI involving the frequency variables, and 4 involving only the spatial variables.

## 4.2 Decomposition of Quaternary Quadratic Forms Having Contragredient Variables

In this section, the quaternary form of order two $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ and quaternary form of order three $P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ in (3.15) will be decomposed into sums of multivariate and multilinear *sub-forms*. Here a sub-form is a form in itself but still part of $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ or $P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$. These decompositions will be used later in this chapter to determine moment invariants of $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ and $P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ by the moment invariants of the sub-forms.

To facilitate the handling of notations, denote $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ the quaternary form of order 2 as in (3.15) by $f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$.

Note that $\boldsymbol{s},\boldsymbol{u}$ are contragredient, with $\boldsymbol{s} = M\boldsymbol{s}'$ and $\boldsymbol{u} = (M^T)^{-1}\boldsymbol{u}'$, and where $M = (A^T)^{-1}$.

Let $\mathbb{M} = \begin{bmatrix} M & 0 \\ 0 & (M^T)^{-1} \end{bmatrix}$, then the two transformations $\boldsymbol{s} = M\boldsymbol{s}'$ and $\boldsymbol{u} = (M^T)^{-1}\boldsymbol{u}'$ can be written as $\begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{u} \end{pmatrix} = \mathbb{M}\begin{pmatrix} \boldsymbol{s}' \\ \boldsymbol{u}' \end{pmatrix}$. Denote by $J = \det M$, and $|J| = |\det M|$

In general, quaternary form $f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ of order 2 has $\binom{4+2-1}{2}$= 10 terms and can be written as:

$$\begin{aligned} f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}) = {} & \mu_{2000}s_1^2 + 2\mu_{1100}s_1s_2 + \mu_{0200}s_2^2 + \\ & 2(\mu_{1010}s_1u_1 + \mu_{1001}s_1u_2 + \mu_{0110}s_2u_1 + \mu_{0101}s_2u_2) + \\ & \mu_{0020}u_1^2 + 2\mu_{0011}u_1u_2 + \mu_{0002}u_2^2 \end{aligned} \tag{4.1}$$

To express $f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ using symbols, let $(a_1,a_2,\alpha_1,\alpha_2)^T$ be a vector of symbols for $f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$. Denote by

$$a_s = a_1s_1 + a_2s_2 \qquad \text{and} \qquad u_\alpha = \alpha_1u_1 + \alpha_2u_2 \tag{4.2}$$

Then

$$f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}) = (a_s + u_\alpha)^2 \tag{4.3}$$

$$= a_s^2 + 2a_s u_\alpha + u_\alpha^2$$

$$= (a_1 s_1 + a_2 s_2)^2 + 2(a_1 s_1 + a_2 s_2)(\alpha_1 u_1 + \alpha_2 u_2) + (\alpha_1 u_1 + \alpha_2 u_2)^2$$

$$= a_1^2 s_1^2 + 2a_1 a_2 s_1 s_2 + a_2^2 s_2^2 +$$

$$2(a_1\alpha_1 s_1 u_1 + a_1\alpha_2 s_1 u_2 + a_2\alpha_1 s_2 u_1 + a_2\alpha_2 s_2 u_2) +$$

$$\alpha_1^2 u_1^2 + 2\alpha_1\alpha_2 u_1 u_2 + \alpha_2^2 u_2^2$$

Where the relationships between the symbols and the actual coefficients are:

$$a_1^2 = \mu_{2000}, \quad a_1 a_2 = \mu_{1100}, \quad a_2^2 = \mu_{0200} \tag{4.4}$$

$$a_1\alpha_1 = \mu_{1010}, \quad a_1\alpha_2 = \mu_{1001}, \quad a_2\alpha_1 = \mu_{0110}, \quad a_2\alpha_2 = \mu_{0101}$$

$$\alpha_1^2 = \mu_{0020}, \quad \alpha_1\alpha_2 = \mu_{0011}, \quad \alpha_2^2 = \mu_{0002}$$

The above discussion has shown that:

**Proposition 4.1**

Form $f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ can be decomposed into:

$$f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}) = f_0(\boldsymbol{s};\boldsymbol{\mu}_0) + 2f_1(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_1) + f_2(\boldsymbol{u};\boldsymbol{\mu}_2) \tag{4.5}$$

Where $f_0(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_0)$ is a binary quadratic form, $f_1(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_1)$ is a binary bilinear form, and $f_2(\boldsymbol{u};\boldsymbol{\mu}_2)$ another binary quadratic form:

$$f_0(\boldsymbol{s};\boldsymbol{\mu}_0) = a_s^2 = (a_1 s_1 + a_2 s_2)^2 \tag{4.6}$$
$$= a_1^2 s_1^2 + 2a_1 a_2 s_1 s_2 + a_2^2 s_2^2$$
$$= \mu_{2000} s_1^2 + 2\mu_{1100} s_1 s_2 + \mu_{0200} s_2^2$$

$$f_1(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_1) = a_s u_\alpha = (a_1 s_1 + a_2 s_2)(\alpha_1 u_1 + \alpha_2 u_2) \tag{4.7}$$
$$= a_1\alpha_1 s_1 u_1 + a_1\alpha_2 s_1 u_2 + a_2\alpha_1 s_2 u_1 + a_2\alpha_2 s_2 u_2$$
$$= \mu_{1010} s_1 u_1 + \mu_{1001} s_1 u_2 + \mu_{0110} s_2 u_1 + \mu_{0101} s_2 u_2$$

$$f_2(\boldsymbol{u};\boldsymbol{\mu}_2) = u_\alpha^2 = (\alpha_1 u_1 + \alpha_2 u_2)^2 \tag{4.8}$$
$$= \alpha_1^2 u_1^2 + 2\alpha_1\alpha_2 u_1 u_2 + \alpha_2^2 u_2^2$$
$$= \mu_{0020} u_1^2 + 2\mu_{0011} u_1 u_2 + \mu_{0002} u_2^2$$

Here $\boldsymbol{\mu}_0 = (\mu_{2000}, \mu_{1100}, \mu_{0200})^T$, $\boldsymbol{\mu}_1 = (\mu_{1010}, \mu_{1001}, \mu_{0110}, \mu_{0101})^T$,

$\boldsymbol{\mu}_2 = (\mu_{0020}, \mu_{0011}, \mu_{0002})^T$, and $\boldsymbol{\mu} = (\boldsymbol{\mu}_0, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2)^T$

It is clear that algebraic invariants of $f_0(\boldsymbol{s};\boldsymbol{\mu}_0)$, $f_1(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_1)$, and $f_2(\boldsymbol{u};\boldsymbol{\mu}_2)$ are also algebraic invariants of $f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$.

Now, given two images $f(\boldsymbol{x})$ and $f'(\boldsymbol{x}')$ such that $f(x) = f'(\boldsymbol{x}')$ if $\boldsymbol{x} = A\boldsymbol{x}'$. The moment generating functions of $f(\boldsymbol{x})$ and $f'(\boldsymbol{x}')$ can be expressed as the decompositions (3.15) and (3.16). Quaternary quadratic forms $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ and $P_2(\boldsymbol{s}',\boldsymbol{u}';\boldsymbol{\mu}')$ are denoted as $f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ and $f(\boldsymbol{s}',\boldsymbol{u}';\boldsymbol{\mu}')$ respectively in this chapter.

Note that $f(\boldsymbol{x})$ and $f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ are two different functions. While they both have the same name $f$, they are applied to different vectors of variables.

The following proposition shows how the components of the decomposition of $f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ as in (4.5) transform.

**Proposition 4.2**

Under the transformations $\boldsymbol{s} = M\boldsymbol{s}'$ and $\boldsymbol{u} = A\boldsymbol{u}'$ **,** where $M = (A^T)^{-1}$, or equivalently $A = (M^T)^{-1}$. Denote $J = \det M$.

i) $f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ is transformed into

$$f\left(\boldsymbol{s}',\boldsymbol{u}';\frac{1}{|J|}\boldsymbol{\mu}'\right) = f_0\left(\boldsymbol{s}';\frac{1}{|J|}\boldsymbol{\mu}'_0\right) + 2f_1\left(\boldsymbol{s}',\boldsymbol{u}';\frac{1}{|J|}\boldsymbol{\mu}'_1\right) + f_2\left(\boldsymbol{u}';\frac{1}{|J|}\boldsymbol{\mu}'_2\right) \tag{4.9}$$

ii) $f_0(\boldsymbol{s};\boldsymbol{\mu}_0)$, $f_1(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_1)$, and $f_2(\boldsymbol{u};\boldsymbol{\mu}_2)$ transform into $f_0\left(\boldsymbol{s}';\frac{1}{|J|}\boldsymbol{\mu}'_0\right)$, $f_1\left(\boldsymbol{s}',\boldsymbol{u}';\frac{1}{|J|}\boldsymbol{\mu}'_1\right)$, and $f_2\left(\boldsymbol{s}';\frac{1}{|J|}\boldsymbol{\mu}'_2\right)$ respectively.

Proof)

For i), By proposition 3.6, $f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ transforms into $f\left(\boldsymbol{s}',\boldsymbol{u}';\frac{1}{|J|}\boldsymbol{\mu}'\right)$. But by applying decomposition (4.5) to $f\left(\boldsymbol{s}',\boldsymbol{u}';\frac{1}{|J|}\boldsymbol{\mu}'\right)$, then $f\left(\boldsymbol{s}',\boldsymbol{u}';\frac{1}{|J|}\boldsymbol{\mu}'\right)$ can be expressed as the sum in (4.9).

For ii), Let $(a_1, a_2, \alpha_1, \alpha_2)^T$ be a vector of symbols for $f(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$, with $\boldsymbol{a} = (a_1, a_2)^T$ and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)^T$. From (4.2) and (4.3) with $a_s = \boldsymbol{a}^T \boldsymbol{s}$ and $u_\alpha = \boldsymbol{\alpha}^T \boldsymbol{u}$, $f(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ can be expressed as:

$$f(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}) = (\boldsymbol{a}^T \boldsymbol{s})^2 + 2(\boldsymbol{a}^T \boldsymbol{s})(\boldsymbol{\alpha}^T \boldsymbol{u}) + (\boldsymbol{\alpha}^T \boldsymbol{u})^2$$

Substitute $\boldsymbol{s} = M\boldsymbol{s}'$ and $\boldsymbol{u} = A\boldsymbol{u}'$ into $f(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$, then $f(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ transforms to:

$$f\left(\boldsymbol{s}', \boldsymbol{u}'; \frac{1}{|J|}\boldsymbol{\mu}'\right) = (\boldsymbol{a}'^T \boldsymbol{s}')^2 + 2(\boldsymbol{a}'^T \boldsymbol{s}')(\boldsymbol{\alpha}'^T \boldsymbol{u}') + (\boldsymbol{\alpha}'^T \boldsymbol{u}')^2$$

Where $\boldsymbol{a}' = M^T \boldsymbol{a}$ and $\boldsymbol{\alpha}' = A^T \boldsymbol{\alpha}$**.** Also $(a'_1, a'_2, \alpha'_1, \alpha'_2)^T$ becomes a vector of symbols for $f\left(\boldsymbol{s}', \boldsymbol{u}'; \frac{1}{|J|}\boldsymbol{\mu}'\right)$, with $\boldsymbol{a}' = (a'_1, a'_2)^T$ and $\boldsymbol{\alpha}' = (\alpha'_1, \alpha'_2)^T$.

Matching the symbolic terms involving variable $\boldsymbol{s}'$ only in the right hand side of the last expression with the terms involving variable $\boldsymbol{s}'$ only in the right hand side of (4.9) gives

$$f_0\left(\boldsymbol{s}'; \frac{\boldsymbol{1}}{|J|}\boldsymbol{\mu}'_0\right) = (\boldsymbol{a}'^T \boldsymbol{s}')^2$$

But substituting $\boldsymbol{s} = M\boldsymbol{s}'$ into $f_0(\boldsymbol{s}; \boldsymbol{\mu}_0) = (\boldsymbol{a}^T \boldsymbol{s})^2$ transforms $f_0(\boldsymbol{s}; \boldsymbol{\mu}_0)$ to $(\boldsymbol{a}'^T \boldsymbol{s}')^2$, which is $f_0\left(\boldsymbol{s}'; \frac{\boldsymbol{1}}{|J|}\boldsymbol{\mu}'_0\right)$ from the last expression. Thus $f_0(\boldsymbol{s}; \boldsymbol{\mu}_0)$ transforms to $f_0\left(\boldsymbol{s}'; \frac{\boldsymbol{1}}{|J|}\boldsymbol{\mu}'_0\right)$.

Similar arguments can be applied to show $f_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$, and $f_2(\boldsymbol{u}; \boldsymbol{\mu}_2)$ transform into $f_1\left(\boldsymbol{s}', \boldsymbol{u}'; \frac{1}{|J|}\boldsymbol{\mu}'_1\right)$, and $f_2\left(\boldsymbol{s}'; \frac{1}{|J|}\boldsymbol{\mu}'_2\right)$ respectively . QED

To facilitate the handling of notations, denote $P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ the quaternary form of order 3 in (3.15) by $g(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$.

In general, quaternary form $g(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ of order 3 has $\binom{4+3-1}{3}$= 20 terms and can be written as:

$$\begin{aligned} g(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}) = {} & \mu_{3000}s_1^3 + 3\mu_{2100}s_1^2s_2 + 3\mu_{1200}s_1s_2^2 + \mu_{0300}s_2^3 + \\ & \mu_{2010}s_1^2u_1 + \mu_{2001}s_1^2u_2 + \mu_{0210}s_2^2u_1 + \mu_{0201}s_2^2u_2 + \\ & 2\mu_{1110}s_1s_2u_1 + 2\mu_{1101}s_1s_2u_2 + \\ & \mu_{1020}s_1u_1^2 + \mu_{1002}s_1u_2^2 + \mu_{0120}s_2u_1^2 + \mu_{0102}s_2u_2^2 + \\ & 2\mu_{1011}s_1u_1u_2 + 2\mu_{0111}s_2u_1u_2 + \\ & \mu_{0030}u_1^3 + 3\mu_{0021}u_1^2u_2 + 3\mu_{0012}u_1u_2^2 + \mu_{0003}u_2^3 \end{aligned} \tag{4.10}$$

To express $g(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ using symbols, let $(a_1,a_2,\alpha_1,\alpha_2)^T$ be a vector of symbols for $g(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$. Denote by

$$a_s = a_1s_1 + a_2s_2 \qquad \text{and} \qquad u_\alpha = \alpha_1u_1 + \alpha_2u_2 \tag{4.11}$$

$$
\begin{aligned}
g(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}) &= (a_s + u_\alpha)^3 \\
&= a_s^3 + 3a_s^2 u_\alpha + 3a_s u_\alpha^2 + u_\alpha^3 \\
&= a_1^3 s_1^3 + 3a_1^2 a_2 s_1^2 s_2 + 3a_1 a_2^2 s_1 s_2^2 + a_2^3 s_2^3 + \\
&\quad 3(a_1^2 \alpha_1 s_1^2 u_1 + a_2^2 \alpha_1 s_2^2 u_1 + a_1^2 \alpha_2 s_1^2 u_2 + a_2^2 \alpha_2 s_2^2 u_2 \\
&\quad + 2a_1 a_2 \alpha_1 s_1 s_2 u_1 + 2a_1 a_2 \alpha_2 s_1 s_2 u_2) + \\
&\quad 3(a_1 \alpha_1^2 s_1 u_1^2 + a_2 \alpha_1^2 s_2 u_1^2 + a_1 \alpha_2^2 s_1 u_2^2 + a_2 \alpha_2^2 s_2 u_2^2 \\
&\quad + 2a_1 \alpha_1 \alpha_2 s_1 u_1 u_2 + 2a_2 \alpha_1 \alpha_2 s_2 u_1 u_2) + \\
&\quad \alpha_1^3 u_1^3 + 3\alpha_1^2 \alpha_2 u_1^2 u_2 + 3\alpha_1 \alpha_2^2 u_1 u_2^2 + \alpha_2^3 u_2^3
\end{aligned}
\tag{4.12}
$$

Where

$$
\begin{aligned}
&a_1^3 = \mu_{3000}\,, \quad a_1^2 a_2 = \mu_{2100}\,, \quad a_1 a_2^2 = \mu_{1200}\,, \quad a_2^3 = \mu_{0300} \\
&a_1^2 \alpha_1 = \mu_{2010}\,, \quad a_1^2 \alpha_2 = \mu_{2001}\,, \quad a_2^2 \alpha_1 = \mu_{0210}\,, \quad a_2^2 \alpha_2 = \mu_{0201}, \\
&a_1 a_2 \alpha_1 = \mu_{1110}\,, \quad a_1 a_2 \alpha_2 = \mu_{1101} \\
&a_1 \alpha_1^2 = \mu_{1020}\,, \quad a_1 \alpha_2^2 = \mu_{1002}, \quad a_2 \alpha_1^2 = \mu_{0120}, \quad a_2 \alpha_2^2 = \mu_{0102}, \\
&a_1 \alpha_1 \alpha_2 = \mu_{1011}\,, \quad a_2 \alpha_1 \alpha_2 = \mu_{0111} \\
&\alpha_1^3 = \mu_{0030}\,, \quad \alpha_1^2 \alpha_2 = \mu_{0021}, \quad \alpha_1 \alpha_2^2 = \mu_{0012}, \quad \alpha_2^3 = \mu_{0003}
\end{aligned}
\tag{4.13}
$$

The above discussion has shown the following:

**Proposition 4.3**

Form $g(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ can be decomposed into

$$
g(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}) = g_0(\boldsymbol{s};\boldsymbol{\mu}_0) + 3g_1(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_1) + 3g_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_2) + g_3(\boldsymbol{u};\boldsymbol{\mu}_3) \tag{4.14}
$$

Where $g_0(\boldsymbol{s};\boldsymbol{\mu}_0)$ is a binary cubic form, $g_1(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_1)$ and $g_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_1)$ are binary trilinear forms, and $g_3(\boldsymbol{u};\boldsymbol{\mu}_2)$ is a binary cubic form:

$$\begin{aligned} g_0(\boldsymbol{s};\boldsymbol{\mu}_0) = a_s^3 &= (a_1 s_1 + a_2 s_2)^3 \\ &= a_1^3 s_1^3 + 3a_1^2 a_2 s_1^2 s_2 + 3a_1 a_2^2 s_1 s_2^2 + a_2^3 s_2^3 \\ &= \mu_{3000} s_1^3 + 3\mu_{2100} s_1^2 s_2 + 3\mu_{1200} s_1 s_2^2 + \mu_{0300} s_2^3 \end{aligned} \tag{4.15}$$

$$\begin{aligned} g_1(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_1) = a_s^2 u_\alpha &= (a_1 s_1 + a_2 s_2)^2 (\alpha_1 u_1 + \alpha_2 u_2) \\ &= a_1^2 \alpha_1 s_1^2 u_1 + a_1^2 \alpha_2 s_1^2 u_2 + a_2^2 \alpha_1 s_2^2 u_1 + a_2^2 \alpha_2 s_2^2 u_2 + \\ &\quad 2a_1 a_2 \alpha_1 s_1 s_2 u_1 + 2a_1 a_2 \alpha_2 s_1 s_2 u_2 \\ &= \mu_{2010} s_1^2 u_1 + \mu_{2001} s_1^2 u_2 + \mu_{0210} s_2^2 u_1 + \mu_{0201} s_2^2 u_2 + \\ &\quad 2\mu_{1110} s_1 s_2 u_1 + 2\mu_{1101} s_1 s_2 u_2 \end{aligned} \tag{4.16}$$

$$\begin{aligned} g_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_2) = a_s u_\alpha^2 &= (a_1 s_1 + a_2 s_2)(\alpha_1 u_1 + \alpha_2 u_2)^2 \\ &= a_1 \alpha_1^2 s_1 u_1^2 + a_1 \alpha_2^2 s_1 u_2^2 + a_2 \alpha_1^2 s_2 u_1^2 + a_2 \alpha_2^2 s_2 u_2^2 + \\ &\quad 2a_1 \alpha_1 \alpha_2 s_1 u_1 u_2 + 2a_2 \alpha_1 \alpha_2 s_2 u_1 u_2 \\ &= \mu_{1020} s_1 u_1^2 + \mu_{1002} s_1 u_2^2 + \mu_{0120} s_2 u_1^2 + \mu_{0102} s_2 u_2^2 + \\ &\quad 2\mu_{1011} s_1 u_1 u_2 + 2\mu_{0111} s_2 u_1 u_2 \end{aligned} \tag{4.17}$$

$$g_3(\boldsymbol{u};\boldsymbol{\mu}_3) = u_\alpha^3 = (\alpha_1 u_1 + \alpha_2 u_2)^3 \tag{4.18}$$

$$= \alpha_1^3 u_1^3 + 3\alpha_1^2\alpha_2 u_1^2 u_2 + 3\alpha_1\alpha_2^2 u_1 u_2^2 + \alpha_2^3 u_2^3$$

$$= \mu_{0030} u_1^3 + 3\mu_{0021} u_1^2 u_2 + 3\mu_{0012} u_1 u_2^2 + \mu_{0003} u_2^3$$

Now, given two images $f(\boldsymbol{x})$ and $f'(\boldsymbol{x}')$ such that $f(x) = f'(\boldsymbol{x}')$ if $\boldsymbol{x} = A\boldsymbol{x}'$. The moment generating functions of $f(\boldsymbol{x})$ and $f'(\boldsymbol{x}')$ can be expressed as the decompositions (3.15) and (3.16). Quaternary cubic forms $P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ and $P_3(\boldsymbol{s}',\boldsymbol{u}';\boldsymbol{\mu}')$ are denoted as $g(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ and $g(\boldsymbol{s}',\boldsymbol{u}';\boldsymbol{\mu}')$ respectively in this chapter.

The following proposition shows how the components of the decomposition of $g(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ as in (4.14) transform.

**Proposition 4.4**

Under the transformations $\boldsymbol{s} = M\boldsymbol{s}'$ and $\boldsymbol{u} = A\boldsymbol{u}'$. Here $M = (A^T)^{-1}$, equivalently $A = (M^T)^{-1}$. Denote $J = \det M$.

i) $g(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ transforms into

$$g\left(\boldsymbol{s}',\boldsymbol{u}';\frac{1}{|J|}\boldsymbol{\mu}'\right) = g_0\left(\boldsymbol{s}';\frac{1}{|J|}\boldsymbol{\mu}'_0\right) + 3g_1\left(\boldsymbol{s}',\boldsymbol{u}';\frac{1}{|J|}\boldsymbol{\mu}'_1\right) + \tag{4.19}$$

$$3\,g_2\left(\boldsymbol{s}',\boldsymbol{u}';\frac{1}{|J|}\boldsymbol{\mu}'_2\right) + g_3\left(\boldsymbol{u}';\frac{1}{|J|}\boldsymbol{\mu}'_3\right)$$

ii) $g_0(\boldsymbol{s};\boldsymbol{\mu}_0)$, $g_1(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_1)$, $g_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_2)$, and $g_3(\boldsymbol{u};\boldsymbol{\mu}_3)$ transform into

$g_0\left(\boldsymbol{s}';\frac{1}{|J|}\boldsymbol{\mu}'_0\right)$, $g_1\left(\boldsymbol{s}',\boldsymbol{u}';\frac{1}{|J|}\boldsymbol{\mu}'_1\right)$, $g_2\left(\boldsymbol{s}',\boldsymbol{u}';\frac{1}{|J|}\boldsymbol{\mu}'_2\right)$, $g_3\left(\boldsymbol{u}';\frac{1}{|J|}\boldsymbol{\mu}'_3\right)$

respectively.

A proof of this proposition 4.4 can obtained in a similar fashion as that of proposition 4.2

## 4.3 Determination of Affine Wigner Moment Invariants from Simpler Forms

As derived in the last chapter, for any images $f(\boldsymbol{x})$ and $f'(\boldsymbol{x}')$, from (3.15) the moment generating functions of $f(\boldsymbol{x})$ and $f'(\boldsymbol{x}')$ can be decomposed into a sum of quaternary forms $P_p(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$'s, and a sum of quaternary forms $P_p(\boldsymbol{s}',\boldsymbol{u}';\boldsymbol{\mu}')$'s respectively. And if $f(\boldsymbol{x})$ is a linear transformation of $f'(\boldsymbol{x}')$, i.e. $f(\boldsymbol{x}) = f'(\boldsymbol{x}')$ whenever $\boldsymbol{x} = A\boldsymbol{x}'$, then from (3.17), $P_p(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}) = P_p\left(\boldsymbol{s}',\boldsymbol{u}';\frac{1}{|J|}\boldsymbol{\mu}'\right)$, where $\boldsymbol{s}$ is contragredient with $\boldsymbol{x}$ and $\boldsymbol{u}$ is cogredient with $\boldsymbol{x}$, and $J = det\, A^{-1}$.

The goal of this chapter is to determine some affine Wigner moment invariant (AWMI) from $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ and $P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$, where $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ and $P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ are denoted by $f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ and $g(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ in this chapter respectively.

Note that any $P_p(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$, which is a quaternary form of order $p$, can be considered as a $p$-linear form, where the $p$ sets of variables are identically equal to $(\boldsymbol{s},\boldsymbol{u})^T$.

The approach as explained in section 3.4.2 can be used to determine AWMI. Let $\Gamma$ be a non empty subset of $\{P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}), P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})\}$, then from (3.17) $\Gamma$ is compatible with linear transformations. And according to proposition 3.7 and 3.8, any algebraic invariant for $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}), P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ can determine an AWMI.

But $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}), P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ are quaternary forms, applying the fundamental theorem for invariants of multivariate or multilinear forms as reviewed in chapter 2 would produce overly complicated invariants, and thus overly complicated AWMI.

This chapter proposes a method to produce simpler algebraic invariants for $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}), P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$, by determining the algebraic invariants for any subset of $\{f_0, f_1, f_2, g_0, g_1, g_2, g_3\}$. Here $f_0, f_1, f_2$ are obtained from decomposition (4.5) for $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$, and $g_0, g_1, g_2, g_3$ are obtained from decomposition (4.14) for$P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$. Simpler algebraic invariants for $P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}), P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ will produce simpler AWMI.

In this chapter, instead of letting $\Gamma$ be any non empty subset of $\{P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}), P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})\}$, let $\Gamma$ be any non empty subset of $\{f_0, f_1, f_2, g_0, g_1, g_2, g_3\}$.

Note that while all $f_0, f_1, f_2$, and $g_0, g_1, g_2, g_3$can be considered as multilinear forms, multilinear forms $f_1, g_1, g_2$ are not multivariate forms.

From propositions 4.2 and 4.4, $\Gamma$ is compatible with linear transformations. Thus if an algebraic invariant of $\Gamma$ can be found, then from proposition 3.7 and 3.8, an AWMI can be determined.

Generally, an algebraic invariant for $\Gamma$ that is a subset of $\{f_0, f_1, f_2, g_0, g_1, g_2, g_3\}$ is simpler than that for $\{P_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}), P_3(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})\}$. It is because all $f_0, f_1, f_2, g_0, g_1, g_2, g_3$ are simpler than $P_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}), P_3(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$. Thus an AWMI determined from $\Gamma$ generally simpler than that from $P_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}), P_3(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$.

To recap, in this chapter AWMI will be determine as follows.

Take any non empty subset $\Gamma$ of $\{f_0, f_1, f_2, g_0, g_1, g_2, g_3\}$. Use the theorem for invariants of multivariate or multilinear forms or other techniques to find an algebraic invariant $K$ for $\Gamma$. Suppose the weight and order of $K$ are $\omega$ and $k$, then by proposition 3.7 and 3.8 the following function $I$ defined as:

$$I = \begin{cases} \dfrac{1}{\mu^{\boldsymbol{\omega+k}}} K & if\ \omega\ is\ even \\ \dfrac{1}{|\mu|^{\boldsymbol{\omega+k}}} |K| & if\ \omega\ is\ odd \end{cases} \tag{4.20}$$

is an affine Wigner moment invariant.

Note that to apply (3.19) or (4.20) to determine a moment invariant from an algebraic invariant $K$, the condition $K$ must satisfy is as stated in (2.25) with the proportional constant $(\det M)^{\omega}$ involving matrix $M$, not involving matrix $A$. This observation is useful in the derivations of a number of AWMI later in this chapter.

In the rest of this chapter, algebraic invariants for different sets $\Gamma$ are derived, and then the corresponding AWMI are obtained based on (4.20).

## 4.4 Affine Wigner Moment Invariants Obtained from Individual Forms

In this section, three affine Wigner moment invariants (AWMI) will be derived from $f(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ and four affine Wigner moment invariants from $g(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$.

**Proposition 4.5**

The following are three affine Wigner moment invariants derived from $(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ :

$$Ip_0 = \frac{2}{\mu^4}(\mu_{2000}\mu_{0200} - \mu_{1100}^2) \tag{4.21}$$

$$Ip_1 = \frac{1}{\mu}(\mu_{1010} + \mu_{0101}) \tag{4.22}$$

$$Ip_2 = 2(\mu_{0020}\mu_{0002} - \mu_{0011}^2) \tag{4.23}$$

Proof)

Take $\Gamma = \{f_0(\mathbf{s};\ \boldsymbol{\mu}_0)\}$, which is compatible with linear transformations because of proposition (4.2). Here, $f_0(\mathbf{s};\ \boldsymbol{\mu}_0)$ as in (4.1) is a binary quadratic. Let $a, b$ two equivalent vectors of symbols for $f_0(\mathbf{s};\ \boldsymbol{\mu}_0)$. By the fundamental theorem for invariants of multivariate forms as stated in section 2.2.5, then $J_0$ as defined below is an algebraic invariant for $f_0(\mathbf{s};\ \boldsymbol{\mu}_0)$.

$$J_0(\boldsymbol{\mu}_0) = (ab)^2 = 2(\mu_{2000}\mu_{0200} - \mu_{1100}^2)^2$$

Moreover $J_0$ is of weight 2 and order 2. From (4.20), $Ip_0$ as defined in (4.21) is a moment invariant defined from $f_0(\mathbf{s}, \mathbf{u};\ \boldsymbol{\mu}_0)$, and is thus an AWMI.

Now, take the linear transformation compatible set of forms $\Gamma = \{f_1(\boldsymbol{s}, \boldsymbol{u};\ \boldsymbol{\mu}_1)\ \}$. Here $f_1(\boldsymbol{s}, \boldsymbol{u};\ \boldsymbol{\mu}_1)$ is a binary bilinear form as in (4.7). From the fundamental theorem for invariants of multilinear forms as stated in section 2.2.5, $J_1$ as defined below is an algebraic invariant.

$$J_1(\boldsymbol{\mu_1}) = a_\alpha = a_1\alpha_1 + a_2\alpha_2$$

Moreover it is of weight 0, i.e. absolute invariant, and of order 1. Thus by (4.20), $Ip_1$ as defined in (4.22) is a moment invariant defined from $f_1(\boldsymbol{s}, \boldsymbol{u};\ \boldsymbol{\mu}_1)$.

Now, $f_2(\boldsymbol{u};\ \boldsymbol{\mu}_2)$ is a binary quadratic form, then similar to the case of $f_0(\boldsymbol{u};\ \boldsymbol{\mu}_0)$, $J_2$ as defined below is an algebraic invariant of order 2 and weight 2 with respect to the transformation matrix $A = (M^T)^{-\mathbf{1}}$.

$$J_2(\boldsymbol{\mu}_2) = (ab)^2 = 2(\mu_{0020}\mu_{0002} - \mu_{0011}^2)$$

To use (4.20) or (3.19) to determine a moment invariant from $J_2$, $J_2$ must be an algebraic invariant with respect to matrix $M$, as defined in (2.25) in chapter 2 and observed in the paragraph right after expression (4.20).

But since $A = (M^T)^{-\mathbf{1}}$ so $\det A = (\det M)^{-1}$, thus by using (2.25) $J_2$ as an algebraic invariant of order 2 and weight 2 with respect to matrix $A$ becomes an algebraic invariant of order 2 and weight -2 with respect to matrix $M$.

Then by (4.20), $Ip_2$ as defined in (4.23) is a moment invariant defined from $f_2(\boldsymbol{u};\ \boldsymbol{\mu}_2)$, and thus an AWMI. QED

Note that, since $f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ is the sum of $f_0(\mathbf{s};\ \boldsymbol{\mu}_0)$, $2f_1(\boldsymbol{s},\boldsymbol{u};\ \boldsymbol{\mu}_1)$, and $f_2(\boldsymbol{u};\ \boldsymbol{\mu}_2)$, thus the sum of $Ip_0$, $2Ip_1$, and $Ip_2$ is also an AWMI.

**Proposition 4.6**

The following are four affine Wigner moment invariants derived from $g(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$:

$$Iq_0 = \frac{1}{\mu^{10}}(\mu_{3000}^2\mu_{0300}^2 - 6\,\mu_{3000}\mu_{2100}\mu_{1200}\mu_{0300} + 4\,\mu_{2100}^3\,\mu_{0300} - 3\,\mu_{2100}^2\mu_{1200}^2 + 4\,\mu_{3000}\mu_{1200}^3) \tag{4.24}$$

$$Iq_1 = \frac{2}{\mu^4}[4(\mu_{2010}\mu_{0210} - \mu_{1110}^2)(\mu_{2001}\mu_{0201} - \mu_{1101}^2) - (\mu_{2010}\mu_{0201} + \mu_{0210}\mu_{2001} - 2\mu_{1110}\mu_{1101})^2] \tag{4.25}$$

$$Iq_2 = 2[4(\mu_{1020}\mu_{1002} - \mu_{1011}^2)(\mu_{0120}\mu_{0102} - \mu_{0111}^2) - (\mu_{1020}\mu_{0102} + \mu_{1002}\mu_{0120} - 2\mu_{1011}\mu_{0111})^2] \tag{4.26}$$

$$Iq_3 = \mu^2(\mu_{0030}^2\mu_{0003}^2 - 6\,\mu_{0030}\mu_{0021}\mu_{0012}\mu_{0003} + 4\,\mu_{0021}^3\,\mu_{0003} - 3\,\mu_{0021}^2\mu_{0012}^2 + 4\,\mu_{0030}\mu_{0012}^3) \tag{4.27}$$

Proof)

$g_0(\boldsymbol{s};\boldsymbol{\mu}_0)$ is a binary cubic form, then as in ([5][4][29][30]), an invariant of weight 6 and order 4 for it is:

$$J_0 = (\mu_{3000}\mu_{0300} - \mu_{2100}\mu_{1200})^2 - 4(\mu_{3000}\mu_{1200} - \mu_{2100}^2)(\mu_{2100}\mu_{0300} - \mu_{1200}^2)$$

By expanding and simplifying $J_0$ can be expressed as:

$$J_0 = \mu_{3000}^2\mu_{0300}^2 - 6\,\mu_{3000}\mu_{2100}\mu_{1200}\mu_{0300} + 4\,\mu_{2100}^3\,\mu_{0300} - 3\,\mu_{2100}^2\mu_{1200}^2 + 4\,\mu_{3000}\mu_{1200}^3$$

By applying (4.20), $Iq_0$ defined as $Iq_0 = \frac{1}{\mu^{10}}J_0$ and as stated in (4.24) is a moment invariant defined from $g_0(\boldsymbol{s};\boldsymbol{\mu}_0)$, and thus an AWMI.

Similarly $g_3(\boldsymbol{u};\boldsymbol{\mu}_3)$ is a binary cubic form, then an invariant of weight -6 with respect matrix $M$ and order 4 for it can be expressed as $J_3$:

$$J_3 = \mu_{0030}^2\mu_{0003}^2 - 6\,\mu_{0030}\mu_{0021}\mu_{0012}\mu_{0003} + 4\,\mu_{0021}^3\,\mu_{0003} - 3\,\mu_{0021}^2\mu_{0012}^2 + 4\,\mu_{0030}\mu_{0012}^3$$

By (4.20), $Iq_3$ defined as $Iq_3 = \mu^2 J_3$ and as stated in (4.27) is a moment invariant defined from $g_3(\boldsymbol{u};\boldsymbol{\mu}_3)$, and thus an AWMI.

Now, an invariant for the binary trilinear form $g_1(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_1)$ is determined. Consider the symbols used for $g_1(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_1)$ as in (4.16). Let $a,\alpha$ and $b,\beta$ are two equivant sets of symbols. Let

$$\Delta = (ab)^2u_\alpha u_\beta = (ab)^2(\alpha_1u_1 + \alpha_2u_2)(\beta_1u_1 + \beta_2u_2) \qquad (4.25.1)$$
$$= (ab)^2\alpha_1\beta_1u_1^2 + (ab)^2(\alpha_1\beta_2 + \alpha_2\beta_1)u_1u_2 + (ab)^2\alpha_2\beta_2u_2^2$$

And let

$$A_{20} = (ab)^2\alpha_1\beta_1\,, \qquad A_{11} = \tfrac{1}{2}(ab)^2(\alpha_1\beta_2 + \alpha_2\beta_1)\,, \qquad A_{02} = (ab)^2\alpha_2\beta_2 \tag{4.25.2}$$

Thus $\Delta$ can be expressed as:

$$\Delta = A_{20}u_1^2 + 2A_{11}u_1u_2 + A_{02}u_2^2 \tag{4.25.3}$$

Similar to the definition of $\Delta$ in (4.25.1), let $\Delta'$ be defined in the vectors of symbols $a', \alpha'$ and $b', \beta'$ obtained under the transformations $\boldsymbol{s} = M\boldsymbol{s}'$, $\boldsymbol{u} = A\boldsymbol{u}'$:

$$\begin{aligned}\Delta' &= (a'b')^2 u'_{\alpha'} u'_{\beta'} \\ &= (a'b')^2\alpha'_1\beta'_1u'^2_1 + (a'b')^2(\alpha'_1\beta'_2 + \alpha'_2\beta'_1)u'_1u'_2 \\ &\qquad + (a'b')^2\alpha'_2\beta'_2u'^2_2\end{aligned} \tag{4.25.4}$$

And let

$$A'_{20} = (a'b')^2\alpha'_1\beta'_1\,, \qquad A'_{11} = \tfrac{1}{2}(a'b')^2(\alpha'_1\beta'_2 + \alpha'_2\beta'_1)\,, \tag{4.25.5}$$

$$A'_{02} = (a'b')^2\alpha'_2\beta'_2$$

And $\Delta'$ can also be expressed as:

$$\Delta' = A'_{20}u'^2_1 + 2A'_{11}u'_1u'_2 + A'_{02}u'^2_2 \tag{4.25.6}$$

Note that $A_{20}$ and $A'_{20}$ while expressed in symbols, they are actually the same function but evaluated at the coefficients of $g_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$ and at the coefficients of $g_1\left(\boldsymbol{s}', \boldsymbol{u}'; \frac{\mathbf{1}}{|J|}\boldsymbol{\mu}'_1\right)$, respectively. Similarly for $A_{11}$ and $A'_{11}$, and for $A_{02}$ and $A'_{02}$.

Now, an invariant of $g_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$ will be determined as a function $J_1$, such that for some $\omega$,

$$J_1(A_{20}, A_{11}, A_{02}) = (detM)^{\omega}\, J_1(A'_{20}, A'_{11}, A'_{02}) \tag{4.25.7}$$

Now, under the transformation $\boldsymbol{u} = A\boldsymbol{u}'$, $\Delta$ in (4.25.3) as a binary quadratic form transforms into

$$\Delta = A''_{20}u'^2_1 + 2A''_{11}u'_1u'_2 + A''_{02}u'^2_2 \tag{4.25.8}$$

Since $\Delta$ as a binary quadratic form, an invariant of weight 2 with respect to $A$ and order 2 is:

$$J_1 = 2(A_{20}A_{02} - A_{11}^2) \tag{4.25.9}$$

Since $J_1$ is an invariant of weight 2, and since $det\ A = (\det M)^{-1}$, the following is obtained:

$$2(A''_{20}A''_{02} - A''^2_{11}) = (det\ M)^{-2}\, 2(A_{20}A_{02} - A_{11}^2) \tag{4.25.10}$$

But from (2.23), $(a'b')^2 = (det\ M)^2\ (ab)^2$, and since $u_\alpha = u'_{\alpha'}$ and $u_\beta = u'_{\beta'}$, thus

$$\Delta' = (det\ M)^2\, \Delta \tag{4.25.11}$$

In the theory of algebraic forms, this essentially says, $\Delta$ is a covariant of order 2 for $g_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$. Substitute (4.25.6) and (4.25.8) into the left and right hand side of (4.25.11) respectively to obtain:

$$A'_{20}u'^2_1 + 2A'_{11}u'_1u'_2 + A'_{02}u'^2_2 \qquad (4.25.12)$$
$$= (det\ M)^2\ (A''_{20}u'^2_1 + 2A''_{11}u'_1u'_2 + A''_{02}u'^2_2)$$

By equating the coefficients of the terms having the same variables in the left and right hand sides of (4.25.12), the following is obtained:

$$A''_{20} = \frac{1}{(\det M)^2}A'_{20} \qquad A''_{11} = \frac{1}{(\det M)^2}A'_{11} \qquad (4.25.13)$$

$$A''_{02} = \frac{1}{(\det M)^2}A'_{02}$$

Substituting expressions in (4.25.13) into (4.25.10) to obtain:

$$(A'_{20}A'_{02} - A'^2_{11}) = (det\ M)^2\ (A_{20}A_{02} - A^2_{11}) \qquad (4.25.14)$$

Now, consider $J_1$ in (4.25.9) as a function $J_1(A_{20}, A_{11}, A_{02})$. Comparing (4.25.14) with (4.25.7) shows that $J_1$ is an invariant of weight 2 and order 2 for $g_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$. Here

$$A_{20} = (ab)^2\alpha_1\beta_1 = (a_1b_2 - a_2b_1)^2\alpha_1\beta_1 \qquad (4.25.15)$$
$$= a_1^2\alpha_1b_2^2\beta_1 + a_2^2\alpha_1b_1^2\beta_1 - 2a_1a_2\alpha_1b_1b_2\beta_1$$
$$= 2(\mu_{2010}\mu_{0210} - \mu^2_{1110})$$

$$A_{02} = (ab)^2\alpha_2\beta_2 = (a_1b_2 - a_2b_1)^2\alpha_2\beta_2 \tag{4.25.16}$$
$$= a_1^2\alpha_2b_2^2\beta_2 + a_2^2\alpha_2b_1^2\beta_2 - 2a_1a_2\alpha_2b_1b_2\beta_2$$
$$= 2(\mu_{2001}\mu_{0201} - \mu_{1101}^2)$$

$$A_{11} = \frac{1}{2}(ab)^2(\alpha_1\beta_2 + \alpha_2\beta_1) = \frac{1}{2}(a_1b_2 - a_2b_1)^2(\alpha_1\beta_2 + \alpha_2\beta_1) \tag{4.25.17}$$
$$= \frac{1}{2}(a_1^2\alpha_1b_2^2\beta_2 + a_2^2\alpha_1b_1^2\beta_2 - 2\,a_1a_2\alpha_1b_1b_2\beta_2 + a_1^2\alpha_2b_2^2\beta_1$$
$$+ a_2^2\alpha_2b_1^2\beta_1 - 2\,a_1a_2\alpha_2b_1b_2\beta_1)$$
$$= \mu_{2010}\mu_{0201} + \mu_{0210}\mu_{2001} - 2\mu_{1110}\mu_{1101}$$

Thus

$$J_1 = 2[4(\mu_{2010}\mu_{0210} - \mu_{1110}^2)(\mu_{2001}\mu_{0201} - \mu_{1101}^2) \tag{4.25.18}$$
$$- (\mu_{2010}\mu_{0201} + \mu_{0210}\mu_{2001} - 2\mu_{1110}\mu_{1101})^2]$$

Since $J_1$ is of weight 2 and order 2, by (4.20), ), $Iq_1$ defined as $Iq_1 = \frac{1}{\mu^4}J_1$ as stated in (4.25) is an AWMI derived from $g_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$.

Now, an invariant for the binary trilinear form $g_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_2)$ is derived. The derivation is in modification of the above derivation of an invariant for $g_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$.

Consider the symbols used for $g_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$ as in (4.17), let $a, \alpha$ and $b, \beta$ are two equivant sets of symbols. Let

$$\Delta = (\alpha\beta)^2a_sb_s = (\alpha\beta)^2(a_1s_1 + a_2s_2)(b_1s_1 + b_2s_2) \tag{4.26.1}$$
$$= (\alpha\beta)^2a_1b_1s_1^2 + (\alpha\beta)^2(a_1b_2 + a_2b_1)s_1s_2 + (\alpha\beta)^2a_2b_2s_2^2$$

Let

$$A_{20} = (\alpha\beta)^2 a_1 b_1 \qquad A_{11} = \tfrac{1}{2}(\alpha\beta)^2 (a_1 b_2 + a_2 b_1) \qquad A_{02} = (\alpha\beta)^2 a_2 b_2 \quad (4.26.2)$$

Thus $\Delta$ can be expressed as

$$\Delta = A_{20} s_1^2 + 2A_{11} s_1 s_2 + A_{02} s_2^2 \quad (4.26.3)$$

Similar to the definition of $\Delta$ in (4.26.1), let $\Delta'$ be defined in the vectors of symbols $a', \alpha'$ and $b', \beta'$ obtained under the transformations $\boldsymbol{s} = M\boldsymbol{s}'$, $\boldsymbol{u} = A\boldsymbol{u}'$:

$$\begin{aligned} \Delta' &= (\alpha'\beta')^2 a'_{s'} b'_{s'} \quad (4.26.4) \\ &= (\alpha'\beta')^2 a'_1 b'_1 s'^2_1 + (\alpha'\beta')^2 (a'_1 b'_2 + a'_2 b'_1) s'_1 s'_2 \\ &\quad + (\alpha'\beta')^2 a'_2 b'_2 s'^2_2 \end{aligned}$$

And let

$$A'_{20} = (\alpha'\beta')^2 a'_1 b'_1 \qquad A'_{11} = \tfrac{1}{2}(\alpha'\beta')^2 (a'_1 b'_2 + a'_2 b'_1) \quad (4.26.5)$$

$$A'_{02} = (\alpha'\beta')^2 a'_2 b'_2$$

And $\Delta'$ can be expressed as:

$$\Delta' = A'_{20} s'^2_1 + 2A'_{11} s'_1 s'_2 + A'_{02} s'^2_2 \quad (4.26.6)$$

Now, an invariant of $g_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_2)$ will be determined as a function $J_2(A_{20}, A_{11}, A_{02})$ such that, for some $\omega$,

$$J_2(A_{20}, A_{11}, A_{02}) = (detM)^{\omega}\, J_2(A'_{20}, A'_{11}, A'_{02}) \quad (4.26.7)$$

Now, under the transformation $\boldsymbol{s} = M\boldsymbol{s}'$, $\Delta$ in (4.26.3) as a binary quadratic form is transformed into

$$\Delta = A''_{20}s'^2_1 + 2A''_{11}s'_1s'_2 + A''_{02}s'^2_2 \tag{4.26.8}$$

Since $\Delta$ as a binary quadratic form, an invariant of weight 2 with respect to $M$ and order 2 is:

$$J_2 = 2(A_{20}A_{02} - A_{11}^2) \tag{4.26.9}$$

With

$$2(A''_{20}A''_{02} - A''^2_{11}) = (det\, M)^2\, 2(A_{20}A_{02} - A_{11}^2) \tag{4.26.10}$$

But from (2.24), $(\alpha'\beta')^2 = (det\, M)^{-2}\,(\alpha\beta)^2$, and since $a_s = a'_{s'}$ and $b_s = b'_{s'}$, thus

$$\Delta' = (det\, M)^{-2}\,\Delta \tag{4.26.11}$$

This essentially says, $\Delta$ is a covariant of order -2 for $g_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$. Substitute (4.26.6) and (4.26.8) into the left and right hand side of (4.26.11) respectively to obtain:

$$A'_{20}s'^2_1 + 2A'_{11}s'_1s'_2 + A'_{02}s'^2_2 \tag{4.26.12}$$
$$= (det\, M)^{-2}\,(A''_{20}s'^2_1 + 2A''_{11}s'_1s'_2 + A''_{02}s'^2_2)$$

By equating the coefficients of the terms having the same variables in the left and right hand sides of (4.26.12), the following is obtained:

$$A''_{20} = (det\, M)^2\, A'_{20} \qquad A''_{11} = (det\, M)^2\, A'_{11} \tag{4.26.13}$$

$$A''_{02} = (det\, M)^2\, A'_{02}$$

Substituting expressions in (4.26.13) into (4.25.10) to obtain:

$$(A'_{20}A'_{02} - A'^{2}_{11}) = (det\, M)^{-2}\,(A_{20}A_{02} - A_{11}^2) \tag{4.26.14}$$

Now, consider $J_2$ in (4.26.9) as a function $J_2(A_{20}, A_{11}, A_{02})$. Comparing (4.26.14) with (4.26.7) shows that $J_2$ is an invariant of weight -2 and order 2 for $g_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_2)$. Here

$$\begin{aligned} A_{20} &= (\alpha\beta)^2 a_1 b_1 = (\alpha_1\beta_2 - \alpha_2\beta_1)^2 a_1 b_1 \\ &= a_1\alpha_1^2 b_1\beta_2^2 + a_1\alpha_2^2 b_1\beta_1^2 - 2a_1\alpha_1\alpha_2 b_1\beta_1\beta_2 \\ &= 2(\mu_{1020}\mu_{1002} - \mu_{1011}^2) \end{aligned} \tag{4.26.15}$$

$$\begin{aligned} A_{02} &= (\alpha\beta)^2 a_2 b_2 = (\alpha_1\beta_2 - \alpha_2\beta_1)^2 a_2 b_2 \\ &= a_2\alpha_2^2 b_2\beta_1^2 + a_2\alpha_1^2 b_2\beta_2^2 - 2a_2\alpha_1\alpha_2 b_2\beta_1\beta_2 \\ &= 2(\mu_{0120}\mu_{0102} - \mu_{0111}^2) \end{aligned} \tag{4.26.16}$$

$$\begin{aligned} A_{11} &= \frac{1}{2}(\alpha\beta)^2(a_1 b_2 + a_2 b_1) = \frac{1}{2}(\alpha_1\beta_2 - \alpha_2\beta_1)^2(a_1 b_2 + a_2 b_1) \\ &= \frac{1}{2}(a_1\alpha_1^2 b_2\beta_2^2 + a_1\alpha_2^2 b_2\beta_1^2 - 2\, a_1\alpha_1\alpha_2 b_2\beta_1\beta_2 + a_2\alpha_1^2 b_1\beta_2^2 \\ &\qquad + a_2\alpha_2^2 b_1\beta_1^2 - 2\, a_2\alpha_1\alpha_2 b_1\beta_1\beta_2) \\ &= \mu_{1020}\mu_{0102} + \mu_{1002}\mu_{0120} - 2\mu_{1011}\mu_{0111} \end{aligned} \tag{4.26.17}$$

Thus

$$J_2 = 2[4(\mu_{1020}\mu_{1002} - \mu_{1011}^2)(\mu_{0120}\mu_{0102} - \mu_{0111}^2) \quad (4.26.18)$$
$$- (\mu_{1020}\mu_{0102} + \mu_{1002}\mu_{0120} - 2\mu_{1011}\mu_{0111})^2]$$

Since $J_2$ is of weight -2 and order 2, by (4.20) $Iq_2$ defined as $Iq_2 = J_2$ as stated in (4.26) is an AWMI derived from $g_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_2)$. QED

Note that, since $g(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ is the sum of $g_0(\boldsymbol{s};\ \boldsymbol{\mu}_0)$, $3g_1(\boldsymbol{s}, \boldsymbol{u};\ \boldsymbol{\mu}_1)$, $3g_2(\boldsymbol{s}, \boldsymbol{u};\ \boldsymbol{\mu}_2)$, and $g_3(\boldsymbol{u};\ \boldsymbol{\mu}_3)$, thus the sum of $Iq_0$, $3Iq_2$, $3Iq_2$, and $Iq_3$ is also an AWMI.

## 4.5 Affine Wigner Moment Invariants Obtained from Simultaneous Forms

In this section, two affine Wigner moment invariants (AWMI) will be determined from $f_0(\boldsymbol{s}; \boldsymbol{\mu})$ and $g_0(\boldsymbol{s}; \boldsymbol{\mu})$, another two affine Wigner moment invariants will be determined from $f_2(\boldsymbol{u}; \boldsymbol{\mu})$ and $g_3(\boldsymbol{u}; \boldsymbol{\mu})$, one affine Wigner moment invariants will be determined from $f_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ and $g_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$, and finally affine Wigner moment invariants will be determined from $f_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$ and $g_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu})$.

**Proposition 4.7**

i) The following $I_{00}$ and $J_{00}$ are AWMI determined from binary quadratic $f_0(\boldsymbol{x}; \boldsymbol{\mu}_0)$ and binary cubic $g_0(\boldsymbol{x}; \boldsymbol{\mu}_0)$ simultaneously.

$$I_{00} = \frac{1}{\mu^7}[\mu_{2000}(\mu_{2100}\mu_{0300} - \mu_{1200}^2) - \mu_{1100}(\mu_{3000}\mu_{0300} - \mu_{2100}\mu_{1200}) + \mu_{0200}(\mu_{3000}\mu_{1200} - \mu_{2100}^2)] \quad (4.28)$$

$$\begin{aligned} J_{00} = \frac{1}{\mu^{11}}[&\mu_{3000}^2\mu_{0200}^3 - 6\mu_{3000}\mu_{2100}\mu_{1100}\mu_{0200}^2 \\ &+ 6\mu_{3000}\mu_{1200}\mu_{0200}(2\mu_{1100}^2 - \mu_{2000}\mu_{0200}) \\ &+ \mu_{3000}\mu_{0300}(6\mu_{2000}\mu_{1100}\mu_{0200} - 8\,\mu_{1100}^3) \\ &+ 9\mu_{2100}^2\mu_{2000}\mu_{0200}^2 - 18\mu_{2100}\mu_{1200}\mu_{2000}\mu_{1100}\mu_{0200} \\ &+ 6\mu_{2100}\mu_{0300}\mu_{2000}(2\mu_{1100}^2 - \mu_{2000}\mu_{0200}) \\ &+ 9\mu_{1200}^2\mu_{2000}^2\mu_{0200} - 6\mu_{1200}\mu_{0300}\mu_{1100}\mu_{2000}^2 \\ &+ \mu_{0300}^2\mu_{2000}^3] \end{aligned} \quad (4.29)$$

ii) The following $I_{23}$ and $J_{23}$ are AWMI determined from binary quadratic $f_2(\boldsymbol{x};\boldsymbol{\mu}_0)$ and binary cubic $g_3(\boldsymbol{x};\boldsymbol{\mu}_0)$ simultaneously.

$$I_{23} = \mu[\mu_{0020}(\mu_{0021}\mu_{0003} - \mu_{0012}^2) - \mu_{0011}(\mu_{0030}\mu_{0003} - \mu_{0021}\mu_{0012}) + \mu_{0002}(\mu_{0030}\mu_{0012} - \mu_{0021}^2)] \quad (4.30)$$

$$J_{23} = \mu[\mu_{0030}^2\mu_{0002}^3 - 6\mu_{0030}\mu_{0021}\mu_{0011}\mu_{0002}^2 \tag{4.31}$$
$$+ 6\mu_{0030}\mu_{0012}\mu_{0002}(2\mu_{0011}^2 - \mu_{0020}\mu_{0002})$$
$$+ \mu_{0030}\mu_{0003}(6\mu_{0020}\mu_{0011}\mu_{0002} - 8\,\mu_{0011}^3)$$
$$+ 9\mu_{0021}^2\mu_{0020}\mu_{0002}^2 - 18\mu_{0021}\mu_{0012}\mu_{0020}\mu_{0011}\mu_{0002}$$
$$+ 6\mu_{0021}\mu_{0003}\mu_{0020}(2\mu_{0011}^2 - \mu_{0020}\mu_{0002})$$
$$+ 9\mu_{0012}^2\mu_{0020}^2\mu_{0002} - 6\mu_{0012}\mu_{0003}\mu_{0011}\mu_{0020}^2$$
$$+ \mu_{0003}^2\mu_{0020}^3]$$

Proof)

For i), from ([29][30][5][4]) there are two simultaneous invariants for binary quadratic $f_0(\boldsymbol{s};\boldsymbol{\mu}_0)$ and binary cubic $g_0(\boldsymbol{s};\boldsymbol{\mu}_0)$ forms. One is of weight 4 and order 3, and another is of weight 6 and order 5. Using (4.20), two moment invariants can be obtained from these two invariants, and these moment invariants are also AWMI. These two AWMI are $I_{00}$ as in (4.28) and $J_{00}$ as in (4.29).

For ii),as in i) there are two simultaneous invariants for $f_2(\boldsymbol{u};\boldsymbol{\mu}_0)$ and $g_3(\boldsymbol{u};\boldsymbol{\mu}_0)$ with respect to matrix $A$. Converting to with respect to matrix $M$, one simultaneous invariant is of weight -4 and order 3, the other is of weight -6 and order 5. Using (4.20) two moment invariants $I_{23}$ and $J_{23}$, which are also AWMI, can be obtained as stated in (4.30) and (4.31) respectively. QED.

**Proposition 4.8**

i) The following $I_{11}$ is an AWMI determined from binary bilinear $f_1(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_1)$ and binary trilinear $g_1(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_1)$ simultaneously.

$$I_{11} = \frac{1}{\mu^4}|\mu_{1010}(\mu_{2001}\mu_{0210} + \mu_{2010}\mu_{0201} - 2\mu_{1110}\mu_{1101}) \tag{4.32}$$

$$- 2\mu_{1001}(\mu_{2010}\mu_{0210} - \mu_{1110}^2)$$

$$+ 2\mu_{0110}(\mu_{2001}\mu_{0201} - \mu_{1101}^2)$$

$$- \mu_{0101}(\mu_{2010}\mu_{0201} + \mu_{2001}\mu_{0210} - 2\mu_{1110}\mu_{1101})|$$

ii) The following $I_{12}$ is an AWMI determined from binary bilinear $f_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$ and binary trilinear $g_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_2)$ simultaneously.

$$I_{12} = \frac{1}{\mu^2}|\mu_{1010}(\mu_{1002}\mu_{0120} + \mu_{1020}\mu_{0102} - 2\mu_{1011}\mu_{0111}) \tag{4.33}$$

$$+ 2\mu_{1001}(\mu_{0120}\mu_{0102} - \mu_{0111}^2)$$

$$- 2\mu_{0110}(\mu_{1020}\mu_{1002} - \mu_{1011}^2)$$

$$- \mu_{0101}(\mu_{1020}\mu_{0102} + \mu_{1002}\mu_{0120} - 2\mu_{1011}\mu_{0111})|$$

Proof)

For i), let $a, \alpha$ be the symbols used to describe $f_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$, and let $b, \beta$ and $c, \gamma$ be equivalent symbols used to describe $g_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$. Thus

$$f_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1) = a_s u_\alpha = (a_1 s_1 + a_2 s_2)(\alpha_1 u_1 + \alpha_2 u_2)$$

And

$$g_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1) = b_s^2 u_\beta = (b_1 s_1 + b_2 s_2)^2(\beta_1 u_1 + \beta_2 u_2)$$

From the fundamental theorem for invariants of multilinear forms as reviewed in section 2.2.5, the following can be verified as a simultaneous invariant for $f_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$ and $g_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$:

$$J_{11} = (bc)^2(\alpha\beta)a_\gamma$$

Indeed, under the transformations $\boldsymbol{s} = M\boldsymbol{s}'$ and $\boldsymbol{u} = A\boldsymbol{u}'$, $J_{11}$ becomes $J'_{11}$:

$$J'_{11} = (b'c')^2(\alpha'\beta')a'_{\gamma'}$$

From (2.23), $(b'c')^2 = (det\, M)^2\,(bc)^2$ . And from (2.24), $(\alpha'\beta') = (det\, M)^{-1}\,(\alpha\beta)$. Also $a_\gamma = a'_{\gamma'}$. Thus $J'_{11}$ as above can be written as:

$$J'_{11} = (det\, M)^2\,(bc)^2(det\, M)^{-1}\,(\alpha\beta)a_\gamma$$
$$= (det\, M)\,J_{11}$$

Thus $J_{11}$ is an invariant of weight 1. Now, expanding $J_{11}$ gives:

$$J_{11} = (b_1c_2 - b_2c_1)^2(\alpha_1\beta_2 - \alpha_2\beta_1)(\gamma_1a_1 + \gamma_2a_2)$$

Expand $J_{11}$ again and use (4.4) and (4.13) to convert the symbols involved into the coefficients. $J_{11}$ becomes a polynomial of degree 3 in the coefficients or of order 3.

Sine $J_{11}$ is of weight 1 and order 3, by (4.20) $I_{11}$ defined as $I_{11} = \frac{1}{\mu^4}|J_{11}|$ is a simultaneous moment invariant for $f_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$ and $g_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$. $I_{11}$ is as stated in (4.32).

For ii), let $a, \alpha$ be the symbols used to describe $f_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$, and let $b, \beta$ and $c, \gamma$ be equivalent symbols used to describe $g_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_2)$. Thus

$$f_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1) = a_s u_\alpha = (a_1 s_1 + a_2 s_2)(\alpha_1 u_1 + \alpha_2 u_2)$$

And

$$g_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_2) = b_s u_\beta^2 = (b_1 s_1 + b_2 s_2)(\beta_1 u_1 + \beta_2 u_2)^2$$

From the fundamental theorem for invariants of multilinear forms as reviewed in section 2.2.5, the following can be verified as a simultaneous invariant for $f_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$ and $g_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_2)$:

$$J_{12} = (\beta\gamma)^2 (ab) c_\alpha$$

Indeed, under the transformations $\boldsymbol{s} = M\boldsymbol{s}'$ and $\boldsymbol{u}' = M^T \boldsymbol{u}$, $J_{12}$ becomes $J'_{12}$:

$$J'_{12} = (\beta'\gamma')^2 (a'b') c'_{\alpha'}$$

Use (2.23) and (2.24), and $c_\alpha = c'_{\alpha'}$ , $J'_{12}$ can be written as:

$$J'_{12} = (det\, M)^{-2} (\beta\gamma)^2 (det\, M)\, (ab) c_\alpha$$

$$= (det\, M)^{-1} J_{12}$$

Thus $J_{12}$ is an invariant of weight -1. Now, expanding $J_{12}$ gives

$$J_{12} = (\beta_1\gamma_2 - \beta_2\gamma_1)^2 (a_1 b_2 - a_2 b_1)(\alpha_1 c_1 + \alpha_2 c_2)$$

Expand $J_{12}$ again and use (4.4) and (4.13) to convert the symbols involved into the coefficients. $J_{12}$ becomes a polynomial of degree 3 in the coefficients or of order 3.

Sine $J_{12}$ is of weight -1 and order 3, by (4.20) $I_{12}$ defined as $I_{12} = \frac{1}{\mu^2}|J_{12}|$ is a simultaneous moment invariant for $f_1(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_1)$ and $g_2(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_2)$. $I_{12}$ is as stated in (4.32). QED.

## 4.6 A Summary of Steps to Derive Affine Wigner Moment Invariants

The following is a summary of the steps to derive affine Wigner moment invariants (AWMI) for two-dimensional images.

1. Given an image $f(\boldsymbol{x})$ that is an affine transformation of image $f'(\boldsymbol{x}')$. This means $f(\boldsymbol{x}) = f'(\boldsymbol{x}')$ whenever $\boldsymbol{x} = A\boldsymbol{x}' + \boldsymbol{b}$, where A is $2 \times 2$ non singular matrix and $\boldsymbol{b}$ is a vector in $\boldsymbol{R^2}$.
2. Determine the Wigner distributions $W(\boldsymbol{x}, \boldsymbol{w})$ and $W'(\boldsymbol{x}', \boldsymbol{w}')$ for $f(\boldsymbol{x})$ and $f'(\boldsymbol{x}')$.
3. Calculate the centroids of Wigner distributions $W(\boldsymbol{x}, \boldsymbol{w})$ and $W'(\boldsymbol{x}', \boldsymbol{w}')$.
4. Decompose the moment generating functions of $W(\boldsymbol{x}, \boldsymbol{w})$ and $W'(\boldsymbol{x}', \boldsymbol{w}')$ into sums of quaternary forms as in (3.15) and (3.16). Here a quaternary form $P_p(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_p)$ of order $p$ is

$$P_p(\boldsymbol{s}, \boldsymbol{u}; \boldsymbol{\mu}_p) = \sum_{k_1+k_2+k_3+k_4=p} \binom{p}{k_1\ k_2\ k_3\ k_4} \mu_{k_1k_2k_3k_4} s_1^{k_1} s_2^{k_2}\, u_1^{k_3} u_2^{k_4}$$

   And $\boldsymbol{s} = (s_1, s_2)^T$ and $\boldsymbol{u} = (u_1, u_2)^T$ are vectors in $\boldsymbol{R^2}$, and $\mu_{k_1k_2k_3k_4}$ are moments of $W(\boldsymbol{x}, \boldsymbol{w})$ with respect to the coordinate system having the origin at the centroid of $W(\boldsymbol{x}, \boldsymbol{w})$.

5. Denote $f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_p) = P_2(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_p)$ and $g(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_p) = P_3(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_p)$. Decompose $f(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_p)$ and $g(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu}_p)$ as in (4.5) and (4.14) to obtain simpler forms $f_0$, $f_1$, $f_2$, and $g_0$, $g_1$, $g_2$, $g_3$.
6. Let $h(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ be any of $f_0$, $f_1$, $f_2$, and $g_0$, $g_1$, $g_2$, $g_3$. Substitute $\boldsymbol{s} = \mathrm{M}\boldsymbol{s}'$ and $\boldsymbol{u} = \mathrm{A}\boldsymbol{u}'$ into $h(\boldsymbol{s},\boldsymbol{u};\boldsymbol{\mu})$ to obtain $h\left(\boldsymbol{s}',\boldsymbol{u}';\frac{\mathbf{1}}{|J|}\boldsymbol{\mu'}\right)$ as in proposition 4.2 part ii) and proposition 4.4 part ii). Here $M = (A^T)^{-1}$ and $J = |\det M|$.
7. Obtain an algebraic invariant $K(\boldsymbol{\mu})$ for $\{f_0, f_1, f_2, g_0, g_1, g_2, g_3\}$.
8. Use (4.20) to obtain an AWMI $I(\boldsymbol{\mu})$**.** Here $I(\boldsymbol{\mu}) = I(\boldsymbol{\mu'})$**,** with $\boldsymbol{\mu}$ and $\boldsymbol{\mu'}$ being vectors of moments of $W(\boldsymbol{x},\boldsymbol{w})$ and $W'(\boldsymbol{x}',\boldsymbol{w}')$ with respect to the coordinate systems centered at the centroids of $W(\boldsymbol{x},\boldsymbol{w})$ and $W'(\boldsymbol{x}',\boldsymbol{w}')$.

The above steps of derivation can be illustrated as in Figure 4.1

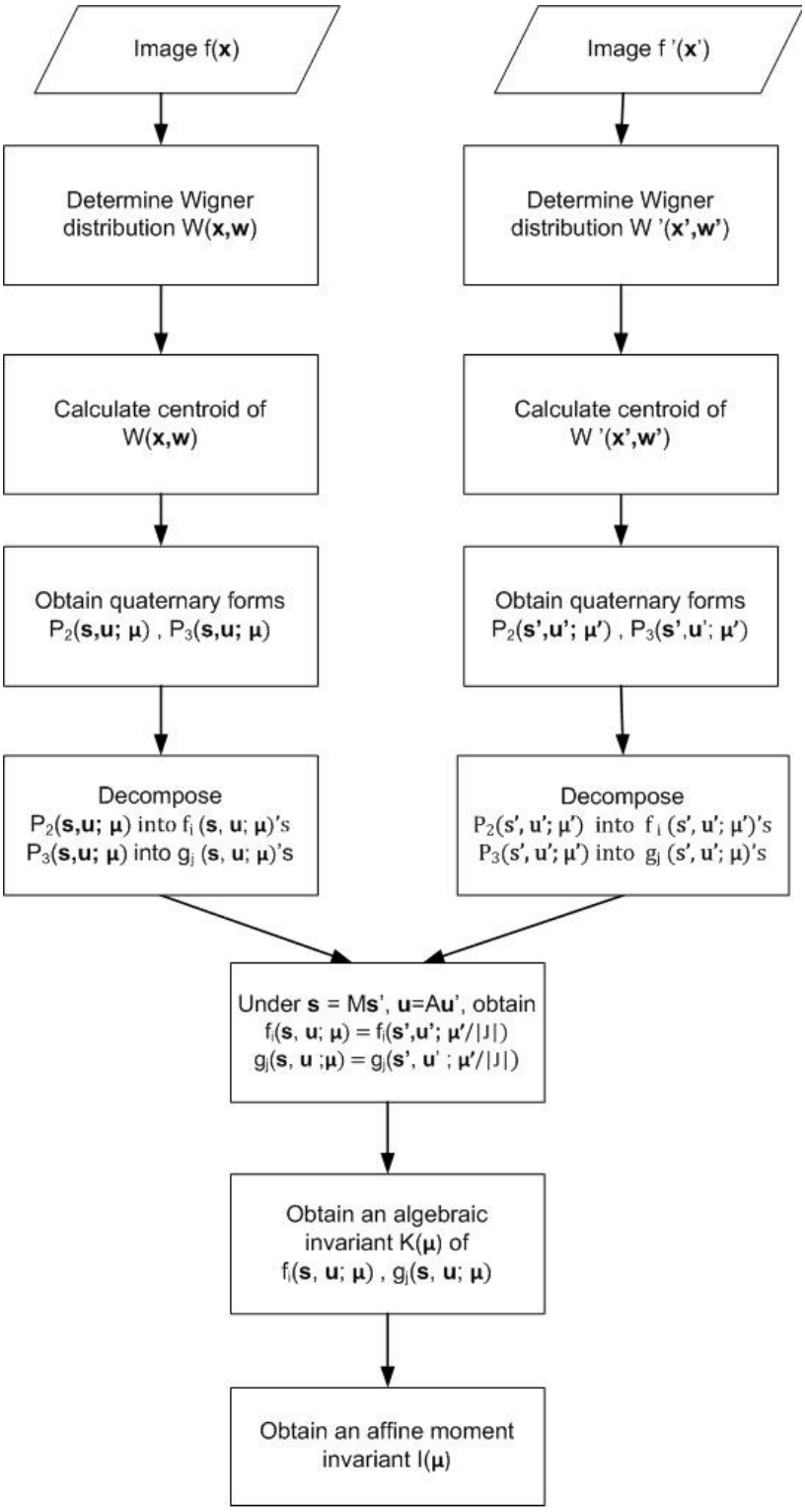


Figure 4.1: Steps in deriving affine Wigner moment invariants

# Chapter 5 – Calculations of Affine Wigner Moment Invariants

## 5.1 Introduction

The affine Wigner moment invariants as derived in chapter 3 and 4 are under the theoretical consideration of continuous Wigner distributions. For practical calculations, the affine Wigner moment invariants need to be calculated from discrete Wigner distributions that are determined from real world images. There are, however, a number of issues in the actual calculation of affine Wigner moment invariants, and in testing if these invariants are indeed invariant, and in assessing if they indeed extract certain texture information from images.

This chapter shows how to calculate the affine Wigner moments from two-dimensional images, and reports two experiments. One experiment is to show these invariants are indeed invariant and showing they indeed extract certain overall texture information from images. Another experiment is to test how these invariants could be used for image retrieval.

The affine Wigner moment invariants as can seen from chapter 4 are functions of the moments of Wigner distributions. For a given real world image, a Wigner distribution can be calculated and it is commonly called a discrete Wigner distribution ([76] [43] [77] [78]). Then the moments calculated based on the discrete Wigner distribution are obtained, and finally the affine Wigner moment invariants are calculated.

The calculation of discrete Wigner distributions, however, suffers from two problems. The first is the aliasing problem that means the discrete and continuous Wigner distributions are not the same on the domain of interest ([79][80][81][82]). The second is the interference problem of negative and positive values of spatial frequencies ([65][83][84][85][86]).

Section 5.2 - 5.3 in this chapter show how to deal with the aliasing problem in the determination of discrete Wigner distributions.

Since the aliasing and interference problems are associated with the choice of domain for the spatial frequency $\boldsymbol{w}$ of any Wigner distribution $W(\boldsymbol{x}, \boldsymbol{w})$. In section 5.2, the domain for $\boldsymbol{w}$ will be clearly derived. Section 5.3 will show how the aliasing and interference problems can be alleviated by using the oversampling by a factor of 2 approach ([76][43]) and by choosing certain appropriate domain for the spatial frequency $\boldsymbol{w}$.

Section 5.4 in this chapter reports an experiment showing how to check if the affine Wigner moment invariants are indeed invariant and comparing the performance of these invariants with that of affine moment invariants in extracting texture information.

## 5.2 Domain of Wigner Distributions

Let $f(\boldsymbol{x})$ be a function defined on $\boldsymbol{x} = (x_1, x_2)^T$ representing any 2D image. In this chapter function $f(\boldsymbol{x})$ is assumed to be a real valued function.

Let $W(\boldsymbol{x},\boldsymbol{w})$ be the Wigner distribution of $f(\boldsymbol{x})$ and where $\boldsymbol{w} = (w_1, w_2)^T$ . When considered as a function of $\boldsymbol{w}$ at some fixed $\boldsymbol{x}$, $W(\boldsymbol{x},\boldsymbol{w})$ is denoted by $W_{\boldsymbol{x}}(\boldsymbol{w})$ . Recall from the definition of Wigner distribution as in (3.3) that $W_{\boldsymbol{x}}(\boldsymbol{w})$:

$$W_{\boldsymbol{x}}(\boldsymbol{w}) = 4\int g_x(u_1,u_2)exp\{-j(u_1 2w_1 + u_2 2w_2)\}du_1 du_2 \tag{5.1}$$

where

$$g_x(s_1,s_2) = f(x_1+s_1, x_2+s_2)f(x_1-s_1, x_2-s_2) \tag{5.2}$$

Let $G_x(w_1,w_2)$ be the Fourier transform of $g_x(u_1,u_2)$, then

$$W_{\boldsymbol{x}}(\boldsymbol{w}) = 4\, G_x(2w_1, 2w_2) \tag{5.3}$$

Now, assume $f(\boldsymbol{x})$ is band limited, and is sampled at the Nyquist rates in both the coordinates $x_1$, and $x_2$ with sampling intervals $T_1$ and $T_2$. Then the non-zero domain for the Fourier transform $F(w_1,w_2)$ of $f(\boldsymbol{x})$ is :

$$D = \left(-\frac{\pi}{T_1}, \frac{\pi}{T_1}\right) \times \left(-\frac{\pi}{T_2}, \frac{\pi}{T_2}\right) \tag{5.4}$$

Where the cut off frequencies for $w_1$ and $w_2$ are $\frac{\pi}{T_1}$ and $\frac{\pi}{T_2}$ ([76][87]).

But if $F(\boldsymbol{w}) = 0$, for $\mathbf{w} \notin D$, then as in ([10]), $W_x(\boldsymbol{w}) = 0$, for $\mathbf{w} \notin D$, for all $\boldsymbol{x}$.

Thus $D$ can be taken as the non-zero domain for $W_{\boldsymbol{x}}(\boldsymbol{w})$, where for all $\boldsymbol{w}$ not in $D$, $W_{\boldsymbol{x}}(\boldsymbol{w}) = 0$.

## 5.3 Discrete Wigner Distributions

### 5.3.1 Aliasing Problem of Discrete Wigner Distribution

It has often been said that the discrete Wigner distribution suffers from the aliasing problem. This section shows explicitly what the aliasing problem actually is for 2-dimensional Wigner distributions.

**Definition**

As in ([76][43]), the Wigner distribution $W_{\boldsymbol{x}}(\boldsymbol{w})$ as in (5.1) can be approximated by the trapezoidal rule to define *the discrete Wigner distribution* as:

$$W_{\boldsymbol{x},d}(\boldsymbol{w}) = 4T_1T_2 \sum_{k_1,k_2=-\infty}^{+\infty} g_{\boldsymbol{x},d}(\boldsymbol{k}) exp\{-2j(k_1w_1T_1 + k_2w_2T_2)\} \tag{5.5}$$

Where $T_{1,}\ T_2$ are the sampling inverals for $x_1$ and $x_2$, and

$$g_{\boldsymbol{x},d}(\boldsymbol{k}) = f(x_1 + k_1T_1, x_2 + k_2T_2)f(x_1 - k_1T_1, x_2 - k_2T_2) \tag{5.6}$$

and $\boldsymbol{k} = (k_1, k_2)'$. Thus

$$W_{\boldsymbol{x},d}(\boldsymbol{w}) = 4T_1T_2\ G_{x,d}(2\theta_1, 2\theta_2) \tag{5.7}$$

Where $G_{x,d}(\theta_1, \theta_2)$ is the Fourier transform of $g_{\boldsymbol{x},d}(\boldsymbol{k})$, and $\theta_i = w_iT_i$, $i = 1, 2$.

Note that $G_{x,d}(\theta_1, \theta_2)$ is periodic with period $\left(\frac{\pi}{T_1}, \frac{\pi}{T_2}\right)$, meaning $G_{x,d}(\theta_1, \theta_2)$ = $G_{x,d}\left(\theta_1 + \frac{\pi}{T_1}, \theta_2\right)$ = $G_{x,d}\left(\theta_1, \theta_2 + \frac{\pi}{T_2}\right)$, for any $\theta_1, \theta_2$.

Thus $W_{\boldsymbol{x},d}(\boldsymbol{w})$ is also periodic with period $\left(\frac{\pi}{T_1},\frac{\pi}{T_2}\right)$, and its fundamental domain can be taken to be:

$$D_1 = \left(-\frac{\pi}{2T_1},\frac{\pi}{2T_1}\right)\times\left(-\frac{\pi}{2T_2},\frac{\pi}{2T_2}\right) \tag{5.8}$$

The following proposition shows how the discrete Wigner distribution is related to the continuous Wigner distribution.

**Proposition 5.1**

$$W_{\boldsymbol{x},d}(\boldsymbol{w}) = \sum_{r_1,r_2=-\infty}^{+\infty} W_{\boldsymbol{x}}\left(w_1 - r_1\frac{\pi}{T_1}, w_2 - r_2\frac{\pi}{T_2}\right) \tag{5.9}$$

Proof)

By the sampling theorem for 2D signals ([88]), the Fourier transform $G_{\boldsymbol{x},d}(\boldsymbol{\theta})$ of the discrete signal $g_{x,d}(\boldsymbol{k})$ is related to the Fourier transform $G_{\boldsymbol{x}}(\boldsymbol{w})$ of the continuous signal $g_x(s_1,s_2)$ by:

$$G_{x,d}(\theta_1,\theta_2) = \frac{1}{T_1T_2}\sum_{r_1,r_2=-\infty}^{+\infty} G_x\left(w_1 - r_1\frac{2\pi}{T_1}, w_2 - r_2\frac{2\pi}{T_2}\right)$$

with $\theta_i = w_iT_i$, $i = 1,2$. Thus

$$G_{x,d}(2\theta_1, 2\theta_2) = \frac{1}{T_1 T_2} \sum_{r_1, r_2 = -\infty}^{+\infty} G_x \left[ 2\left(w_1 - r_1 \frac{\pi}{T_1}\right), 2\left(w_2 - r_2 \frac{\pi}{T_2}\right) \right]$$

Substituting the above expression into (5.7) gives

$$W_{\boldsymbol{x},d}(\boldsymbol{w}) = 4 \sum_{r_1, r_2 = -\infty}^{+\infty} G_x \left[ 2\left(w_1 - r_1 \frac{\pi}{T_1}\right), 2\left(w_2 - r_2 \frac{\pi}{T_2}\right) \right]$$

Using (5.3) for each term in the above summation gives (5.9). QED

The above proposition says that $W_{\boldsymbol{x},d}(\boldsymbol{w})$ is the sum of $W_{\boldsymbol{x}}(\boldsymbol{w})$ and all its duplicates shifted by $\left(r_1 \frac{\pi}{T_1}, r_2 \frac{\pi}{T_2}\right)$, where $r_1, r_2$ are integers. The domains of all the duplicates are the domain D of $W_{\boldsymbol{x}}(\boldsymbol{w})$ shifted by $\left(r_1 \frac{\pi}{T_1}, r_2 \frac{\pi}{T_2}\right)$.

For $W_{\boldsymbol{x},d}(\boldsymbol{w})$ to be equal to $W_{\boldsymbol{x}}(\boldsymbol{w})$ for some $\boldsymbol{w}$, then the domain $D_1$ of $W_{\boldsymbol{x},d}(\boldsymbol{w})$ must be a subset of the domain $D$ of $W_{\boldsymbol{x}}(\boldsymbol{w})$. But $D$ overlaps with the domains of $W_{\boldsymbol{x}}\left(w_1 - r_1 \frac{\pi}{T_1}, w_2 - r_2 \frac{\pi}{T_2}\right)$ where $r_1, r_1 = -1,0,1$. Thus

$$W_{\boldsymbol{x},d}(\boldsymbol{w}) = \sum_{k_1, k_2 = -1,0,1} W_{\boldsymbol{x}}\left(w_1 - r_1 \frac{\pi}{T_1}, w_2 - r_2 \frac{\pi}{T_2}\right) \tag{5.10}$$

Thus $W_{\boldsymbol{x},d}(\boldsymbol{w})$ is not related to $W_{\boldsymbol{x}}(\boldsymbol{w})$ alone, and thus cannot be used to determine $W_{\boldsymbol{x}}(\boldsymbol{w})$ , for $\boldsymbol{w}$ in $D_1$. This is called the aliasing problem.

### 5.3.2 Discrete Wigner Distribution of Oversampled Sample

To correct the aliasing problem, it has been suggested to sample the underlying signal by a factor of at least 2 ([76][43]). The following shows how using oversampling will overcome the aliasing problem for 2-dimensional Wigner distributions.

**Definition**

The integral defining the Wigner distribution $W_{\boldsymbol{x}}(\boldsymbol{w})$ can now be approximated by the trapezoidal rule with intervals $\frac{T_1}{2}$ and $\frac{T_2}{2}$ to define the *discrete Wigner distributio*n as:

$$W_{\boldsymbol{x},d}(\boldsymbol{w}) = 4\frac{T_1}{2}\frac{T_2}{2}\sum_{k_1,k_2=-\infty}^{+\infty} g_{\boldsymbol{x}}(\boldsymbol{k})exp\left\{-2j\left(k_1 w_1 \frac{T_1}{2} + k_2 w_2 \frac{T_2}{2}\right)\right\} \tag{5.11}$$

Thus

$$W_{\boldsymbol{x},d}(\boldsymbol{w}) = T_1 T_2 \sum_{k_1,k_2=-\infty}^{+\infty} g_{\boldsymbol{x},d}(\boldsymbol{k})exp\{-j(k_1 w_1 T_1 + k_2 w_2 T_2)\} \tag{5.12}$$

Where $\boldsymbol{x} = (m_1 T_1, m_2 T_2)'$, and

$$g_{x,d}(\boldsymbol{k}) = f\left(x_1 + k_1\frac{T_1}{2}, x_2 + k_2\frac{T_2}{2}\right) f\left(x_1 - k_1\frac{T_1}{2}, x_2 - k_2\frac{T_2}{2}\right) \tag{5.13}$$

Thus

$$W_{\boldsymbol{x},d}(\boldsymbol{w}) = T_1 T_2 G_{\boldsymbol{x},d}(\theta_1, \theta_2) \tag{5.14}$$

Where $G_{\boldsymbol{x},d}(\theta_1, \theta_2)$ is the Fourier transform of $g_{x,d}(\boldsymbol{k})$ , and $\theta_i = w_i T_i$ , $i = 1,2$.

It is clear that $G_{\boldsymbol{x},d}(\theta_1, \theta_2)$ is periodic with period $(2\pi, 2\pi)$, thus $W_{\boldsymbol{x},d}(\boldsymbol{w})$ is periodic with period $\left(\frac{2\pi}{T_1}, \frac{2\pi}{T_2}\right)$.

Using intervals $\frac{T_1}{2}$ and $\frac{T_2}{2}$ instead of $T_1, T_2$ in the trapezoidal rule to define the discrete Wigner distribution as in (5.12) is said to oversample by a factor of 2.

The following proposition shows how the discrete Wigner distribution is related to the continuous Wigner distribution, in the case of oversampling by a factor of 2.

**Proposition 5.2**

$$W_{\boldsymbol{x},d}(\boldsymbol{w}) = \sum_{r_1,r_2=-\infty}^{+\infty} W_{\boldsymbol{x}}\left(w_1 - r_1\frac{2\pi}{T_1}, w_2 - r_2\frac{2\pi}{T_2}\right) \tag{5.15}$$

Proof)

The sampling theorem for 2D signal applied to $G_{\boldsymbol{x},d}(\boldsymbol{w})$ and $G_{\boldsymbol{x}}(\boldsymbol{w})$ with $T_i^{'} = \frac{T_i}{2}$, $i = 1,2$, gives:

$$G_{x,d}(\theta_1, \theta_2) = \frac{1}{T_1^{'}T_2^{'}} \sum_{r_1,r_2=-\infty}^{+\infty} G_x\left(w_1^{'} - r_1\frac{2\pi}{T_1^{'}}, w_2^{'} - r_2\frac{2\pi}{T_2^{'}}\right)$$

with $\theta_i = w_i^{'}T_i^{'}$, $i = 1, 2$. But $\theta_i = w_i T_i$ , thus $w_i^{'} = 2w_i$.

$$G_{x,d}(\theta_1, \theta_2) = \frac{4}{T_1T_2} \sum_{r_1,r_2=-\infty}^{+\infty} G_x\left[2\left(w_1 - r_1\frac{2\pi}{T_1}\right), 2\left(w_2 - r_2\frac{2\pi}{T_2}\right)\right]$$

Substituting the above into (5.14) gives:

$$W_{x,d}(\boldsymbol{w}) = 4\sum_{r_1,r_2=-\infty}^{+\infty} G_x\left[2\left(w_1 - r_1\frac{2\pi}{T_1}\right), 2\left(w_2 - r_2\frac{2\pi}{T_2}\right)\right]$$

Using (5.3) for each term in the above summation gives (5.15). QED

In (5.15), the domains of $W_{\boldsymbol{x}}\left(w_1 - r_1\frac{2\pi}{T_1}, w_2 - r_2\frac{2\pi}{T_2}\right)$'s are $D$ shifted by $\left(r_1\frac{2\pi}{T_1}, r_2\frac{2\pi}{T_2}\right)$, and they don't overlap because they they are shifted by a multiple of $\frac{2\pi}{T_1}$ and $\frac{2\pi}{T_2}$ in each direction $x_1$ and $x_2$. Thus for all $\boldsymbol{w}$ in $D$:

$$W_{\boldsymbol{x},d}(\boldsymbol{w}) = W_{\boldsymbol{x}}(\boldsymbol{w}) \tag{5.16}$$

This means $W_{\boldsymbol{x},d}(\boldsymbol{w})$ as calculated by the trapezoidal rule in (5.12) is exactly $W_{\boldsymbol{x}}(\boldsymbol{w})$ in their same domain $D$. There is no aliasing in this case.

### 5.3.3 Calculation of Discrete Wigner Distributions

In ([43]) a procedure has derived to calculate the discrete Wigner distributions for the one-dimensional case using oversampling. The following is a generalization of this procedure to the two-dimensional case.

Let $f_1(x_1, x_2)$ of an image of size $M_1 \times M_2$ whose discrete Wigner distribution needs to be calculated. Let $f(x_1, x_2)$ be an image of size $(2M_1 - 1) \times (2M_2 - 1)$ interpolated from image $f_1(x_1, x_2)$, and $L = \max(M_1, M_2)$. Image $f(x_1, x_2)$ is considered as a sample image oversampled by a factor of 2.

Then $g_{\boldsymbol{x}}(k_1, k_2) = 0$ as defined in (5.6) for $|k_1| \geq L$ or $|k_2| \geq L$. The Fourier transform $G_{\boldsymbol{x},d}(\theta_1, \theta_2)$ of $g_{\boldsymbol{x}}(k_1, k_2)$ can be written as:

$$G_{\boldsymbol{x},d}(\theta_1,\theta_2) = \sum_{k_1,k_2=-(L-1)}^{L-1} g_{\boldsymbol{x}}(\boldsymbol{k}) exp\{-j(k_1\theta_1 + k_2\theta_2)\} \tag{5.17}$$

The above form of Fourier transform can't be calculated directly by the discrete Fourier transform as implemented in Matlab. The reason is that $k_1$, $k_2$ start from the negative number $-(L-1)$, while the discrete Fourier transform as implemented in Matlab having $k_1$, $k_2$ start from 0.

A way to compute $G_{\boldsymbol{x},d}(\theta_1,\theta_2)$ is to re-arrange the above sum so that discrete Fourier transforms as implemented in Matlab can be used.

**Proposition 5.3**

The Fourier transform $G_{\boldsymbol{x},d}(\theta_1,\theta_2)$ can be rewritten as:

$$G_{\boldsymbol{x},d}(\theta_1,\theta_2) = A + B + C + D + E \tag{5.18}$$

where

$$\begin{aligned} A &= \sum_{k_2=0}^{L-1}\sum_{k_1=0}^{L-1} g_{\boldsymbol{x}}(\boldsymbol{k}) exp\{-j(k_1\theta_1 + k_2\theta_2)\} \\ &\quad + \sum_{k_2=-(L-1)}^{0}\sum_{k_1=-(L-1)}^{0} g_{\boldsymbol{x}}(\boldsymbol{k}) exp\{-j(k_1\theta_1 + k_2\theta_2)\} \\ &= 2\,Real\left\{\sum_{k_2=0}^{L-1}\sum_{k_1=0}^{L-1} g_{\boldsymbol{x}}(\boldsymbol{k}) exp\{-j(k_1\theta_1 + k_2\theta_2)\}\right\} \end{aligned} \tag{5.19}$$

$$B = \sum_{k_2=0}^{L-1} \sum_{k_1=-(L-1)}^{0} g_{\boldsymbol{x}}(\boldsymbol{k}) exp\{-j(k_1\theta_1 + k_2\theta_2)\} + \sum_{k_2=-(L-1)}^{0} \sum_{k_1=0}^{-(L-1)} g_{\boldsymbol{x}}(\boldsymbol{k}) exp\{-j(k_1\theta_1 + k_2\theta_2)\}$$

$$= 2\, Real\left\{ \sum_{k_2=0}^{L-1} \sum_{k_1=0}^{L-1} g_{\boldsymbol{x}}(-k_1, k_2) exp\{jk_1\theta_1 - k_2\theta_2\} \right\} \quad (5.20)$$

$$C = -\left[ \sum_{k_1=0}^{L-1} g_{\boldsymbol{x}}(k_1, 0) exp\{-jk_1\theta_1\} + \sum_{k_1=-(L-1)}^{0} g_{\boldsymbol{x}}(k_1, 0) exp\{-jk_1\theta_1\} \right]$$

$$= -2Real\left\{ \sum_{k_1=0}^{L-1} g_{\boldsymbol{x}}(k_1, 0) exp\{-jk_1\theta_1\} \right\} \quad (5.21)$$

$$D = -\left[ \sum_{k_2=0}^{L-1} g_{\boldsymbol{x}}(0, k_2) exp\{-jk_2\theta_2\} + \sum_{k_2=-(L-1)}^{0} g_{\boldsymbol{x}}(0, k_2) exp\{-jk_2\theta_2\} \right]$$

$$= -2Real\left\{ \sum_{k_1=0}^{L-1} g_{\boldsymbol{x}}(0, k_2) exp\{-jk_2\theta_2\} \right\} \quad (5.22)$$

$$E = g_{\boldsymbol{x}}(0,0) \quad (5.23)$$

Proof)

Expression (5.17) can be re-arranged into (5.18) using the fact that

$g_{\boldsymbol{x}}(k_1, k_2) = g_{\boldsymbol{x}}(-k_1, -k_2)$. QED

Using (5.18)-(5.23), the discrete Wigner distribution as defined and derived in (5.11)-(5.14) can be determined, provided $T_1, T_2$ are known.

### 5.3.4 Actual Domain of Frequencies Used for Calculations of Wigner Distributions

Assume that the image is sampled at the Nyquist rate with $T_1 = T_2 = 1$. From (5.18)-(5.23), discrete Fourier transforms (DFT) can be used to obtain $G_{\boldsymbol{x},d}(\theta_1, \theta_2)$ with the domain for $\theta_1$ and $\theta_2$ being $(0, 2\pi)$. To obtain the discrete Wigner distribution $W_{\boldsymbol{x},d}(\boldsymbol{w})$, as derived in (5.14), the periodicity of DFT can be used to arrive at $G_{\boldsymbol{x},d}(\theta_1, \theta_2)$ for $\theta_1$ and $\theta_2$ in $(-\pi, \pi)$.

This approach, however, creates a problem of interference between the negative and positive values of $\theta_i$, $i = 1,2$. And it makes many moments involving odd order of the frequency coordinates zero and makes many AWMI zero and thus useless. To side step this problem, the domain of $\theta_1$ and $\theta_2$ is now taken to be $(0,2\pi)$ and $W_{\boldsymbol{x},d}(\boldsymbol{w})$ can be re-defined accordingly using the periodicity of $G_{\boldsymbol{x},d}(\theta_1, \theta_2)$.

A choice of a finite domain for the frequency variables however creates a problem for calculating the 9 AWMI involving frequency variables. It is because when an affine transform between two images is unknown, then it is not possible to determine the transform of the domain of the frequency variables, so these 9 AWMI cannot be calculated and compared. A more detailed explanation of this issue is reported later in Experiment 2 in this chapter. Further research is needed to solve this problem.

## 5.4 Experiment 1

A simulated experiment has been carried out in Matlab to verify empirically if the AWMI as derived in this dissertation are indeed affine invariant, and if they are, then compare their performance in extracting texture information with that of AMI . Thus besides the derived AWMI were to be calculated, the four AMI listed as $\psi_1, \psi_2, \psi_3, \psi_4$ in ([4]) were also to be calculated and compared.

An input image was read in and resized to an image considered as an oversampled image. This oversampled image was then sampled by a factor of two to arrive at an image of interest whose AMI and AWMI were to be calculated. The four AMI were calculated according to the formulas given in ([4]) as reproduced in (2.34)-(2.37). Discrete Wigner distributions $W_{\boldsymbol{x},d}(\boldsymbol{w})$ were calculated at every $\boldsymbol{x}$ using the domain $(0,2\pi)$. The AWMI as derived in this dissertation were then calculated.

The calculations of AMI and AWMI for the affine transformed version of the image of interest were carried out as follows.

Call the image of interest $f'(\boldsymbol{x}')$. Its affine transformed image $f(\boldsymbol{x})$ was generated by calculating $\boldsymbol{x} = A\boldsymbol{x}' + \boldsymbol{b}$, and setting $f(\boldsymbol{x}) = f'(\boldsymbol{x}')$. Here matrix $A$ and vector $\boldsymbol{b}$ represent the affine transform. The four AMI for $f(\boldsymbol{x})$ were then calculated. And for AWMI, let $W'_{\boldsymbol{x}',d}(\boldsymbol{w}')$ be the Wigner distribution of $'(\boldsymbol{x}')$ , $W_{\boldsymbol{x},d}(\boldsymbol{w})$ was generated by: for $\boldsymbol{x}$ with $\boldsymbol{x} = A\boldsymbol{x}' + \boldsymbol{b}$ calculating $\boldsymbol{w} = M\boldsymbol{w}'$ and setting $W_{\boldsymbol{x},\boldsymbol{d}}(\boldsymbol{w}) = W'_{\boldsymbol{x}',d}(\boldsymbol{w}')$. Here $M = (A^T)^{-1}$. The AWMI for $f(\boldsymbol{x})$ were then calculated.

Each AMI of $f(\boldsymbol{x})$ was adjusted by dividing itself by $|detA|^w$. Similarly for each AWMI. Here $w$ is the weight of the algebraic invariant used to determine the associated AMI or AWMI. The adjustment is necessary because $f(\boldsymbol{x})$ was discretely generated, not a continuous function.

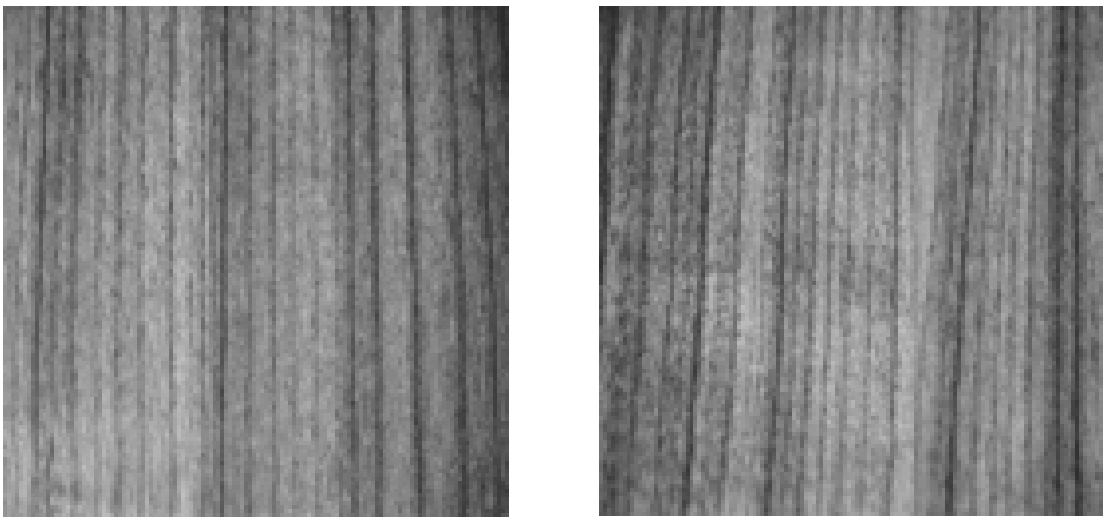

Figure 5.1: Input images T04_17 (Left) and T04_18 (Right)

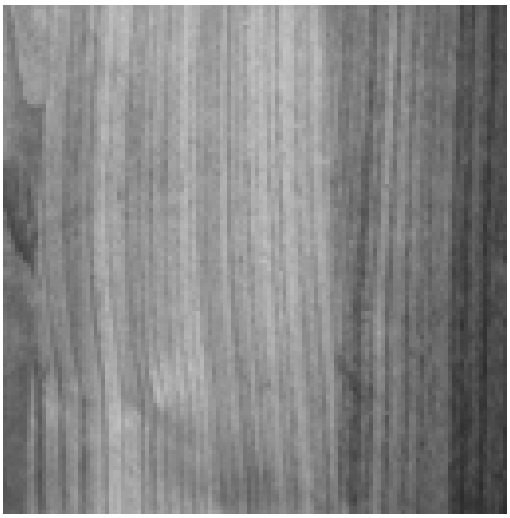

Figure 5.2: Input image T04_03

Three images from the UIUC database of texture images were used in this experiment. These are images T04_17, T04_18 and T04_03 as displayed in Figure 5.1 and Figure 5,2. The three input images were resized so that the images of interest were all of size 65x65. The AMI and AWMI for all three images have been calculated and displayed in Table 1 under the big column labeled *Invariants*. $\psi_1$ and $\psi_2$ are two AMI for the binary quadratic and cubic forms. $\psi_3$ and $\psi_4$ are the two AMI simultaneoulsy for the quadratic and cubic forms. The AWMI in Table 1 were calculated using formulas given in the Appendix. Here the AMI and AWMI for an image were calculated with the origin being the centroid of the image and Wigner distribution respectively.

All the AMI and the AWMI for the three images as calculated here were indeed affine invariant in this experiment. This means the AMI and AWMI of the images of interest were calculated and turned out exactly the same as the AMI and AWMI of the affine transformed versions of these images. Different matrices and translation vectors defining different affine transformations have been tried, and the calculated AMI and AWMI remained unchanged.

Table 1. AMI and AWMI and their rates of change

| | | Invariants | | | Rates of Change | |
|---|---|---|---|---|---|---|
| | | T04_17 | T04_18 | T04_03 | T04_18 | T04_03 |
| AMI | $\psi_1$ | 5.08E-02 | 5.11E-02 | 4.37E-02 | 0.01 | 0.14 |
| | $\psi_2$ | 8.30E-10 | 3.11E-11 | -9.34E-11 | 0.96 | 1.11 |
| | $\psi_3$ | -6.69E-07 | 7.50E-07 | -2.46E-06 | 2.12 | 2.68 |
| | $\psi_4$ | 4.17E-07 | 5.71E-08 | 4.90E-07 | 0.86 | 0.17 |
| AWMI | $Ip_0$ | 9.60E-09 | 9.73E-09 | 6.88E-09 | 0.01 | 0.28 |
| | $Ip_1$ | -7.52E+00 | -9.89E-01 | -2.13E+00 | 0.87 | 0.72 |
| | $Ip_2$ | -2.70E+15 | -2.53E+15 | -2.89E+15 | 0.06 | 0.07 |
| | $Iq_0$ | 1.62E-29 | 3.77E-30 | 2.47E-29 | 0.77 | 0.52 |
| | $Iq_1$ | -7.56E+11 | -6.38E+11 | -3.94E+11 | 0.16 | 0.48 |
| | $Iq_2$ | -3.27E+31 | -3.33E+29 | 5.49E+31 | 0.99 | 2.68 |
| | $Iq_3$ | 3.28E+47 | 2.70E+47 | 5.74E+47 | 0.18 | 0.75 |
| | $I_{00}$ | -1.62E-19 | 2.32E-21 | -1.09E-19 | 1.01 | 0.33 |
| | $J_{00}$ | 8.35E-27 | 1.27E-27 | 5.75E-27 | 0.85 | 0.31 |
| | $I_{23}$ | -1.52E+31 | -1.33E+31 | -2.26E+31 | 0.13 | 0.49 |
| | $J_{23}$ | 1.57E+46 | 1.21E+46 | 2.56E+46 | 0.23 | 0.63 |
| | $I_{11}$ | 6.01E-02 | 6.46E-01 | 5.78E-01 | 9.75 | 8.62 |
| | $I_{12}$ | 4.05E+08 | 5.59E+08 | 5.99E+09 | 0.38 | 13.79 |

Now the performance of AMI and AWMI can be compared. It has been expected from the introduction of this paper that AMWI can be used to better capture image texture than can AMI. The reason is that some AWMI are defined on both the space and spatial frequency coordinates through Wigner distributions while AMI are defined on the space coordinates only. And Wigner distributions have been used to analyze images' texture, so AWMI could be.

To compare the performance, the rates of change of each AMI and AWMI for images T04_18 and image T04_03 with respect to image T04_17 were calculated. For example for Ip0, from Table 1, the rate of change for T04_18 with respect to T04_17 is (9.73E-09-9.60E-09)/9.60E-09 = 0.01. Columns T04_08 and T04_3 under the big column labelled *Rates of Change* of Table 1 give all the rates of change for all invariants for images T04_18 and T04_03.

It can be seen that for T04_18 the largest rate of change among the AMI belongs to $\psi_3$ with value 2.12. The largest rate of change among the AWMI belongs to $I_{11}$ with value of 9.75. Thus an AWMI that is $I_{11}$ differentiates T04_18 better than any of the four AMI. $I_{11}$ as seen from the Appendix involves moments of both the space coordinates and first order spatial frequency coordinates. Thus the spatial frequencies help in differentiating the textured image T04_18 from T04_17.

Also for T04_03 the largest rate of change among AMI belongs also to $\psi_3$ with value 2.68. The largest rate of change among AWMI are $I_{12}$ with value of 13.79 and $I_{11}$ with value of 8.62. Thus some AWMI differentiate T04_03 from T04_17 better than all AMI. $I_{12}$ and $I_{11}$ involve the moments of both the space and spatial frequency coordinates. Thus the spatial frequencies again help in differentiating the textured images.

Thus in this experiment, the AWMI derived in this paper capture the texture information better than all the four AMI being considered.

## 5.5 Experiment 2

The last experiment was basically a simulation to verify if the derived AWMI are indeed affine invariant. The calculations of AWMI were based on the assumption that the affine transformation transforming one image to another is known. The result of the experiment was that AWMI are indeed affine invariant as theoretically shown in this dissertation.

In this experiment, called experiment 2, the invariance of AWMI was tested for the case when the affine transformation transforming one image to another is unknown.

This experiment 2 was formulated as an image retrieval problem of retrieving images similar to query images. Here two images are similar to each other if one image is an affine transform of the other.

For any image, its AWMI will be calculated and the Euclidean distance between certain logarithms of AWMI will be used for retrieval, with smaller distances to mean images being more similar.

For comparisons, AMI and the SIFT method will also be tested in this image retrieval experiment. The performance of AWMI, AMI, SIFT methods will be compared based on their precision-recall curves. The more the precision-recall curve is to the top right of the graph the better performance the underlying method has.

## An issue in calculating AWMI

Recall that AWMI as derived in this chapter 4 consists of 13 invariants, 9 of these invariants involve the moments of the frequency variables. The other 4 invariants involve only the spatial moments and are $Ip_0$, $Iq_0$, $Iq_0$, $I_{00}$ , and $J_{00}$.

In using AWMI for image retrieval there is a problem in calculating the 9 AWMI involving the moments of frequency variables. The problem is that in comparing AWMI for two images that are affine transform of each other, the domains for the frequency variables are unknown because the underlying affine transformation is unknown. The problem for the 9 AWMI involving the frequency variables can be seen clearly as follows:

For two images $f'(\boldsymbol{x}')$ and $f(\boldsymbol{x})$, with $f(\boldsymbol{x})$ being an affine transform of $f'(\boldsymbol{x}')$, then as in (3.11) but for the case of discrete variables, the two Wigner distributions are the same for related spatial and frequency variables:

if $\boldsymbol{x} = A\boldsymbol{x}'$, and $\boldsymbol{w} = M\boldsymbol{w}'$, then $W'(\boldsymbol{x}', \boldsymbol{w}') = W(\boldsymbol{x}, \boldsymbol{w})$.

Here the 2x2 non-singular matrix $A$ represents the affine transformation, and matrix $M = (A^T)^{-1}$. The AWMI for $f'(\boldsymbol{x}')$ and $f(\boldsymbol{x})$ are calculated based on the moments of Wigner distributions $W'(\boldsymbol{x}', \boldsymbol{w}')$ and $W(\boldsymbol{x}, \boldsymbol{w})$.

The domain for the spatial variable $\boldsymbol{x}'$ is given in image $f'(\boldsymbol{x}')$. This means $\boldsymbol{x}'$ is in rectangle $(L'_1, H'_1) \times (L'_2, H'_2)$, where the four number $L'_1, H'_1$, and $L'_2, H'_2$ are known.

The domain for the frequency variable $\boldsymbol{w}'$ in calculating the 9 AWMI involving frequency variables is taken to be $(0,2\pi) \times (0,2\pi)$ in this dissertation.

Thus with the domains for $\boldsymbol{x}'$ and $\boldsymbol{w}'$ are known, the moments of $W'(\boldsymbol{x}', \boldsymbol{w}')$ can be calculated, so can the AWMI for image $f'(\boldsymbol{x}')$.

Now there is a problem calculating AWMI for $f(\boldsymbol{x})$, which is an affine transform of $f'(\boldsymbol{x}')$.

The domain for the spatial variable $\boldsymbol{x}$ is given in image $f(\boldsymbol{x})$. This means $\boldsymbol{x}$ is in $(L_1, H_1) \times (L_2, H_2)$, where the four number $L_1, H_1$, and $L_2, H_2$ are known, and $(L_1, H_1) \times (L_2, H_2)$ is an affine transform of $\left(L_1', H_1'\right) \times \left(L_2', H_2'\right)$.

But the domain for $\boldsymbol{w}$ is not given in image $f(\boldsymbol{x})$, and must be inferred from matrix $A$ that represents the transformation transforming $f'(\boldsymbol{x}')$ to $f(\boldsymbol{x})$.

Thus in order to calculate the 9 AWMI involving frequency variables for image $f(\boldsymbol{x})$, matrix $A$ needs to be estimated from the two images $f'(\boldsymbol{x'})$ to $f(\boldsymbol{x})$. This estimation could leads to a less accurate AWMI for image $f(\boldsymbol{x})$.

For the 4 AWMI not involving the frequency variables, the domain for the frequency variables for image $f'(\boldsymbol{x}')$ can be taken to be $(0,2\pi) \times (0,2\pi)$ so that the domain for the frequency variables for image $f(\boldsymbol{x})$ needs to be estimated as above.

But a more convenient approach is to consider the domain for the frequency variables for both images $f'(\boldsymbol{x})$ and $f(\boldsymbol{x})$ to be the whole plane. Then from (3.9) it can be seen that these 4 AWMI are just 4 AMI for image $f^2(\boldsymbol{x})$.

AMI for any image $f(\boldsymbol{x})$ do not have this issue of unknown domain because only the spatial variables are used and are readily available in the given images.

## Retrieving Images

Images from the benchmark database of images VOC 2011 will be used for this experiment. A number of images from the database of images were selected and resized to 95x95 gray scale images. Each of these gray scale images was then randomly affine transformed to a random number of images. Altogether there are 200 gray scale images that were randomly affine transformed from a number of images selected from the VOC 2011 database. Thus, each image $I$ from the 200 images has a random number of other images that are affine transforms of $I$. These 200 images serve as a database of images for this image retrieval experiment.

In Figure 5.3, in the left is an image from the VOC 2011 database, in the right is a gray scale image obtained from an affine transformation of the image in the left. The image in the right is also an image in the database of 200 images used for this image retrieval experiment.

Figure 5.4 shows two other images in the database of 200 images used for this image retrieval experiment.

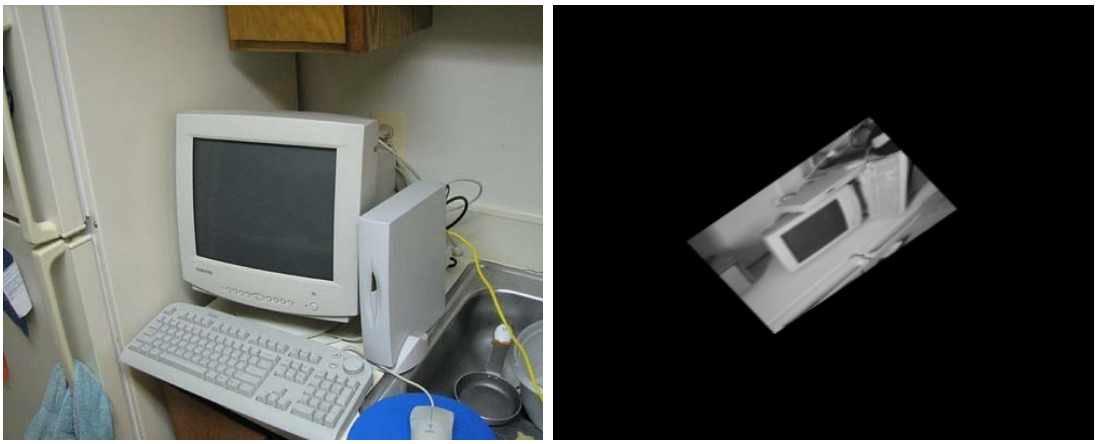

Figure 5.3: An image from the VOC 2011 database (Left), and its affine transformed image in grayscale (Right)

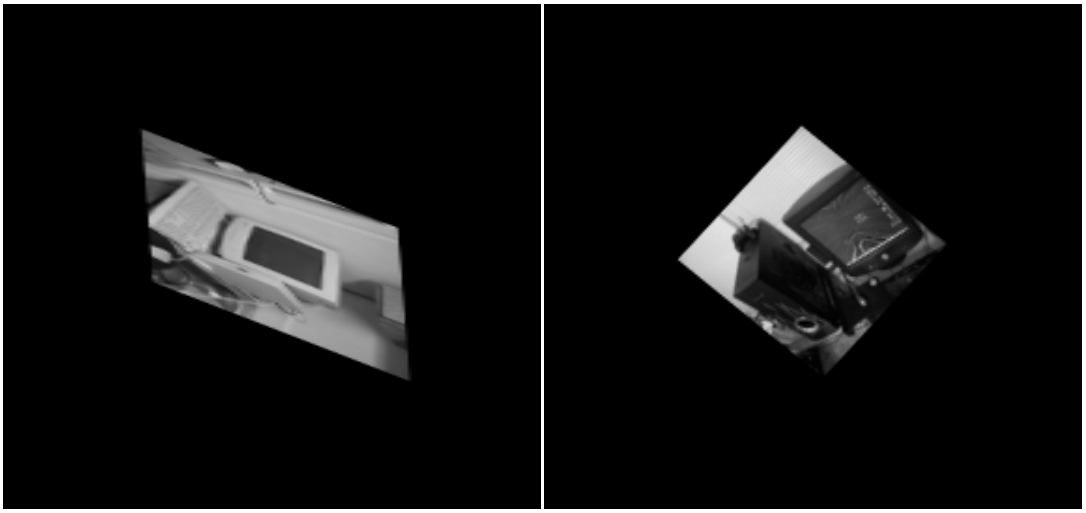

Figure 5.4: Two gray scale images in the database of 200 images used in the experiment

To test the AWMI, AMI, and SIFT methods on image retrieval, arbirary image $Q$ called query image is taken from the database $DB$ of 200 images, then retrieve images from $DB$ that are similar to $Q$, where similarity was defined differently for different methods.

For AMI, let $A_Q$ be the AMI calculated for the query image $Q$, and let $A_I$ be the AMI for any image $I$ in $DB$ . Let the similarity between $Q$ and $I$ be the Euclidean distance between $log\left[1 + abs\left(A_Q\right)\right]$ and $log[1 + abs(A_I)]$. The smaller the distance, the more similar image $Q$ and image $I$ is considered to be. Images in $DB$ with small distances in generall will be retrieved and considered as similar to $Q$. The specific form of logarithm as mentioned was used to bring all the four AMI into smaller and positive scale.

For the SIFT method, some trials and errors have been tested to find how best to retrieve images. A simple measure of defining similarity between two images is chosen to be the number of matched points, and is used in this experiment. Images $I$ in $DB$ that has high number of matched points with $Q$ in general will be retrieved.

For AWMI, image $Q$ and any image $I$ in $DB$ are considered similar if the AWMI for $Q$ and $I$ are close together, where $I$ could be an affine transform of $Q$. Since for the 9 AWMI involving frequency variables for $Q$ and $I$ cannot be compared when the affine transform is not known as explained earlier, some way to estimate the affine trasnform between any $Q$ and $I$ is necessary.

Assume that an affine transform represented by a 2x2 matrix $A$ transforming $I$ to $Q$ can be estimated by some method. Note that, as explained in chapter 3, an affine transform for the purpose of calculating AWMI and AMI can be represented by a 2x2 matrix $A$, without the need of a 2x1 translation vector.

Given the matrix $A$, the AWMI for $Q$ and $I$ can be computed and compared. The domain for the frequency variables in the case of $Q$ could be taken to be $(0,2\pi) \times (0,2\pi)$, and the domain for the frequency variables in the case of $I$ is the affine transform of $(0,2\pi) \times (0,2\pi)$ under the matrix $A^{-1}$.

The Euclidean distance of logarithm of the same form as used for AMI above can be used for AWMI. Image $I$ in the $DB$ having smaller distance is considered to be similar to $Q$ and in general will be retrieved.

For simple calculations, however, the AWMI for $I$ under the transformation represented by $A$ are not calculated. Instead, matrix A is used to transform $I$ to another image $I'$ and the AWMI of $I'$ are calculated. The rationale for doing this is that under matrix $A$ the domain of the frequency variables for $I'$ is the affine transform of the domain of the frequency variables for $I$, and thus is $(0,2\pi) \times (0,2\pi)$. Thus $I'$ and $Q$ have the same domain for the frequency variables and thus make the calculations less complicated.

Also for the testing purpose, the 4 AWMI involving only the spatial variables will also be calculated using the domain $(0,2\pi) \times (0,2\pi)$ when calculating the Wigner distributions. Because, if using the whole plane as the domain for the frequency variables, these four AWMI become four AMI for $f^2(\boldsymbol{x})$, thus do not serve the testing purpose.

## Estimation of Affine Transformations

As explained above, computing AWMI to check if one image is an affine transform of another image requires an estimation of the affine transformation that transforms the images. For the purpose of calculating AWMI, an affine transform is represented by a non-singular 2x2 matrix $A$.

The database of 200 images considered in this experiment consists of images like the two images in Figure 5.4. Each of the images is a parallelogram within a black background, and the parallelogram has four corners that could be exploited for the estimation. Indeed, affine transformations map corners and straight lines in one image to corners and straight lines in another image. Thus the existence of these four corners in an image suggests the following method of estimation of the matrix $A$ transforming one image to another.

Given any two images $Q$ and $I$, a matrix $A$ needs to be estimated to transform $I$ to $Q$.

- Use Harris detector algorithm to detect four points of interest in $I$ and four points of interest in $Q$. It is assumed that matrix $A$ will map the four points in $I$ to

four points in $Q$. It is expected that the four detected points are the four corners, and thus the four corners in one image will be mapped into the four corners of the other image.

- Assume the detected four points in each image are the corners, then number them 1, 2, 3, 4 such that edges (1,2), (2,3), (3,4), and (1,4) are the four edges of the parallelogram.

- For each edge of the parallelogram in image $I$, consider the two end points of that edge. There is always an affine transform that maps these two end points to the two other end points of an edge in image $Q$. The matrix $A$ consisting of four unknown entries will then be estimated from the correspondence $\boldsymbol{x} = A\boldsymbol{x}'$, where end point $\boldsymbol{x}'$ in $I$ is mapped to end point $\boldsymbol{x}$ in $Q$.

- The last step generates many affine transformations from $I$ to $Q$. Select the affine transformation that has the highest mutual information measure. For each affine transformation, let $I'$ be the transform of $I$. The mutual information essentially measures the degree of agreement between $I'$ and $Q$. Mutual information was found to be effective in assessing the estimation of affine transformations ([89]) .

## Results

The calculations of AWMI depend on the calculations of Wigner distributions, which have a number of issues such as cross interference term, efficiency, and stability. There are a number of methods proposed in the literature to deal with these issues ([90], [91], [92]). In this experiment, the calculation of Wigner distribution is based on the derivation given in section 5.3, which doesn't address all the computing issues and thus may not produce very accurate and fast AWMI.

The following are some sample results of the distances between a query image and two other images, with one image being an affine transform of the query and the other image not.

Table 2: Sample Results of Calculations

| | AWMI | AMI | Number of SIFT matches |
|---|---|---|---|
| Image is an affine transform of the query | 1.4599 | .0001 | 9 |
| Image is not an affine transform of the query | 7.1011 | .0812 | 7 |

The results as displayed in Table 2 mean the following.

For the image that is an affine transform of the query image, the distance between the two sets of AWMI for the two images is 1.4599, the distance between the two sets of AMI is .0001, and the number of matched points between two images is 9.

For the image that is not an affine transform of the query image, the distance between the two sets of AWMI for the two images is 7.1011, the distance between the two sets of AMI is .0812, and the number of matched points between two images is 7.

In this example, the AWMI and AMI distances for the case one image is an affine transform of the other are smaller than those for the case one image is not an affine transform of the other. Similarly, the number of matched points obtained from the SIFT method are larger for the case of affine transform than those for the case of not affine transform.

To assess the performance of these AWMI, AMI, and SIFT methods, the precision-recall curves for retrieving images under these methods were determined and compared.

The precision is a rate to measure the precision of the retrieval method, and was defined as the number of images retrieved and relevant over the number of images retrieved. The recall is a rate to measure the relevance of the retrieval method, and was defined as the number of images retrieved and relevant over the number of images relevant to the query.

By varying the percentage of number of images retrieved for each query image, different precision and recall rates could be obtained and used for constructing the precision-recall curve. Table 3 shows the average rates of precision and recall for different percentages of images retrieved from the database of 200 images.

Table 3: Average Precision and Recall Rates for AWMI, AMI, and SIFT methods

| | | 2% Retrieved | 5% Retrieved | 7.5% Retrieved | 10% Retrieved | 15% Retrieved |
|---|---|---|---|---|---|---|
| AWMI | Precision | 1.0 | .83 | .74 | .58 | .46 |
| | Recall | .28 | .55 | .66 | .77 | .88 |
| AMI | Precision | 1.0 | 1.0 | .93 | .77 | .52 |
| | Recall | .27 | .69 | .87 | .98 | 1.0 |
| SIFT | Precision | .92 | .63 | .48 | .38 | .29 |
| | Recall | .26 | .44 | .47 | .54 | .61 |

The precision and recall rates in Table 3 were used in constructing the precision-recall curves as displayed in Figure 5.5.

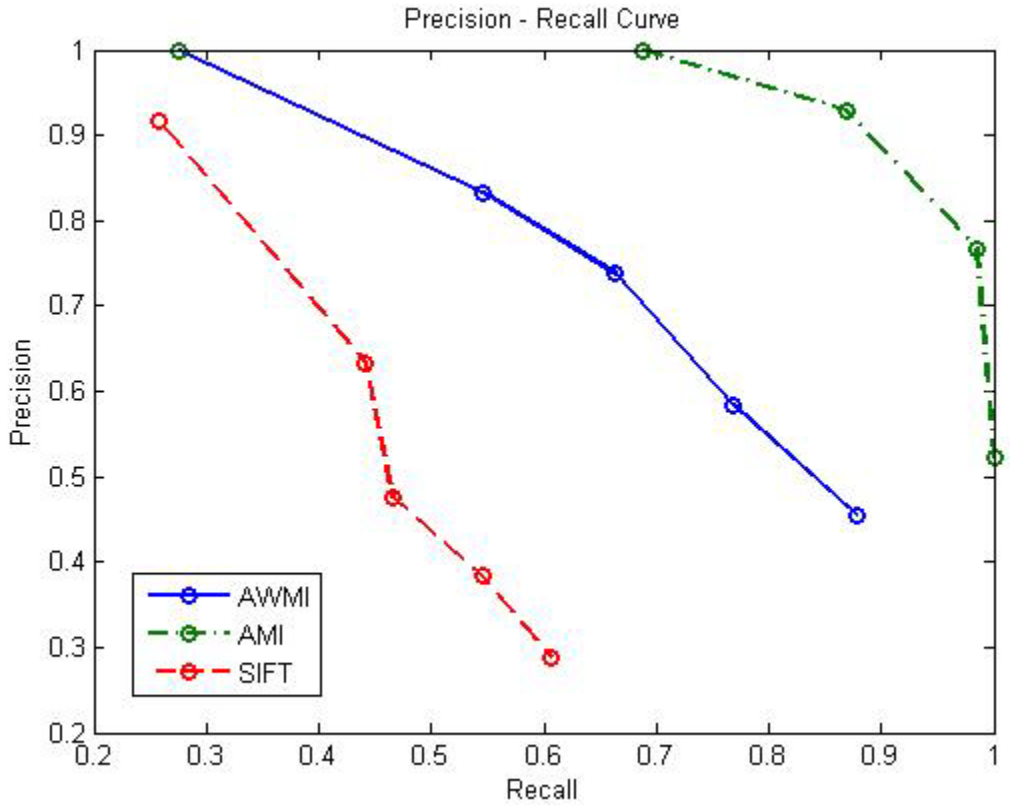


Figure 5.5: The Precision-Recall curves for the three methods AMI, AWMI, and SIFT methods used in image retrieval.

From Figure 5.5, the precision-recall curve for the AMI method is close to the top right corner of the coordinate system, the curve for the AWMI is in the middle, and the curve for the SIFT method is close to the bottom left corner.

These curves generally indicate that for a fixed recall rate, the precision rate for the SIFT method is lowest, the precision for the AWMI is the second lowest, and that for the AMI is highest.

Thus the AMI has the highest performance in image retrieval, then the AWMI second highest, and then the SIFT method.

The above is a tentative conclusion about the performance of the three methods in retrieving images.

There are a number of issues, however, related to these methods that should be considered and improved for better retrieval.

For the SIFT method:

- This method was used for retrieving affine transformed images in this experiment, but actually it was not fully affine invariant. A generalized version of SIFT called the ASIFT method was proposed in the literature and claimed to be affine invariant. Probably the ASIFT method could have a better performance in retrieving affine transformed images.

- The number of matched points between two images was used as a similarity measure in this experiment, after some trial and error evaluation. A better measure could be used to improve the performance of the precision-recall curves, but probably couldn't improve a lot to overtake the performance of AWMI.

For the AWMI method:

- This method depends on an accurate estimation of affine transformation in order to be able to calculate the 9 AWMI that involve the frequency variables. Inaccurate estimation of the transformation and also inaccurate calculations of Wigner distributions propagate to inaccurate calculations of AWMI, and thus could lower the retrieval performance.

- If using only 4 of the AWMI that involve only spatial variables, which are invariants $Ip_0, Iq_0, Iq_0, I_{00}$ , and $J_{00}$, then these AWMI are the AMI for image $f^2(\boldsymbol{x})$. This means AWMI are really AMI plus some more invariants involving the frequency variables.

For the AMI and AWMI method:

- A common problem of AMI and AWMI is that some invariants are large and some are small and thus in different scales. They need to be rescaled appropriately to obtain a meaningful distance. In this experiment, the logarithm

as described above was used but may not be the best rescaling approach. Mahalanobis distance was tried but didn't produce good answers, probably because these moment invariants are not linearly independent.

- It has been claimed that AMI are not linearly independent of each other ([1]). Much effort has been to modify and devise linearly independent invariants such as complex moment invariants. It would be interesting to see how the mechanism to derive invariants as described in this dissertation could be applied to different types of moment invariants.

# Chapter 6 – Conclusions

## 6.1 Summary of Dissertation Results

In this dissertation a method has been proposed to derive features from two dimensional images such that these features capture some shape and texture information of objects in images and they are invariant when the images are affine transformed. These features are called affine Wigner moment invariants (AWMI).

Results obtained in this dissertation have been presented in chapter 3, 4, and 5.

In chapter 3, AWMI have been defined and derived following the approach of deriving affine moment invariants (AMI) from algebraic invariants. Specifically, AWMI have been defined as functions of the moments of the Wigner distribution of the image's intensity. And a relationship has been derived relating a certain algebraic invariant to an AWMI. Based on this relationship, if the required algebraic invariant is known then an AWMI can be obtained.

The results presented in chapter 3 generalize the results obtained for AMI as presented in the literature ([5][4][1]). The generalization is possible thanks to the properties of Wigner distributions, and thanks to the general concepts of algebraic invariants defined not only for multivariate forms but also for multilinear forms.

In chapter 4, the algebraic invariants required for the calculations of AWMI have been derived, so that AWMI can be determined from these algebraic invariants based on the relationship established in chapter 3. A total of 13 AWMI have been determined.

The algebraic invariants derived in chapter 4 are not only for multivariate forms, but also for multilinear forms that are not multivariate forms. Compared to the approach of deriving AMI, the algebraic invariants required for the determination of AMI are just for the multivariate forms only. Thus deriving AWMI requires a more extensive use of the theory of algebraic forms than deriving AMI does.

A summary of steps to derive AMI are given in Figure 2.1. And a summary of steps to derive AWMI are given in Figure 4.1

Chapter 5 presents a method to calculate AWMI from real world two dimensional images. And it reports one simulated experiment to test for the affine invariance of AWMI and another experiment to show how AWMI could be used for image retrieval.

There are altogether 13 AWMI derived in this dissertation, and 9 of them involving frequency variables. There is an issue in calculating and comparing these 9 AWMI for two images that are affine transform of each other: knowledge of the affine transform needs to be known. The reason is that the domain of the frequency variables cannot be determined as explained in Experiment 2.

The simulated experiment reported in this chapter shows that the 13 AWMI are indeed affine invariant and extract texture information better than AMI. The other experiment shows that using AWMI for image retrieval requires estimation of affine transformations and accurate calculations of Wigner distributions. This experiment shows the AWMI have an acceptable performance compared with the AMI and SIFT methods.

## 6.2 Further Research

The outcome of research carried out in this dissertation is a new method to derive a number of features called AWMI from two-dimensional images. These features are calculated from the moments of Wigner distributions, and they capture certain information about shapes and textures in images and are affine invariant.

There are, however, a number of possible research issues that could be further investigated. The following are some issues.

Out of 13 AWMI derived in this dissertation, 4 AWMI involve only the spatial variables and 9 AWMI involving the frequency variables. The calculation and comparison of these 9 AWMI for two images that are affine transform of each other require the affine transform to be known so that the domain of the frequency variables can be determined. Further research is needed to find ways to solve this issue.

The derivation of AWMI derived in this dissertation is a generalization of the derivation of AMI. And it shows a unique relationship between moments and the coefficients of algebraic forms so that the theory of algebraic forms can be applied, 3.17, 3.29 -3.36. Further generalization thus could be possible for other types of moments.

A more systematic treatment of the method to derive algebraic invariants for a multivariate form from simpler forms as presented in chapter 5 may be needed. The method presented in chapter 5 only considers the cases of quaternary forms of order 2 and 3, where the variables in the forms are contragredient. Further research could investigate the possibility of generalizing the method to more general forms. Also only 13 AWMI have been derived in chapter 5, more AWMI for the quaternary forms of order 2 and 3, however, could be found.

AWMI as derived in this dissertation are for two-dimensional images, a future research direction could be to generalize AWMI from two-dimensional to three-dimensional. Memory storage and speed of calculations in the case of three dimensions could be a problem. Computing algorithms expressed in a parallel fashion may be needed.

AWMI have been derived based on Wigner distributions in order to extract some of the information the distributions extract. Holograms, however, record better information about real world scenes through recording both the intensities and phases of the scenes ([57][58]). Thus for pattern recognition tasks, it could be useful to extract features from holograms, which can be represented as Fresnel or more generally as linear canonical transforms ([57][62] [63]). It may be worthwhile then to investigate if AMI and AWMI calculated from holograms could be used as features, or some other types of features are needed for holograms.

# References


[1] J. Flusser, T. Suk, and B. Zitová, *Moments and Moment Invariants in Pattern Recognition*. John Wiley and Sons, 2009, p. 312.

[2] M. Nixon and A. Aguado, *Feature Extraction & Image Processing*, Second. Academic Press, 2008.

[3] T. H. Reiss, *Recognizing Planar Objects Using Invariant Image Features*. Springer-Verlag, 1993.

[4] T. H. Reiss, "The Revised Fundamental Theorem of Moment Invariants," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 13, no. 5, pp. 830-834, 1991.

[5] M.-K. Hu, "Visual Pattern Recognition by Moment Invariants," *IRE Transactions on Information Theory*, vol. 8, no. 2, pp. 179-187, 1962.

[6] J. Zhang and T. Tan, "Brief review of invariant texture analysis methods," *Pattern Recognition*, vol. 35, no. 3, pp. 735-747, Mar. 2002.

[7] J. a. Montoya-Zegarra, J. P. Papa, N. J. Leite, R. da Silva Torres, and A. X. Falcão, "Learning How to Extract Rotation-Invariant and Scale-Invariant Features from Texture Images," *EURASIP Journal on Advances in Signal Processing*, vol. 2008, pp. 1-16, 2008.

[8] L. Cohen, "Time-Frequency Distributions - A Review," *Proceedings of the IEEE*, vol. 77, no. 7, pp. 941-981, Jul. 1989.

[9] L. Cohen, *Time-Frequency Analysis: Theory and Applications (Prentice Hall Signal Processing Series)*. Prentice Hall, 1994.

[10] T. Claasen and W. F. G. Mecklenbrauker, "The Wigner Distribution – A Tool for Time-Frequency Signal Analysis, Part I: Continuous Time Signals," *Philips J. Res.*, vol. 35, pp. 217-250, 1980.

[11] R. T. Reed and H. Wechsler, "Texture Segmentation and Organization Using the Wigner Distribution."

[12] J. Hormigo and G. Cristóbal, "Image segmentation using the Wigner-Ville distribution," *Advances in Imaging and Electron Physics*, vol. 131, pp. 65-80, 2004.

[13] J.-M. Morel and G. Yu, “ASIFT: A New Framework for Fully Affine Invariant Image Comparison,” *SIAM Journal on Imaging Sciences*, vol. 2, no. 2, 2009.

[14] C. Schmid and R. Mohr, “Local grayvalue invariants for image retrieval,” *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 19, no. 5, pp. 530-535, 1997.

[15] T. Tuytelaars and L. Van Gool, “Content-based image retrieval based on local affinely invariant regions,” in *Int. Conf. on Visual Information Systems*, 1999, pp. 493-500.

[16] T. Tuytelaars and K. Mikolajczyk, “Local Invariant Feature Detectors: A Survey,” *Foundations and Trends in Computer Graphics and Vision*, vol. 3, no. 3, pp. 177-280, 2007.

[17] G. D. Lowe, “Distinctive image features from scale-invariant key points,” *International Journal of Computer Vision*, vol. 60, no. 2, pp. 91-110, 2004.

[18] P. L. E. Ekombo, N. Ennahnahi, M. Oumsis, and M. Meknassi, “Application of affine invariant Fourier descriptor to shape- based image retrieval,” *Journal of Computer Science and Network Security*, vol. 9, no. 7, pp. 240-247, 2009.

[19] D. Zhang and G. Lu, “Enhanced Generic Fourier Descriptors for Object-Based Image Retrieval,” *International Conference on Acoustics, Speech, and Signal Processing (ICASSP 2002)*, 2002.

[20] M. Mercimek, K. Gulez, and T. V. Mumcu, “Real object recognition using moment invariants,” *Sadhana*, vol. 30, no. 6, pp. 765-775, Dec. 2005.

[21] G. Shen, L. Jiang, and G. Zhang, “An Image Retrieval Algorithm Based on Color Segment and Shape Moment Invariants,” *2009 Second International Symposium on Computational Intelligence and Design*, pp. 517-521, Dec. 2009.

[22] A. Choksuriwong, H. Laurent, and B. Emile, “Comparison of Invariant Descriptors for Object Recognition,” *IEEE Proceedings*, 2005.

[23] H. Schulz-Mirbach, H. Burkhardt, and S. Siggelkow, “Using Invariant Features for Content Based Data Retrieval,” *First NOBLESSE Workshop on Nonlinear Methods in Model-Based Image Interpretation, Lausanne Switzerland*, no. September, 1996.

[24] B. Potocnik, “Assessment of Region-Based Moment Invariants for Object Recognition,” *Proceedings ELMAR 2006*, no. June, pp. 27-32, Jun. 2006.

[25] J. Flusser, “Moment Invariants in Image Analysis,” *Computica*, pp. 196-201, 2006.

[26] J. Flusser and T. Suk, “Affine moment invariants：A new tool for character recognition,” *Pattern Recognition Letters*, vol. 15, no. 4, pp. 433-436, Apr. 1994.

[27] J. Flusser and T. Sue, “Pattern Recognition by Affine Moment Invariants,” *Pattern Recognition*, vol. 26, no. 1, pp. 167-174, 1993.

[28] A. G. Mamistvalov, “N-Dimensional Moment Invariants and Conceptual Mathematical Theory of Recognition N-Dimensional Solids,” *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 20, no. 8, pp. 819-831, 1998.

[29] H. W. Turnbull, *The theory of determinants, matrices and invariants*. Blackie, 1950, p. 368.

[30] O. E. Glenn, *A Treatise on the Theory of Invariants*. Ginn and Company, 1915.

[31] L. E. Dickson, *Algebraic invariants*. J. Wiley & Sons, inc., 1914, p. 100.

[32] E. B. Elliott, *An introduction to the algebra of quantics*. Nabu Press, 1913.

[33] D. Hilbert, *Theory of Algebraic Invariants*. Cambridge Mathematical Library, 1993.

[34] P. J. Olver, *Classical Invariant Theory*. Cambridge University Press, 1999.

[35] M. Hamermesh, *Group theory and its application to physical problems*. Courier Dover Publications, 1989, p. 509.

[36] B. Sturmfels, *Algorithms in Invariant Theory (Texts and Monographs in Symbolic Computation)*, Second Edi. Springer, 2008.

[37] R. Goodman and N. R. Wallach, *Symmetry, Representations, and Invariants*. Springer, 2009.

[38] C. Procesi, *Lie Groups An Approach through Invariants and Representation*. Springer, 2006.

[39] W. Martin and P. Flandrin, “Wigner-Ville spectral analysis of nonstationary processes,” *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 33, no. 6, pp. 1461-1470, Dec. 1985.

[40] S. Qian and D. Chen, “Understanding the Nature of Signals Whose Power Spectra Change with Time,” *IEEE Signal Processing Magazine*, 1999.

[41] P. Flandrin, *Time-Frequency/Time-Scale Analysis, Volume 10 (Wavelet Analysis and Its Applications)*. Academic Press, 1998.

[42] F. Hlawatsch and G. F. Boudreaux-Bartels, "Linear and quadratic time-frequency signal representations," *IEEE Signal Processing Magazine*, vol. 9, no. 2, pp. 21-67, Apr-1992.

[43] S. Qian and D. Chen, *Joint-Time Frequency Analysis: Methods and Applications*. Prentice Hall, 1996.

[44] L. Debnath, "Recent Developments in the Wigner-Ville Distribution and Time-Frequency Signal Analysis," *PINSA*, vol. 68, no. 1, pp. 35-56, 2002.

[45] M. Tucerryan and A. Jain, "Texture Analysis," in *The Handbook of Pattern Recognition and Computer Vision*, 2nd ed., C. H. Chen, L. F. Pau, and P. S. P. Wang, Eds. World Scientific Publishing Co., 1998, pp. 207-248.

[46] R. M. Haralick, K. Shanmugam, and I. Dinstein, "Textural Features for Image Classification," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 3, no. 6, pp. 610-621, Nov. 1973.

[47] T. R. Reed and H. Wechsler, "Segmentation of textured images and Gestalt organization using spatial/spatial-frequency representations," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 12, no. 1, pp. 1-12, 1990.

[48] P. Vautrot, N. Bonnet, and M. Herbin, "Comparative study of different spatial/spatial-frequency methods (Gabor filters, wavelets, wavelets packets) for texture segmentation/classification," *Proceedings of 3rd IEEE International Conference on Image Processing*, pp. 145-148, 1996.

[49] H. Yu, M. Li, H.-J. Zhang, and J. Feng, "Color Texture Moments for Content Based Image Retrieval," *Proceedings of International Conference on Image Processing.*, 2002.

[50] S. Aksoy and R. M. Haralick, "Textural features for image database retrieval," *Proceedings. IEEE Workshop on Content-Based Access of Image and Video Libraries*, pp. 45-49, 1998.

[51] F. Dupont, M. Y. Zhu, C. Odet, and R. Goutte, "Texture classification in the Wigner-Ville Domain," *Proceedings ICSP*, Aug. 1996.

[52] M. Mellor, B.-W. Hong, and M. Brady, "Locally rotation, contrast, and scale invariant descriptors for texture analysis.," *IEEE transactions on pattern analysis and machine intelligence*, vol. 30, no. 1, pp. 52-61, Jan. 2008.

[53] C. Ballester and M. Gonzalez, “Affine Invariant Texture Segmentation and Shape From Texture by Variational Methods,” *Journal of Mathematical Imaging and Vision*, vol. 9, no. 2, pp. 141-171, 1998.

[54] M. J. Bastiaans and T. Alieva, “Wigner distribution moments in fractional Fourier transform systems.,” *Journal of the Optical Society of America. A, Optics, image science, and vision*, vol. 19, no. 9, pp. 1763-73, Sep. 2002.

[55] D. Dragoman, “Applications of the Wigner Distribution Function in Signal Processing,” *EURASIP Journal on Advances in Signal Processing*, vol. 2005, no. 10, pp. 1520-1534, 2005.

[56] C. D. Ruberto and A. Morgera, “Moment-Based Techniques for Image Retrieval,” *2008 19th International Conference on Database and Expert Systems Applications*, pp. 155-159, Sep. 2008.

[57] W. J. Goodman, *Introduction to Fourier Optics*. Englewood, Colorado: Roberts & Company, 2005.

[58] T. Kreis, *Handbook of Holographic Interferometry: Optical and Digital Methods*. Germany: Wilet-VCH, 2005.

[59] K. Curtis, L. Dhar, A. Hill, W. Wilson, and M. Ayres, *Holographic Data Storage: From Theory to Practical Systems*. Wiley, 2010.

[60] D. Psaltis and W. G. Burr, “Holographic Data Storage,” *Computer*, 1998.

[61] S. Berkovich, “Ultimate Irreversibility in the Universe: Continuous Holographic Recording of Every Event and Biological Memory as a Part of It.” 2007.

[62] H. M. Ozaktas, Z. Zalevsky, and M. A. Kutay, *The Fractional Fourier Transform: with Applications in Optics and Signal Processing (Wiley Series in Pure and Applied Optics)*. Wiley-Blackwell, 2001.

[63] A. Stern, “Why is the Linear Canonical Transformation so Little Known,” in *AIP Conference, 5th International workshop on Information Optics*, 2006.

[64] L. D. Jacobson and H. Wechsler, “Joint spatial/spatial-frequency representation☆,” *Signal Processing*, vol. 14, no. 1, pp. 37-68, Jan. 1988.

[65] M. Y. Zhu, F. Peyrin, and R. Goutte, “The Use of a Two Dimensional Hilbert Transform for Wigner Analysis of 2-Dimensional Real Sinals,” *Signal Processing*, vol. 19, pp. 205-220, 1990.

[66] B. Hunt, *The Geometry of some special arithmetic Quotients*. Springer, 1996.

[67] K. Mikolajczyk, T. Tuytelaars, C. Schmid, A. Zisserman, and J. Matas, "A Comparison of Affine Region Detectors," *International Journal of Computer Vision*, 2006.

[68] P. Saad and N. Rusli, "Feature Extraction of Trademark Images Using Geometric Invariant Moment and Zernike Moment - A Comparison," *Jurnal Penyelidikan dan Pendidikan Kejuruteraan*, 2004.

[69] I. Sommer, O. Müller, F. S. Domingues, O. Sander, J. Weickert, and T. Lengauer, "Moment invariants as shape recognition technique for comparing protein binding sites.," *Bioinformatics (Oxford, England)*, vol. 23, no. 23, pp. 3139-46, Dec. 2007.

[70] R. T. Reed and H. Wechsler, "Texture Analysis and Clustering Using the Wigner Distribution," *IEEE Proceedings*, pp. 770-772, 1988.

[71] J. Hormigo, G. Crist, L. Cohen, L. Galleani, and B. Suter, "Texture discrimination using the Wigner-Ville distribution," *Proceedings of SPIE*, pp. 155-162, 2003.

[72] R. B. Okombi-Diba, J. Miyamichi, and K. Shoji, "Segmentation of Spatially Variant Image Textures," *IEEE Proceedings*, 2002.

[73] G. Salmon, "On Quaternary Cubics," *Philosophical Transactions Of The Royal Society Of London*, vol. 150, pp. 229-239, 1860.

[74] J. Williamson, "The Complete System of Two Quaternary Quadratics," *American Journal of Mathematics*, vol. 51, no. 4, pp. 565-576, 1929.

[75] W. L. Edge, "The Discriminant of A Cubic Surface," *Proceedings of the Royal Irish Academy, Section A*, vol. 80, no. 1, pp. 75-78, 1980.

[76] T. Claasen and W. F. G. Mecklenbrauker, "The Wigner Distribution – A Tool for Time-Frequency Signal Analysis, Part II: Discrete Time Signals," *Philips J. Res.*, vol. 35, pp. 276-300, 1980.

[77] F. Peyrin and R. Prost, "A unified definition for the discrete-time, discrete-frequency, and discrete-time/Frequency Wigner distributions," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 34, no. 4, pp. 858-867, Aug. 1986.

[78] D. Wu and J. M. Morris, “Discrete Cohen’s class of distributions,” *Proceedings of IEEE-SP International Symposium on Time- Frequency and Time-Scale Analysis*, pp. 532-535, 1994.

[79] J. O’Toole, M. Mesbah, and B. Boashash, “A Discrete Time and Frequency Wigner-Ville Distribution: Properties and Implementation,” *International Symposium on Digital Signal Processing and Communication Systems, Australia*, 2005.

[80] T. Claasen and W. F. G. Mecklenbrauker, “The aliasing problem in discrete-time Wigner distributions,” *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 31, no. 5, pp. 1067-1072, Oct. 1983.

[81] A. H. Costa and F. G. Boudreaux-Bartels, “An overview of aliasing errors in discrete-time formulations of time-frequency representations,” *IEEE Transactions on Signal Processing*, vol. 47, no. 5, pp. 1463-1474, May 1999.

[82] B. Boashash, “Note on the Use of the Wigner Distribution for Time-Frequency Signal Analysis,” *IEEE Transactions on Acoustics, Speech, and Signal Processing*, vol. 36, no. 9, pp. 1518-1521, 1988.

[83] D. Ping, P. Zhao, and B. Deng, “Cross-terms suppression in Wigner-Ville distribution based on image processing,” *Proceedings of the 2010 IEEE International Conference on Information and Automation, Harbin, China*, no. 1, pp. 2168-2171, 2010.

[84] F. Hlawatsch, “Interference Terms in the Wigner Distribution,” in *International Conference on Digital Signal Processing, Florence, Italy*, 1984.

[85] W. J. Williams, “Reduced interference distributions: biological applications and interpretations,” *Proceedings of the IEEE*, vol. 84, no. 9, pp. 1264-1280, 1996.

[86] L. Stankovic, “Auto-term representation by the reduced interference distributions: a procedure for kernel design,” *IEEE Transactions on Signal Processing*, vol. 44, no. 6, pp. 1557-1563, Jun. 1996.

[87] J. G. Proakis and D. G. Manolakis, *Digital signal processing*. Pearson Prentice Hall, 2007, p. 1084.

[88] S. J. Lim, *Two-Dimensional Signal and Image Processing*. Prentice Hall, 1989.

[89] P. W. J. Pluim, J. B. A. Maintz, and A. M. Viergever, “Mutual information based registration of medical images: a survey,” *IEEE Transactions on Medical Imaging*, vol. XX, 2003.

[90] M. Sun, C. C. Li, L. N. Sekhar, and R. J. Sclabassi, “Elimination of cross-components of the discrete pseudo Wigner distribution via image processing,” *International Conference on Acoustics, Speech, and Signal Processing*, pp. 2231-2233, 1989.

[91] S. R. Sundaram and K. M. M. Prabhu, “Numerically Stable Algorithm for Computing Wigner-Ville Distribution,” *IEE Proceedings online*, 1996.

[92] K. M. M. Prabhu and R. Shanmuga Sundaram, “Fast algorithm for pseudodiscrete Wigner–Ville distribution using moving discrete Hartley transform,” *IEE Proceedings - Vision, Image, and Signal Processing*, vol. 143, no. 6, p. 383, 1996.