Ordinary Voltage Graphs In Pseudosurfaces And Derived Cellular Homology With Applications To Graph Embeddability

by Steven Schluchter

B.S. in Mathematics, May 2006, Saint Mary's College of California
M.S. in Mathematics, Mathematics, June 2008, California State University, East Bay

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Lowell Abrams
Associate Professor of Mathematics

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Steven Schluchter

Dissertation Research Committee:

Lowell Abrams, Associate Professor of Mathematics, Dissertation Director

Yongwu Rong, Professor of Mathematics, Committee Member

Paul Kainen, Professor of Mathematics, Georgetown University, Committee Member

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Abstract

Ordinary Voltage Graphs In Pseudosurfaces And Derived Cellular Homology With Applications To Graph Embeddability

We develop a global homological understanding of cellular graph embeddings in orientable and nonorientable surfaces and pseudosurfaces from local intersection data. We apply our new theorems to show that homological data of a graph embedding described as the derived graph of an ordinary voltage graph embedding with voltage group A can be derived by considering cosets of specially constructed subgroups of A.

For a prime $p > 5$, we apply our theorems, while making distinctly homological arguments, to show that there is no embedding in the torus of a Generalized Petersen Graph of the form $GP(2p, 2)$ that can be derived from an ordinary voltage graph embedding, even though we can produce embeddings of such graphs in the torus. We conclude that no free group action on $GP(2p, 2)$ embedded in the torus extends to a cellular automorphism of the torus.

For distinct primes $p, q \geq 5$, we use similar argumentation to prove corresponding conclusions about all embeddings of the Cartesian product of two circle graphs C_p and C_q in Dyck's surface.

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List of Symbols

$(G{:}I)_v^a$ The component of $(G{:}I)^\alpha$ containing v^a.

$(G{:}X)_v^a$ The component of $G{:}X)^\alpha$ containing v^a.

$(P{:}I)_v^a$ The component of $(P{:}I)^\alpha$ containing v^a.

1_A The identity element of A.

$[\cdot]$ The projection map $\mathcal{C}(G) \to \vec{B}\ (G \to P)$.

$[\cdot]_2$ The projection map $Z(G) \to Z(G)/B(G)$.

$[m]$ The set of all positive integers less than a positive integer m.

$\alpha \to A$ The voltage assignment to the darts of G.

$\bar{\gamma}(G)$ The nonorientable genus of G.

$\bar{S}_w^b(z)$ The z-region (relative to the 1-chains in the fiber over z that contains w^b.

$\beta_i(F)$ The i^{th} Betti number of a topological space or a cellular complex F.

$\chi(F)$ The Euler characteristic of a pseudosurface or cellular complex F.

$\gamma(G)$ The orientable genus of G.

$\Gamma_z(z_1, \ldots, z_m)$ The z-graph relative to $z_1, \ldots, z_m$.

$\hat{W}_v^a$ The set of consecutive lifts of a closed walk W containing W_v^a.

$\langle G, \alpha \to A \rangle$ An ordinary voltage graph.

$\langle G \to P, \alpha \to A \rangle$ An ordinary voltage graph embedding.

$\mathcal{C}(G)$ The oriented cycle space of G.

$\mathcal{E}(G)$ The $\mathbb{Z}_2$-vector space whose basis elements are the edges of G.

$\dim_K(Y)$ The dimension of a vector space or module Y over a field or ring K.

KB The Klein bottle.

∂I $\sum_{f \in I} \partial f$ for $I \subset F(G \to P)$ or $I \in C_2(P)$.

∂ The boundary map defined on $\mathbb{Z}$ or $\mathbb{Z}_2$ cellular chain complexes.

$\vec{B}\,(G \to P)$ The oriented boundary space of $G \to P$.

$\vec{C}_0\,(P)$ The $\mathbb{Z}$-module of formal linear combinations of vertices of G.

$\vec{C}_1\,(P)$ The $\mathbb{Z}$-module of formal linear combinations of edges of G.

$\vec{C}_2\,(P)$ The $\mathbb{Z}$-module of formal linear combinations of faces of $G \to P$.

A A group (not necessarily abelian).

$A'(w)$ The local voltage group of closed walks in M' based at $w \in V(M')$.

$A(v)$ The local voltage group of closed walks based at $v \in V(G)$.

$A^{\veebar}(w, y, z)$ The set of net voltages of walks in M' that begin at w and end at w or y but do not transversely cross $G{:}z$.

$A^{\veebar}(w, z)$ The local voltage group of closed walks in M' based at w that do not transversely cross $G{:}z$.

$B(G)$ The cut space (or coboundary space) of G.

$B(G \to P)$ The boundary space of $G \to P$.

$C_0(P)$ The $\mathbb{Z}_2$-vector space of formal linear combinations of vertices of G.

$C_1(P)$ The $\mathbb{Z}_2$-vector space of formal linear combinations of edges of G.

$C_2(P)$ The $\mathbb{Z}_2$-vector space of formal linear combinations of faces of elements of $F(G \to P)$.

$D(G)$ The darts on the edges of G.

$E(G)$ The set of edges of G.

e^a The (directed) edge $(e, a) \in E(G) \times A$.

$F(G \to P)$ The set of faces of $G \to P$.

G A finite and connected multigraph.

$G{:}I$ The 1-skeleton of $P{:}I$.

$G{:}X$ The subgraph of G induced by $X \subset E(G)$ or $X \in \mathcal{E}(G)$.

$G{:}X$ The subgraph of G induced by edges in a set X or $X \in \mathcal{E}(G)$

$G \to P$ A proper embedding of G in P.

$G \to S$ A cellular embedding of G in S.

G^α The derived graph of an ordinary voltage graph.

$G^\alpha \to P^\alpha$ The derived embedding of an ordinary voltage graph embedding.

$G_1 \times G_2$ The Cartesian product of two graphs G_1 and G_2.

G_v^a The component of G^α containing v^a.

$GP(n, k)$ The Generalized Petersen Graph with parameters n and k.

$h(d)$ The head vertex of a dart d.

$H_1(F)$ The first homology group of a pseudosurface or cellular complex F.

P A pseudosurface.

$P{:}I$ The subcomplex of P induced by $I \subset F(G \to P)$ or $I \in C_2(P)$.

P_v^a The component of P^α containing v^a.

$R(z)$ A ribbon neighborhood of a circle graph $G{:}z$.

S A surface.

$S \triangle z$ The result after ungluing S along an orientation-preserving circle $G{:}z$.

S^2 The sphere.

$star(v)$ The $\mathbb{Z}_2$-sum of edges joining v to other vertices.

$star_+(v)$ The $\mathbb{Z}_2$-sum of edges incident to v.

T The torus.

$t(d)$ The tail vertex of dart d.

$U^*(v)$ A vertex-star neighborhood of a vertex v.

$V(G)$ The set of vertices of G.

v^a The vertex $(v, a) \in V(G) \times A$.

W A walk in G.

W_v^a The lift of a walk W that is based at v, which begins at v^a.

$Z(G)$ The cycle space of G.

Chapter 1

Introduction

1.1 The homological approach to graph embeddability

An embedding of a graph G in a topological space X is a continuous injection from G into X. A cellular graph embedding can be thought of as a combinatorial object that encodes a cellular decomposition of the topological space in which the graph is embedded. The study of cellular graph embeddings has been approached this way from within the settings of different combinatorial objects. The combinatorial objects themselves, while encoding graph embeddings, do not easily provide a global picture of the embedding that is encoded: a rotation scheme encodes a cellular embedding in terms of cyclic orderings of edges appearing in small neighborhoods of each vertex, therefore encoding a global picture in terms of local pictures; a cycle-double-cover can be used to describe how discs are to be glued to a graph in order to produce a cellular 2-complex K in terms of the boundaries of its 2-cells. The encoded graph is sometimes a minor mystery all by itself: if all one understands about K is a list of the edges of subgraphs of G that bound faces (be they cycle covers or not), then a global understanding of the combinatorial structure of the 1-skeleton of K is not immediately clear. Similar difficulties arise from considering graph embeddings encoded in the form of rotation schemes. Moreover, in the case of a cycle-double-cover it is not immediately clear that $|K|$, the carrier of K, is homeomorphic to a surface; there could be vertices of the graph that are not contained in neighborhoods

homeomorphic to discs.

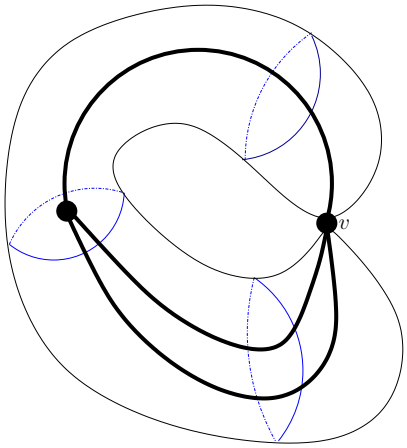


Figure 1.1: A 2-complex K that is not homeomorphic to a surface. The topological space $|K|$ is called a pinched sphere or a spindle surface. The vertex v is not contained in any open set that is homeomorphic to a disc.

A *pseudosurface without boundary* X is the carrier of a simplicial 2-complex in which each 1-simplex is a face of two 2-simplices. Figure 1.1 is an example of a pseudosurface without boundary with a cellular decomposition induced by a graph embedding. The points x' that are not contained in a neighborhood homeomorphic to $\mathbb{R}^2$ are called *pinchpoints*, each pinchpoint being the result of a finite number of points of a surface being identified. A *pinch* is an identification of two points of a surface or pseudosurface. Figure 1.1 is also an example of a *face-connected* cellular 2-complex: a 2-complex K such that for any two faces f_1, f_2 of K, there is a sequence of faces $f_1 = f'_1 \dots f'_k = f_2$ such that consecutive faces share at least one common boundary edge. For a cellular complex K, let $\chi(K)$ denote the Euler characteristic of F, which is defined

$$\chi(K) = \sum_{i \geq 0} k_i,$$

where k_i is the number of cells of dimension i in K. We let $\beta_i(K)$ denote the i^{th} Betti number of K. Theorem 1.1 is a well known result.

1.1 Theorem. [12, Theorem 2.44] For any cellular 2-complex K,

$$\chi(K) = \sum_i (-1)^i \beta_i(K).$$

We also need Theorem 1.2 for our discussion.

1.2 Theorem. For any compact face-connected pseudosurface P with h handles, c crosscaps, and p pinches needed to produce P from a surface S with h handles and c crosscaps,

$$\chi(P) = 2 - 2h - c - p.$$

Proof. Let S_1 denote any compact connected surface without boundary, and let P^2 and T denote the projective plane and the torus, respectively. The connected sums $S_1 \# P^2$ and $S_1 \# T$ have Euler characteristic $\chi(S_1) - 1$ and $\chi(S_1) - 2$, respectively. Inductively, it follows that if S_1 is a surface with h handles and c crosscaps, then $\chi(S) = 2 - 2h - c$ since the Euler characteristic of the sphere is 2.

Assume that P is cellularly decomposed by a graph embedding G and recall that $\chi(P)$ is an invariant of the cellular decompositions of P. Let $Q \colon S \to P$ denote the quotient map that identifies the points of S needed to produce P. Consider the cellular decomposition of S given by $Q^{-1}(G)$. Since identifying two vertices of S lowers the Euler characteristic by 1, by an inductive argument, we see that $\chi(P) = 2 - 2h - c - p$ since there are p total idenfications needed to produce P from S. □

1.3 Corollary. For any compact face-connected pseudosurface P with h handles, c crosscaps, and p pinches $\beta_1(P) = 2h + c + p$.

Proof. This follows from Theorems 1.1, 1.2, and the fact that, as vector spaces over $\mathbb{Z}_2$, $H_0(P)$ and $H_2(P)$ both have rank 1. □

Approached through the lens of cellular homology, the cellular embeddability of a connected graph in a sphere can thus be understood by asking if one can find a set of boundary walks of faces such that the edges of the graph are double covered by these boundary walks

and the resulting face-connected 2-complex has a trivial first homology group. In a more general connected pseudosurface P, *the homological approach to graph embeddability* is the pursuit of criteria that treat embeddability of a graph in P as being equivalent to the availability of boundary walks that double cover the edges of G in such a way that the resulting 2-complex K has homological data that uniquely identifies K as being homeomorphic to P. By Corollary 1.3, the presence of any handle, crosscap, or pinchpoint in any pseudosurface implies that the pseudosurface will have a nontrivial first homology group. Thus, as we see from Theorem 1.2 and Corollary 1.3, a face-connected pseudosurface with trivial first homology group has Euler characteristic 2, the Euler characteristic of the sphere. Note that this condition does not depend on the directions of the edges induced by the boundary walks: it is a consequence of there being only one face-connected 2-complex of Euler characteristic 2 (up to homeomorphism) that an orientation on a face induced by its boundary walk also induces an orientation on the entire sphere. This approach to embeddability in the sphere, (stated equivalently as an approach to planarity) can be traced back to the planarity criteria of Whitney and MacLane, though they both were approaching the question of embeddability in a more explicitly combinatorial manner that we now describe.

Let G denote a graph, and let $V(G)$ and $E(G)$ denote the vertex and edge sets of G, respectively. Also let $|V(G)|$, $|E(G)|$ and $c(G)$ denote the number of vertices, edges, and connected components of G, respectively. Whitney's theorem on planarity requires the definition of a *combinatorial dual graph* G^* of G, which is defined on the same edge set as G and has the property that for any $H \subset G$, the complementary edge set of H forms a subgraph H^* of G^* such that

$$|E(H)| - |V(H)| + c(H) = |V(G^*)| - c(G^*) + c(H^*). \tag{1.4}$$

If G is assumed to be connected and we let $H = G$, then the existence of a connected combinatorial dual graph G^* would mean that G and G^* satisfy the relation

$$|V(G)| - |E(G)| + |V(G^*)| = 2,$$

the Euler characteristic of the sphere. By Theorem 1.2, since 2 is the only positive summand in $2 - 2h - c - p$, the sphere is the only face-connected surface of Euler characteristic 2. Moreover, G^* can be used to produce an embedding of G in the sphere such that G^* is isomorphic to the *topological dual* $G^\perp$ of the embedded G; $G^\perp$ has a vertex set that is the set of faces of the embedding, and edge set $E(G^\perp) = E(G)$ with the incidence of edges and vertices of $G^\perp$ being the same as that of the faces and edges of the embedding of G. Whitney's theorem on planarity [21] formalizes the necessity of the existence of a combinatorial dual for a graph to be planar.

1.5 Theorem. [21]**(Whitney, 1932)** A graph is planar if and only if it has a combinatorial dual graph. Furthermore, if G is connected and embedded in the plane, then the topological dual graph of the embedded G is also a combinatorial dual graph of G.

In a similar vein, and more explicitly algebraic in nature, is MacLane's planarity criterion [15]. This result requires some further definitions. If we think of the edges of G as being a basis for a $\mathbb{Z}_2$-vector space $\mathcal{E}(G)$ and the edge sets of circles (connected 2-regular graphs) as being generators of a subspace of $\mathcal{E}(G)$ called the *cycle space*, denoted $Z(G)$, then a *2-basis* B of $Z(G)$ is a basis with the property that each edge is a summand of at most two of the elements of B.

1.6 Theorem. [15]**(MacLane, 1937)** A graph G is planar if and only if the cycle space of G has a 2-basis relative to the edges of G.

If the cycle space of G does indeed have a 2-basis, then the basis elements can be used to produce the facial boundaries of a plane embedding. Theorem 1.7 is a well-known result.

1.7 Theorem. [5, Theorem 1.9.5] For a graph G

$$\dim_{\mathbb{Z}_2}(Z(G)) = |E(G)| - |V(G)| + c(G).$$

Let $f(G)$ denote the number of faces of an embedding of G. Observe again that the Euler characteristic of the sphere becomes apparent after observing that all but one of the

facial boundaries of the sphere are accounted for in the 2-basis, as

$$|E(G)| - |V(G)| + 1 = f(G) - 1. \tag{1.8}$$

Thus we see the connection between the approaches of Whitney and MacLane in Equation 1.8 and Theorem 1.2: they both provide for elements of the cycle space to which discs can be glued to form a face-connected 2-complex of Euler characteristic 2, thus forming a 2-complex homeomorphic to a sphere.

If one were to take the homological approach to graph embeddability when considering embeddability of connected graphs in face-connected pseudosurfaces P satisfying $\dim_{\mathbb{Z}_2}(H_1(P)) = 1$, then we see from Corollary 1.3 and Theorem 1.2 that P must have Euler characteristic 1. Thus, by Theorem 1.2, one must consider the projective plane and the spindle surface (see Figure 1.1) and ask how one can most easily distinguish between the two from a combinatorial or algebraic standpoint. Lefschetz [13] in 1965 introduced the *umbrella property*, [13, Property (C)] that combinatorially encodes the exclusion of pinch-points in terms of conditions imposed on the embedding. Phrased a bit more explicitly, the umbrella property essentially states that all of the edges of G incident to any particular vertex v in $V(G)$ can be arranged in a cyclic order induced by an orientation on a single disc centered at v. Lefschetz uses the combinatorial language of rotation systems, which we shall formulate explicitly in Chapter 2.

It should be noted that Lefschetz makes a slight error in [14] in a different presentation of his result: a minor arithmetic mistake in a calculation related to the Euler characteristic of a nonorientable surface was noted though not identified by Bruhn and Diestel in [4]. A graph with Betti number R that is cellularly embedded in a nonorientable surface with q crosscaps has $R-q+1$ distinct faces whose facial boundary walks (called loops by Lefschetz) are independent over $\mathbb{Z}$. Lefschetz writes $R - q$ in place of $R - q + 1$ in the statement of [14, Theorem 3.3].

Whitney's dual-graph point of view of the homological approach to graph embeddability was generalized by Edmonds [8], also in 1965. Since the function taking a cellularly em-

bedded graph G to its topological dual $G^{\perp}$ is an involution if and only if G is cellularly embedded in a surface S, one can say (as Edmonds does) that the two graphs G and $G^{\perp}$ have a *duality with respect to a polyhedral surface embedding (in S)*. This notion does not carry over to embeddings in pseudosurfaces, and Edmonds does not comment on embeddability of graphs in pseudosurfaces.

1.9 Theorem. [8] **(Edmonds, 1965)** A one-to-one correspondence between the edges of two connected graphs is a duality with respect to some surface S if and only if, for each vertex v of each graph, the edges which meet v in the graph of v form in the other graph a subgraph H which is connected and has an even number of edge ends to each of its vertices (where if an edge meets v at both ends, its image in H is counted twice).

1.10 Theorem. [8] **(Edmonds, 1965)** A necessary and sufficient condition for a connected graph G to have a polyhedral surface embedding in a surface S of Euler characteristic $\chi(S)$ is that it has an edge correspondence with another graph G' for which:

1. the conditions of Theorem 1.9 are satisfied, and
2. $|V(G)| - |E(G')| + |V(G')| = \chi(S)$.

Edmonds notes at the end of his results that the distinction between orientable and nonorientable surfaces is made with a choice of how the edges are used in the boundary walks of the faces of the embedding of G in S. The directions of the edges as they appear in the facial boundary walks are needed to distinguish an orientable embedding from a nonorientable embedding of the same graph/topological-dual graph pair. Merely knowing the edge bijection between a graph and its topological dual is not enough by itself, as Figure 1.2 makes plainly clear.

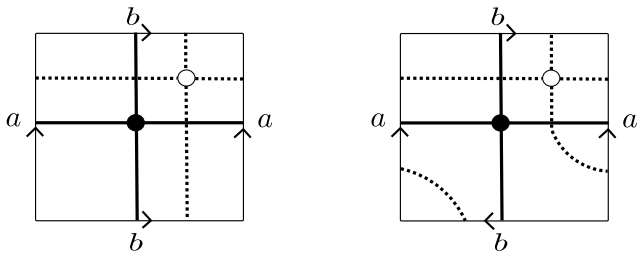


Figure 1.2: The bouquet of two loops cellularly embedded in both the torus and the Klein bottle. The figure on the right is the embedding in the Klein bottle. In both embeddings the topological dual graph is to the bouquet of two loops.

We see that in order to distinguish between embeddings in the Klein bottle, the torus, and other face-connected pseudosurfaces of Euler characteristic 0 and less, whose first homology group is dimension 2 or higher, one must consider data from the boundary walks themselves and not just the knowledge of whether or not a given graph embedding satisfies Lefschetz's umbrella property. We must know how the edges are used in the boundary of the same or different faces; the (non)orientability of a surface or pseudosurface is not guaranteed by knowing just the Euler characteristic as it would be for surfaces of Euler characteristic 2 or 1.

Also of note in the homological approach in the study of graph embeddings in pseudosurfaces is a 1987 paper by Mohar[17]. Mohar weakens the definition of MacLane's 2-basis, defining a *2-set* $B(G)$ of G to be a subset of $Z(G)$ such that each edge of G appears in at most two elements of $B(G)$, and he defines the *codimension of* B *in* G, which Mohar denotes $\delta(B(G))$, to be the dimension of the quotient $Z(G)/B(G)$. The distinction between Mohar's 2-set and MacLane's 2-basis is that the facial boundary walks of a graph embedded in a pseudosurface without boundary are not necessarily 2-basis, but they are a 2-set. Define the *orientable genus of* G and the *nonorientable genus of* G, denoted $\gamma(G)$ and $\bar{\gamma}(G)$, to be the minimum number of handles of an orientable surface in which G has an embedding, and the minimum number of crosscaps of a nonorientable surface in which G has an embedding, respectively. Mohar defines the *order of a graph* G to be

$$\omega(G) = min\left\{2\gamma(G), \bar{\gamma}(G)\right\}.$$

One should note that the order of the graph is the minimum codimension of $B(G)$ in $Z(G)$, where $B(G)$ can be taken to mean the $\mathbb{Z}_2$ subspace generated by elements $z \in Z(G)$ such that the edges of z bound a face f of a cellular embedding of G and each edge appearing in z is contained in the boundary of two distinct faces of the embedding.

1.11 Theorem. [17] **(Mohar, 1987)** If $B(G)$ is a 2-set of cycles of G, then

$$\omega(G) \leq 2\delta(B(G)).$$

Theorem 1.11 is accompanied by an infinite class of examples of graphs such that: for any (integer) $d \geq 0$ there is a graph G_d to which is associated a 2-set $B_d(G)$ such that $\delta(B_d(G)) = d$ and $\omega(G_d) = 2d$. The significance of the infinite family of examples is that for any d, there exists a graph embeddable in an orientable surface of genus d but not in a nonorientable surface with d crosscaps, thus verifying that the bound in Theorem 1.11 is tight for an infinite family of graphs. Also noteworthy about these examples is that each G_d does form the 1-skeleton of a 2-complex K that is homeomorphic to a sphere with d point identifications. Thus we see that embeddability in a pseudosurface with pinchpoints is a worthy question by itself.

Abrams and Slilaty expanded upon Whitney's dual-graph strain in the homological approach to the study of graph embeddability in 2003 [1] and again in 2006 [2]. In these two papers the definition of combinatorial dual graph is made algebraic, and it is weakened in order to capture the embeddability of graphs in surfaces and pseudosurfaces other than the sphere, first for the projective plane in [1] and then for all surfaces and pseudosurfaces in [2]. Informal definitions are given here. Let $B(G)$ be the subspace of $\mathcal{E}(G)$ generated by

$$\{star(v) : v \in V(G)\},$$

where $star(v)$ is being used to denote the $\mathbb{Z}_2$-sum of edges joining v to other vertices. In light of these definitions, a combinatorial dual of G can be understood to be a graph G^* together with an edge bijection $f\colon E(G^*) \to E(G)$ such that $f(B(G^*)) = Z(G)$. This is something pointed out in [1]. To see this, one can consider Theorem 1.7 and Equation

(1.4), and observe that the dimensions of $Z(G)$ and $B(G^*)$ match. Also in [1] Abrams and Slilaty introduce the notion of an *algebraic dual* of G, which is a weakening of the notion of a combinatorial dual graph. An *algebraic dual* of G is another graph G' together with an edge bijection $f\colon E(G') \to E(G)$ satisfying $f(B(G')) \subset Z(G)$. For a vertex $v \in V(G)$, let $star_+(v)$ denote the set of all edges incident to v, and observe that an algebraic dual does not have to satisfy the condition that for each vertex $v \in V(G')$, the induced subgraph $G : f(star_+(v))$ is connected, which would make it a *component-split* algebraic dual. A topological dual $G^\perp$ of a cellularly-embedded graph G in a pseudosurface is a component-split algebraic dual. If G' is an algebraic dual of G, then it is a consequence that G is an algebraic dual of G'. To see why this is true, we need Theorem 1.12.

1.12 Theorem. [5, Theorem 1.9.4] As subspaces of $\mathcal{E}(G)$, $Z(G)$ and $B(G)$ are orthogonal complements of eachother.

By Theorem 1.12, we may decompose $\mathcal{E}(G)$ and $\mathcal{E}(G')$ as

$$\mathcal{E}(G) = B(G) \bigoplus Z(G),$$

$$\mathcal{E}(G') = B(G') \bigoplus Z(G'),$$

and note that if $f(B(G')) \subset Z(G)$, then it follows that $f^{-1}(B(G)) \subset Z(G')$. However, as Figure 1.3 illustrates, while the embedded graph G is an algebraic dual of $G^\perp$ it is not necessarily a component-split algebraic dual of $G^\perp$.

In [1] the algebraic dual is used to establish a homological embeddability criteria for nonplanar 2-connected graphs in the projective plane. In [2] this approach is broadened to all surfaces and pseudosurfaces. The most novel thing about the results of [1] and [2] is the use of the intersections of circles of G to distinguish between embeddings of G in different surfaces and pseudosurfaces. Using $\mathbb{Z}_2$ coefficients we lose the ability to capture the orientability of a surface by using just cellular homology: we have

$$H_1(Torus) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \cong H_1(Klein\ bottle).$$

It is in the use of intersection data associated to transverse crossings of circles that we see elements of the cycle space of an embedded graph being identified with specific topological features of a surface (handles, crosscaps) that is not the sphere. This use of intersection data is expanded upon in [2] where cellular embeddability of connected graphs in pseudosurfaces is considered. The notion of one circle of an embedded graph transversely crossing another has no meaning if one is trying to consider intersections at pinchpoints, as saying that one circle transversely crosses another requires that the transversal crossing be contained in a neighborhood homeomorphic to a disc. Thus, it is in the topological dual of the properly embedded graph that Abrams and Slilaty find intersection data that corresponds to the pinchpoints of P, which they develop in [2].

Recall that in Theorem 1.2, c is the number of crosscaps of S, and h is the number of handles. We let the matrices I_c and J_{2h} denote the $c \times c$ identity matrix and the block-diagonal matrix with h blocks of the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively. As described in [7, p. 221] there is a nondegenerate bilinear intersection form $\langle \cdot, \cdot \rangle_2 \colon H_1(S) \times H_1(S) \to \mathbb{Z}_2$, called "the intersection form." When an appropriate basis is chosen for this intersection form we can produce a block-diagonal matrix representation

$$\left(\begin{array}{c|c} I_c & 0 \\ \hline 0 & J_{2h} \end{array}\right). \tag{1.13}$$

This matrix representation can be said to associate 1-chains with the longitude and meridian of each handle and the core of each Möbius band. Of course, since the topological dual of a cellularly-embedded graph is also cellularly embedded, one can find such a basis of $H_1(S)$ by considering the 1-chains of $Z(G)$ or $Z(G^{\perp})$.

We need one more definition before we formulate Abrams's and Slilaty's theorem concerning an intersection form related to graph embeddings in pseudosurfaces. A *proper embedding* of a graph in a pseudosurface P is a cellular embedding $\phi \colon G \to P$ such each

pinchpoint x of P is in the image of $V(G)$.

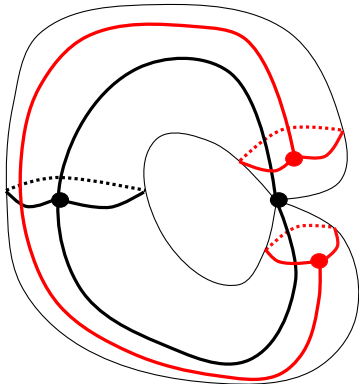


Figure 1.3: A properly embedded graph G in the pinched sphere, and its topological dual $G^{\perp}$, colored red. Both circles of $G^{\perp}$ are contractible to the pinchpoint of the pinched sphere.

As said earlier, the notion of transversality of crossings is inapplicable at pinchpoints. However, we may still algebraically capture the relationship between subraphs of an embedded graph and pinchpoints by considering the topological dual of an embedded graph. Observe in Figure 1.3 that there are two circles in the topological dual of G that are contractible to the pinchpoint. Thus, we may associate the pinchpoint with either of those two circles. As in [2], where the authors use intersection data concerning a component-split algebraic dual G', the intersection in [7, p. 221] mentioned above can be reformulated and extended to be an intersection form that describes the topological features of P; we may chose an independent set of elements representing different classes of $Z(G')/B(G)$ and which produces a matrix representation of the bilinear form like that of (1.13). This independent set can be extended by including homologically independent 1-chains that induce Eulerian subgraphs that contract to pinchpoints. This produces an intersection form with block-diagonal matrix representation

$$\left(\begin{array}{c|c|c} I_c & 0 & 0 \\ \hline 0 & J_{2h} & 0 \\ \hline 0 & 0 & 0_p \end{array}\right), \tag{1.14}$$

where 0_p is the $p \times p$ matrix of zeros. For each $i \in \{1, 2, \ldots, p\}$, the i^{th} row and i^{th}

column of 0_p can be said to capture the intersection properties of a 1-chain $z_i \in Z(G^\perp)$ of G' representing a nonzero equivalence class in $Z(G^\perp)/B(G)$, which is distinct from the homology classes represented by all other such z_i and induces a subgraph that contracts to a pinchpoint.

To fully understand Theorem 1.16, we must understand the degeneracy and self orthogonality of a symmetric bilinear form. Consider Proposition 1.15.

1.15 Proposition. Let $\langle \cdot, \cdot \rangle$ be a symmetric bilinear form on a $\mathbb{Z}_2$-vector space V. The sets

$$D(V) := \{v \in V \ : \ \langle v, x \rangle = 0, \text{ for all } x \in V\}$$

and

$$O(V) := \{v \in V \ : \ \langle v, v \rangle = 0\}$$

are subspaces of V with $D(V) \subset O(V)$.

Proof. Omitted. □

For any symmetric bilinear form $\langle \cdot, \cdot \rangle$ we define the *degeneracy* of $\langle \cdot, \cdot \rangle$ to be the dimension of $D(V)$ and the *self orthogonality* of $\langle \cdot, \cdot \rangle$ to be the dimension of $O(V)$. In the case of the intersection form appearing in (1.14), we see that the intersection product has *degeneracy* p and *self orthogonality* $2h + p$. This degeneracy follows from the fact that the 1-chains algebraically corresponding to the rows and columns of 0_p induce contractible Eulerian subgraphs, which therefore can be said to induce subgraphs transversely crossing no other Eulerian subgraph H of $G^\perp$ when brought into general position with H. To understand why the self orthogonality is $2h + p$, observe that each of the 1-chains inducing a contractible Eulerian subgraph has a zero intersection product with any representative of any equivalence class of $Z(G^\perp)/B(G)$. This explains the addend p. To understand the addend $2h$, consider the longitude and meridian of a handle. The longitude does not intersect a curve γ that is homotopic to, and in general position with, the longitude. A similar conclusion follows for the meridian.

Observe in Figure 1.3 that if an edge e of G occurs twice in the boundary of a face, then

the corresponding dual edge $e^\perp$ is a loop. Note further that $e^\perp$ appears twice in a facial-boundary walk. For each $v \in V(G^\perp)$, the edges of $star_+(v)$ induce a subgraph of G whose edges appear once or twice in the boundary walk of v. Thus, the question of embeddability of a graph in a pseudosurface P can be interpreted as the question of the existence of a component-split algebraic-dual graph G' and facial-boundary walks corresponding to each $star_+(v)$ for $v \in V(G')$ that satisfy certain properties. If both of these things exist, then Theorem 1.16 guarantees that these facial boundary walks can be used in the production of a cellular 2-complex $K(G, G')$ which is guaranteed to be homoeomorphic to P.

1.16 Theorem. [2] **(Abrams and Slilaty, 2006)** A connected graph G properly embeds in a closed, face connected pseudosurface P with p pinches, h handles, and c crosscaps if and only if G has a component-split algebraic dual G' such that

1. G' is connected,
2. $dim\,(Z(G)/B(G')) = 2h + c + p$, and
3. Eulerian walks can be chosen in the construction of $K(G, G')$ so that $\langle \cdot, \cdot \rangle_{K(G,G')}$ has degeneracy p and self-orthogonality $2h + p + c - min(1, c)$.

Formulated in 1986, but for a much different application, is the notion of *binary matroid semiduality* appearing in [19]. Tarsi's paper is mentionable here because the definition of an algebraic-dual graph is a special case of the definition of one binary matroid being semi-dual to another. The cycle matroid of a graph is denoted $M(G)$, and the dual of a matroid $M(X)$ on a set X is denoted $M^*(X)$; it is widely known among matroid theorists that for a plane graph G and its topological dual $G^\perp$, the matroids $M^*(G)$ and $M(G^\perp)$ are isomorphic. Working within the context of binary matroids (matroids representable as matrices over $\mathbb{Z}_2$, as all graphic matroids are), Tarsi weakens the definition of the dual of a matroid, formulating the notion of binary matroid semi-duality in order to make some progress on the famous Cycle-Double-Cover Conjecture, proving the validity of the conjecture for graphs with a Hamiltonian path.

1.17 Definition. [19]**(Tarsi, 1985)** A binary matroid $\mathcal{M}_1$ is defined to be *semi-dual* to another binary matroid $\mathcal{M}_2$ on the same underlying set if every cocycle of $\mathcal{M}_1$ is a cycle of $\mathcal{M}_2$.

The definition of an algebraic dual graph G^* of G is a special case of Tarsi's definition of binary matroid semi-duality. To understand this consider that in matroid theory it is the edge sets of vertex stars of G^* that generate all cocycles of $M(G^*)$. Thus the bijection $f\colon E(G^*) \to E(G)$ satisfying $f(B(G)) \subset Z(G)$ makes the binary matroid $M(G^*)$ semi-dual to $M(G)$. For a full treatment of graphic matroids and their duals see Chapters 1 and 2 of [18]. Matroid theory will not be discussed any further in this dissertation.

Returning to MacLane's point of view, which considers a choice of facial-boundary walks as part of an embeddability criteria and no dual graph of any kind, we return to the earlier-mentioned work of Bruhn and Diestel [4]. The significance of that paper begins with the fact that the umbrella condition imposed by Lefschetz in [13] and [14] is made explicitly algebraic in the definitions of a *sparse family of walks* and a *locally sparse family of walks*. As one may infer, the definition of a locally sparse family of walks is stronger than the other. If the faces of a 2-complex have a sparse family of boundary walks, then the 2-complex can have a pinchpoint vertex that occurs more than once in a single facial-boundary walk. A 2-complex having a locally sparse family of walks that bound faces cannot have such a pinchpoint or any other pinchpoint. For example, Figure 1.1 contains an example of a 2-complex in which the facial-boundary walks are sparse but not locally sparse. The proof of the crucial [4, Lemma 7] employs a procedure of Edmonds found in [8] to show that a sparse family of facial-boundary walks of a 2-complex can be made locally sparse if it isn't already so.

In keeping with Lefschetz' generalizations of MacLane's theorem, Bruhn and Diestel use $\mathbb{Z}$ coefficients for cellular homology for Theorem 1.19. To understand the statements of the results we recall the language of cellular homology over $\mathbb{Z}$: we consider the *oriented* cycle space $\mathcal{C}(G)$ and the *oriented dimension* of a set of facial-boundary walks $\mathcal{W}$, where each walk $W \in \mathcal{W}$ can be interpreted as a sum over $\mathbb{Z}$ of the (directed) edges appearing in W counting multiplicity. We must also consider the codimension of $\mathcal{W}$ in $\mathcal{C}(G)$, which

is the dimension of $\mathcal{C}(G)/\langle\mathcal{W}\rangle$. One may also note the counterpart in $Z(G)$, that is, the codimension of $\mathcal{W}$ in $Z(G)$, in which the coefficients of each walk are reduced modulo 2 and the edges are not directed. Note, as we discussed earlier in our discussion of the work of Abrams and Slilaty, that in Theorem 1.18, below, the orientability of the surface in which the graph is embedded is not determined just by considering the codimension of $\langle\mathcal{W}\rangle$ in $Z(G)$, hence the absence of the mention of orientablity.

Since not all embeddings of a graph G in a surface S are cellular, not all classes of $H_1(S)$ are represented by elements of the (non)oriented cycle space of an embedded G. Thus, the codimension of the subspace of $\mathcal{C}(G)$ or $Z(G)$ generated by the boundary walks of faces that are homeomorphic to discs may be less than the dimension of $H_1(S)$, hence Theorem 1.18. The *Euler genus* of a surface is $2-\chi(S)$, which by Corollary 1.3 is the rank of $H_1(S)$.

1.18 Theorem. [4]**(Bruhn and Diestel, 2009)** For every integer $\epsilon \geq 0$, a graph G can be embedded in some surface of Euler genus at most ϵ iff there is a sparse family of closed walks in G whose codimension in $Z(G)$ is at most ϵ.

As stated earlier, the *oriented dimension* of a family of walks is needed to distinguish between embeddings in orientable surfaces and nonorientable surfaces.

1.19 Theorem. [4]**(Bruhn and Diestel, 2009)** Let S be any surface, and let ϵ denote its Euler genus. Let G be any loopless graph, and let k denote the number of its components.

1. If S is orientable, then G can be embedded in S iff G has a double cover by a locally sparse family $\mathcal{W}$ of closed walks whose oriented dimension is at most $|\mathcal{W}|-k$ and which has codimension at most ϵ.

2. If G is connected and S is not orientable, then G can be embedded in S iff there is a sparse family $\mathcal{W}$ of closed walks in G whose codimension in $\mathcal{C}(\mathcal{G})$ is at most $\epsilon-1$.

Bruhn and Diestel go so far as to conjecture that in Part 1 of Theorem 1.19 the hypothesis of $\mathcal{W}$ being locally sparse cannot be replaced with $\mathcal{W}$ being sparse. One reason to see why this could be true is that Edmonds' procedure (as used in [4, Lemma 7]) creates new facial boundary walks in G, but does not result in a new pseudosurface with a different Euler

characteristic. Thus, the removal of a pinchpoint can be seen to coincide with the addition of a crosscap in order to preserve the Euler characteristic of a topological space, per the arithmetic of Theorem 1.2.

1.2 An analysis of this history

If mathematical results are to be distinguished by whether they help to build a theory or solve a problem, then the results of this chapter definitely appear to be the former and not the latter. A first reading of the theorems in this chapter may leave the reader thinking that the homological approach to the study of graph embeddings is useless because one must consider all possible ways to glue discs to the graph in order to determine the surfaces and pseudosurfaces in which the graph is embeddable. Indeed, the more general a theorem of this sort is, the more difficult it seems to apply it; of course a graph can be cellularly embedded in a certain pseudosurface P if and only if the right sort of boundary walks for 2-cells can be found in G. It seems too difficult to apply these kinds of criteria, and yet these results do suggest an approach to deciding whether or not a graph can be properly embedded in P. One could attempt to describe a graph embedding in terms of assigning a homology class to each 1-chain of G. This is the spirit of this dissertation's approach to graph embeddability.

It is worth pointing out that a homotopy-oriented approach to the question of graph embeddability has been used to prove some very interesting results. For example, in [20] Thomassen proved that if a graph embedding in an orientable surface is such that each facial boundary has fewer edges than each noncontractible cycle does, then the embedding is a minimum genus embedding.

1.3 The contributions of this dissertation

Here, we state the contributions of this dissertation according to the chapters in which they appear.

The more substantial results of Chapter 2 concern the use of intersection theory to obtain a global homological understanding of a graph embedding in terms of local intersection properties.

- Theorem 2.9 relates the span of classes of $H_1(S)$ represented by certain kinds of 1-chains to related face-connected sub-2-complexes of S.

- We show how intersection theory can be used to derive a global homological understanding of a graph embedding in terms of local intersection data. Theorems 2.19 and 2.20 relate the row space of certain matrix representations of $\mathbb{Z}$ or $\mathbb{Z}_2$ intersection forms to independence of a set of classes of $H_1(S)$. Theorem 2.21 and Corollary 2.23 show how, using intersection theory, a homologically independent set of 1-chains satisfying specific conditions can be extended to an even larger set of homologically independent 1-chains.

- Remark 2.25 explains how the aforementioned results that concern intersection theory for graphs embedded in surfaces can be used to treat proper embeddings of graphs in pseudosurfaces.

- Lemma 2.27 shows how intersection theory can be used to prove that in a Klein bottle, there is a crossing of an orientation-preserving nonseparating circle and an orientation-reversing circle.

To have an understanding of the contributions of the remainder of this dissertation, the reader must have some understanding of ordinary voltage graphs and ordinary voltage graph embeddings. Consider a surface S, a pseudosurface P, and graph embeddings $G \to S$ and $G \to P$, which induce cellular decompositions of S and P, respectively. An ordinary voltage graph embedding in a pseudosurface contains an encoding of a *derived graph* G^α, and a *derived embedding* $G^\alpha \to P^\alpha$ in the form of an assignment of elements of a finite group A to the directed edges of G in a base embedding denoted $\langle G \to P, \alpha \to A \rangle$. The graph G^α and the pseudosurface P^α are covering spaces of G and P, respectively. Similar notation is used in connection to an ordinary voltage graph embedding in a surface.

Chapter 3 is expository. There, the definition of a set of consecutive lifts of a closed walk W of a closed walk is formulated and the known voltage group action on the derived graph is extended to the sets of consecutive lifts of W.

Chapter 4 builds a theory that is directed toward understanding the derived embedding in terms of induced subcomplexes of derived surfaces and pseudosurfaces corresponding to cosets of carefully constructed subgroups of A.

- Lemmas 4.2, 4.5, and 4.6 establish bijections between cosets of certain subroups of A and components of fibers over special induced subcomplexes of P and G. Lemma 4.7 establishes a bijection between another subgroup of A and sets of consecutive lifts of a closed walk that single cover a component of the fibers over a connected subgraph of G. These Lemmas are used to prove Theorem 4.8, which counts the number of components of P^α and components of the fiber over connected induced subcomplexes of the 2-complex induced by $G \to P$.

- For base embeddings in a surface, Archdeacon's medial-graph enhancement to an ordinary voltage graph embedding in a surface is used to prove Lemma 4.21, which establishes another correspondence between cosets of another specially constructed subgroup and regions of S^α bounded by orientation-preserving circles forming the fiber over a nonseparating circle in $G \to S$.

The results of Chapter 5 develop a homological understanding of the derived embedding in terms of intersection data and the cosets of the specially-constructed subgroups developed in Chapter 4.

- For a base embedding in a pseudosurface, Proposition 5.1 establishes an upper bound for the dimension in $H_1(P^\alpha)$ of the span of homology classes represented by the 1-chains forming the fiber over a homologically trivial 1-chain inducing a connected subgraph of G.

The remaining results of Chapter 5 concern base embeddings in surfaces.

- For a homologically trivial 1-chain z inducing a circle in S, Proposition 5.3 establishes the dimension in $H_1(S^\alpha)$ of the span of the homology classes represented by the elements of the fiber over z. Construction 5.2 describes a graph that allows us to chose a basis B, and Remark 5.4 explains how B can be extended to a set that is independent in $H_1(S^\alpha)$ and is twice as large as B.

- For a homologically nontrivial 1-chain z inducing a nonseparating orientation-reserving circle, Proposition 5.6 establishes that if the 1-chains in the fiber over z induce orientation-reversing circles, then they are homologically independent. Corollary 5.7 establishes the dimension in $H_1(S^\alpha)$ of the span of the homology classes represented by the elements of the fiber over z. In the case that the 1-chains in the fiber over z induce orientation-preserving circles, Construction 5.15 describes a graph in such a way that allows us to determine the dimension in $H_1(S^\alpha)$ of the span of the homology classes represented by elements of the fiber over z. Remark 5.17 describes the incidences of the edges and vertices of this graph, which allows us to determine a basis B, and Remark 5.18 explains how B can be extended to a set that is independent in $H_1(S^\alpha)$ that is twice as large as B.

- For a homologically nontrivial 1-chain z inducing a nonseparating orientation-preserving circle. Corollary 5.21 shows that no single element in the fiber over z is homologically trivial. Construction 5.22 describes a graph in such a way that establishes the dimension in $H_1(S^\alpha)$ of the span of the homology classes represented by the elements of the fiber over z. Remark 5.24 describes the incidences of the edges and vertices of this graph, which allows us to determine a basis B, and Remark 5.25 explains how B can be extended to a set that is independent in $H_1(S^\alpha)$ that is twice as large as B.

Chapter 6 concerns applications of the existing ordinary voltage graph theory and the results of this dissertation to the study of graph embeddability in surfaces.

- We show in Theorem 6.9 that for distinct primes $p, q > 5$, the Cartesian product of two circle graphs C_p and C_q has no embedding in Dyck's (3-crosscap) surface that can be derived from an ordinary voltage graph embedding, while we can produce cellular

embeddings of these graphs in Dyck's surface.

- Similarly, we show in Theorem 6.16 that for a prime $p > 5$ that there is no embedding of the Generalized Petersen Graph of the form $GP(2p, 2)$ in the torus that can be derived from an ordinary voltage graph embedding, while using an adaptation a technique borrowed from Xuong, [11, Lemma 3.4.8] we can produce cellular embeddings of these graphs in the torus.

- The existing theory allows us to conclude that there is no free action of any group on the aforementioned graphs of the form $C_p \times C_q$ embedded in Dyck's surface or the aforementioned graphs of the form $GP(2p, 2)$ embedded in the torus that extends to a cellular automorphism.

Chapter 2

Graphs, Graph Embeddings, and Cellular Homology

This chapter gives necessary background information and introduces some new theorems as well.

2.1 Necessary definitions and theorems

For the purposes of this dissertation $G = (V, E)$ will denote a finite and connected multigraph, which may include loops and parallel edges, and for an integer $m > 0$, $[m]$ will denote $\{1, 2, \ldots, m\}$. An edge is a *link* if it is not a loop. For $v \in V(G)$, let $star(v)$ denote the set of links incident to v, and let $star_+(v)$ denote the set of all edges incident to v. A path in G is a sequence of vertices and edges $v_1 e_1 v_2 e_2 \ldots e_{l-1} v_l$ such that the v_i are all distinct and the edge e_i connects vertices v_i and v_{i+1}. Given a subgraph H of G, an H-path is a path that meets H at its end vertices and only at its end vertices. Given two disjoint subgraphs H_1, H_2 of G, an H_1-H_2-path is a path that has one of its end vertices in H_1, the other end vertex in H_2, and no other vertices in $H_1 \cup H_2$. Let $D(G)$ denote the set of darts (directed edges) on the edges of G. To each dart $d \in D$ is associated a *head vertex* $h(d)$ and a *tail vertex* $t(d)$; we say that two distinct darts on the same edge are *opposites* of each other. We will call one dart on e the *positively directed dart* and the other the *negatively*

directed dart. A *walk* W in G is a sequence of darts $d_1 d_2 \dots d_m$ such that $h(d_b) = t(d_{b+1})$ for all $b \in [m-1]$. If $h(d_m) = t(d_1)$, then we say that W is a *closed walk*. The notion of H-walk is defined by analogy with H-path. We define the operation of *subdivision of an edge* e to be the replacement of e with a path of length 2. We also define the *subdivision of a graph* to be the operation of subdividing each edge of the graph.

Let S denote a compact connected surface without boundary. A *pseudosurface* is the geometric realization of a finite simplicial 2-complex in which every 1-simplex is a subface of at least one 2-simplex. Every surface is by definition a pseudosurface. An example of a pseudosurface that is not a surface appears in Figure 2.3. We will let P denote a pseudosurface without boundary, which has the additional constraint that every 1-simplex is a subface of exactly two 2-simplices. If we think of a graph as a topological space, then we may define a graph embedding in a surface (or pseudosurface) to be a continuous one to one function $i\colon G \to S$ (or $i\colon G \to P$). A *cellular embedding* of G in S is an embedding that subdivides S into 2-cells. A *proper embedding* of G in a pseudosurface P is an embedding of G that subdivides P into 2-cells such that the image of the vertex set of G necessarily contains the pinchpoints of P and all boundary components of P are contained in the image of G. The regions of the complement of an embedded G in P (or S), are called *faces* of the embedding. We will let $G \to S$ and $G \to P$ denote a cellular embedding of G in S and a proper embedding of G in P, respectively. We let $F(G \to P)$ denote the set of faces of $G \to P$. Since $G \to P$ induces a cellular decomposition, P (and S) can be thought of as a cellular chain complex $\{C_k(P), \partial\}$, where $C_0(P)$ and $C_1(P)$ are the $\mathbb{Z}_2$-vector spaces of formal linear combinations of elements of $V(G)$ and $E(G)$ respectively. We let $C_2(P)$ denote the $\mathbb{Z}_2$-vector space of formal linear combinations of faces of $G \to P$. We similarly denote the associated chain complex and subcomplexes for $G \to S$. We let $\partial\colon C_k(P) \to C_{k-1}(P)$ denote the usual cellular boundary map. Note that the fact that we are using $\mathbb{Z}_2$ coefficients means that for $f \in C_2(P)$, ∂f can be expressed as a sum of all edges appearing once in a boundary walk of f, regardless of any orientation on the edges bounding f. We will sometimes use $F(G \to P)$ in place of $C_2(P)$. The graph G can also be thought of as a cellular 1-complex with $\mathbb{Z}_2$-vector spaces $C_1(G)$ and $C_0(G)$.

For $I \subset F(G \to P)$ let ∂I denote $\sum_{f \in I} \partial f$, treating I as if it were an element of $C_2(P)$, and let I^c denote the complement of I in $F(G \to P)$. Let $\vec{C}_0(G)$ and $\vec{C}_1(G)$ denote the $\mathbb{Z}$-modules of formal linear combinations of elements of $V(G)$ and $D(G)$, respectively. We will let $\vec{C}_2(G \to P)$ denote the $\mathbb{Z}$-module of formal linear combinations of faces of $G \to P$. We will use the notation $\left\{\vec{C}_i(P), \partial\right\}$ to denote the associated chain complex over $\mathbb{Z}$. Note that the boundary map is denoted ∂ in both $\left\{\vec{C}_i(P), \partial\right\}$ and $\{C_i(P), \partial\}$.

For $I \subset F(G \to P)$ let $P{:}I$ denote the induced sub-2-complex of P consisting of the faces of I and all vertices and edges bounding the faces of I. Let $G{:}I$ denote the induced subgraph of the embedded G consisting of the vertices and edges bounding the faces of I. Even though P has no boundary edges, $P{:}I$ may contain boundary edges. For $X \subset E(G)$ or $X \in C_1(G)$ let $G{:}X$ denote the subgraph of G consisting of edges of X and all vertices of G to which the edges of X are incident. If $G{:}X$ is connected, then we will say that X is connected. For $\vec{z} \in \vec{C}_1(G)$ let $G{:}\vec{z}$ denote the directed graph whose edges are directed according to the coefficient of each edge e with a nonzero coefficient: a positive coefficient means that it is the positive dart whose direction is imposed on e. For $I \in \vec{C}_2(P)$, let $P{:}I$ denote the induced nonoriented subcomplex consisting of the faces with nonzero coefficients appearing in I and all nonoriented subfaces.

Two proper or cellular embeddings of G are said to be isomorphic if there is a bijection between their faces that preserves the incidence of faces and edges.

A subgraph of G is called a *circle* if it is a connected 2-regular graph. Let $Z(G)$ denote the subspace of $C_1(G)$ with generating set

$$\{z_i : \ G{:}z_i \text{ is a circle in } G\}.$$

The subspace $Z(G)$ is called the *cycle space* of G. We will let $\mathcal{C}(G)$ denote the submodule of $\vec{C}_1(G)$ with generating set

$$\left\{d \in \vec{C}_1(G) : \ G{:}d \text{ is an oriented circle in } G\right\}.$$

The $\mathbb{Z}$-module $\mathcal{C}(G)$ is called the *oriented cycle space of* G. Observe that $Z(G)$ and $\mathcal{C}(G)$

are the kernels of the boundary map ∂ when ∂ is applied to $C_1(G)$ and $\overrightarrow{C}_1(G)$, respectively.

Given a spanning tree T of G, each edge e of $G \setminus T$ when adjoined to T creates a unique circle C_e. When considered as a 1-chain, the sum of the edges of C_e is called a *fundamental cycle of G with respect to T*. Let $\beta_1(G)$ denote the dimension of $Z(G)$, which is also the same as the dimension of $\mathcal{C}(G)$. Per Theorem 1.7 we have $\beta_1(G) = |E(G)| - |V(G)| + 1$, which is the number of fundamental edges of G with respect to any spanning tree T. Let $B(G)$ denote the subspace of $C_1(G)$ with generating set

$$\{star(v) : v \in V(G)\}.$$

The subspace $B(G)$ is called the *cut space* or the *bond space* of G. If e is an edge of T, $T \setminus \{e\}$ is a disjoint union of two connected components, T_1 and T_2. The edges of G that join T_1 and T_2 are denoted D_e, and are called a *fundamental cut of G with respect to T*. When considered a 1-chain,

$$D_e = \sum_{v \in V(T_1)} star(v) = \sum_{v \in V(T_2)} star(v).$$

For a more complete treatment of the cycle and cut spaces of G (as $\mathbb{Z}_2$-vector spaces), the reader is referred to [5, §1.9].

Recall that a face walk from face f_a to face f_b is a sequence of faces whose first element is f_a, whose last last element is f_b, and such that any two consecutive faces share a common boundary edge. A face-connected component of P is induced by a maximal $I \subset F(G \to P)$ such that $P : I$ is a face connected 2-complex. We define a *face-block partition* of $F(G \to P)$ to be a partition $\{B_1, B_2, \ldots, B_k\}$ such that $P{:}B_i$ is a face-connected component for all $i \in [k]$. The next two lemmas are easy to prove using the structure of cellular 2-complexes. Their proofs are omitted.

2.1 Lemma. If $I \subset F(G \to P)$ is such that $P{:}I$ is connected, then $G{:}I$ is connected.

2.2 Lemma. If $I \subset F(G \to P)$ is such that ∂I is connected, then $P{:}I$ is connected.

For the remainder of this dissertation P will be assumed to be a face-connected pseudo-surface having no boundary edges. For $G \to P$ and $v \in V(G)$, let $U^*(v)$ denote the subspace of P homeomorphic to a set of open discs that are all identified at a single point (which will be v if v is mapped to a pinchpoint by $G \to P$), whose intersection with G consists of the vertex v and the ends of the edges of $star_+(v)$. We also require that $U^*(v) \setminus G{:}star_+(v)$ is a disjoint union of open sets in P. The set $U^*(v)$ will be referred to as a *vertex-star neighborhood of* v. Each one of the discs appearing in $U^*(v)$ is called an *umbrella of* v.

Given $G \to S$ and following [11, §3.2], if one thickens the embedded graph such that the vertices become discs and the edges become rectangular strips glued to the discs, one produces what is called a *band decomposition* of the surface S. The *0-bands* are the discs, the *1-bands* are the rectangular strips, and the *2-bands* are discs glued to the 1-bands and the 0-bands. If one of the two possible orientations on any given 1-band joining two 0-bands is consistent with both of the orientations induced by ρ on the joined 0-bands (see [11, Figures 3.13, 3.14] for enlightening diagrams), then it is said that the given 1-band is *orientation preserving* (type-0), else it is *orientation reversing* (type-1). An edge e of an embedded graph may be designated an orientation-reversing (type-1) edge to mean that its associated 1-band is orientation reversing and an *orientation-preserving edge* otherwise. It may help the reader to think of an orientation-reversing edge as being "twisted".

For each $v \in V(G)$ and dart d with tail vertex v, let $\rho\colon D \to D$ be the permutation that takes d to the next dart in the cyclic order of darts with tail vertex v appearing in the same umbrella as d. The order of darts that follows the order induced by ρ is called the *rotation on* v. A rotation on a vertex v is a list of the form $v : d_i d_j d_k \ldots$. The permutation ρ combinatorially encodes $G \to P$ and is called a rotation scheme on G. Following [11, §3.2] if e is an orientation-reversing edge of $G \to P$, then the rotation scheme of G features a 1 superscript above all occurrences of a dart on e. A rotation scheme can be denoted by listing all of the rotations on all of the vertices of G. See [11, Figure 3.15] for an elucidating figure.

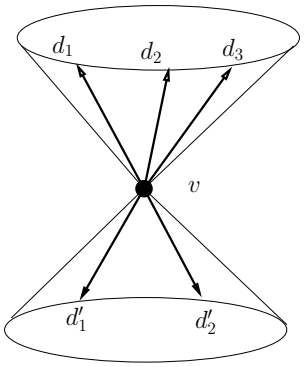


Figure 2.1: A vertex-star neighborhood of a pinchpoint vertex v. The rotation on v is denoted $v{:}d_1d_2d_3$; $d'_1d'_2$. This particular vertex-star neighborhood has two umbrellas.

For a fixed vertex v of G, consider $U^*(v)$, and let the *corners of* $G \to P$ *at* v refer to the components of $U^*(v) \setminus G{:}star_+(v)$. We define the *medial graph* to be the graph whose vertices are the edges of G and having an edge joining any two vertices that correspond to the two edge ends bounding a corner. The medial graph is clearly 4-regular; as Archdeacon [3, page 114] points out, each face creates two adjacencies for each edge in its boundary. Figure 2.2 shows an example of a proper (cellular) graph embedding and its corresponding medial graph. For a medial graph of a graph cellularly embedded in a surface, note that each face of the medial graph embedding falls into one of two categories, those which correspond to the vertices of G and those which correspond to the faces of $G \to P$.

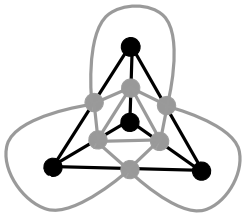


Figure 2.2: A cellularly embedded graph G in the sphere and the corresponding medial graph, colored gray.

For $G \to P$, a connected 2-regular subgraph is called an orientation-reversing circle if it has a regular neighborhood homeomorphic to a Möbius band, and is called an orientation-preserving if it has a regular neighborhood homeomorphic to an annulus. A walk W having an odd number of orientation-preserving edges and the property that any two consecu-

tive darts $d_i d_{i+1}$ of W are directed into and out of the same umbrella of v is called an orientation-reversing walk; replacing the qualifier "odd" with "even" would make W an orientation-preserving walk. An example of an orientation-preserving walk in a graph properly embedded in a pseudosurface is given in Figure 2.3.

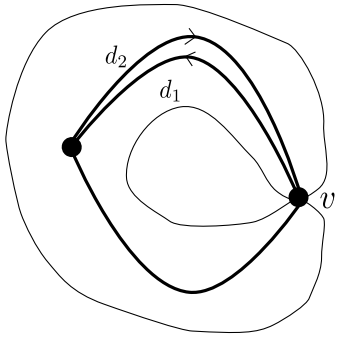


Figure 2.3: Two consecutive darts of a closed walk that are directed into and out of the same umbrella at v.

Following the discussion on [11, page 111], one may reverse the orientation on a 0-band without changing the embedding that corresponds to the decomposition and the rotation scheme: the rotation on that vertex v is reversed (the rotation on v is then given by ρ^{-1}) and the orientation type of each link incident to v is changed. Archdeacon in [3] calls this process a local sign switch. Two band decompositions of a surface S are considered to be equivalent if they differ by a sequence of local sign switches.

Let $\hat{P}$ denote the surface that is created by releasing all of the pinches of P. We will let $\hat{G}$ denote the 1-skeleton of $\hat{P}$ and call $\hat{G} \to \hat{P}$ the *pinchpoint released graph embedding* of $G \to P$. Figure 2.4 illuminates this definition.

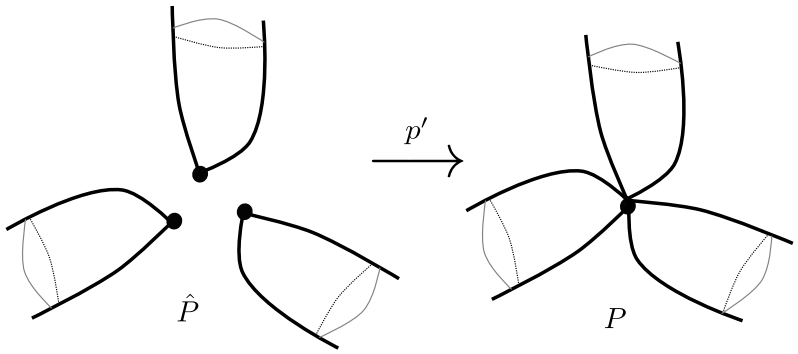


Figure 2.4: A pinchpoint vertex v of $G \to P$ whose vertex star neighborhood contains three umbrellas, and whose preimage under a quotient map p' is three distinct vertices contained in three nonintersecting umbrellas.

2.1.1 Cellular homology

Let $B(G \to P)$ denote the subspace of $Z(G)$ with generating set

$$\{\partial f : f \in C_2(G \to P)\},$$

and let $\vec{B}(G \to P)$ denote the subspace of $\vec{C}_1(G)$ generated by

$$\left\{\partial f : f \in \vec{C}_2(G \to P)\right\}.$$

In the literature $B(G \to P)$ is called the *boundary space* of $G \to P$. Note that $Z(G)/B(G \to P)$ is the first homology group $H_1(P)$ with $\mathbb{Z}_2$ coefficients. We will let $[\cdot]_2 \colon Z(G) \to Z(G)/B(G \to P)$ denote the projection map and, and we will let $[\cdot]$ also denote the corresponding projection map $\mathcal{C}(G) \to \mathcal{C}(G)/\vec{B}(G \to P)$. If $[z] = [0]$ or $[\vec{z}] = [0]$, then we say that z or $\vec{z}$ is *homologically trivial*.

2.1.2 2-complexes

2.3 Lemma. If I is a nonempty proper subset of $F(G \to P)$, then for all nonempty proper subsets T of I, $\partial T \neq \partial I$.

Proof. Let Π_I and Π_T denote face-block partitions of I and T, respectively. Since T is a nonempty proper subset of I, there is a block T_h of Π_T properly contained in a corresponding

block I_k of Π_I. A face walk beginning at a face of T_h and ending at a face of $I_k \setminus T_h$ must contain two consecutive faces f_γ and f_δ such that f_γ is a face of T_h and f_δ is not. Let e denote a common boundary edge of f_γ and f_δ and note that e appears in ∂T_h but not in ∂I_k. The conclusion follows. □

2.4 Corollary. For a subset I of $F(G \to P)$, $\partial I \neq 0$ if and only if I is a nonempty proper subset of $F(G \to P)$.

2.5 Theorem. For $I \subset F(G \to P)$, $\partial I = \partial(I^c)$.

Proof. By Corollary 2.4 the result is trivial if $I = \emptyset$ or $I = F(G \to P)$.

If $e \in \partial I$, then e is an edge appearing in the boundary of $P{:}I$. Since P is a pseudosurface containing no boundary edges, $e \in \partial I^c$. A similar argument completes the inclusion in the other direction. □

Let $z \in Z(G)$ have the property that $G{:}z$ is a circle whose edges intersect at most one umbrella of each $v \in V(G{:}z)$. Let $R(z)$ denote a connected set of points of P properly containing $G{:}z$ and satisfying:

1. the vertices of $G{:}z$ are the only possible boundary points of $R(z)$,
2. the only whole edges contained in $R(z)$ are those of $G{:}z$, and
3. the only edge ends contained in $R(z)$ are those that are incident to the vertices v of $G{:}z$ and are contained in the same umbrellas of the vertices of $G{:}z$.

We will call $R(z)$ a *ribbon neighborhood of* z. Note from Figure 2.5 that $R(z)$ need not be open in P. Observe that a ribbon neighborhood of z is homeomorphic to an annulus or to a Möbius band (both not containing their respective boundaries). Observe also that in order for z to have a ribbon neighborhood, it is both a necessary and sufficient condition that the edges of z intersect only one umbrella of $U^*(v)$ for each $v \in V(G{:}z)$.

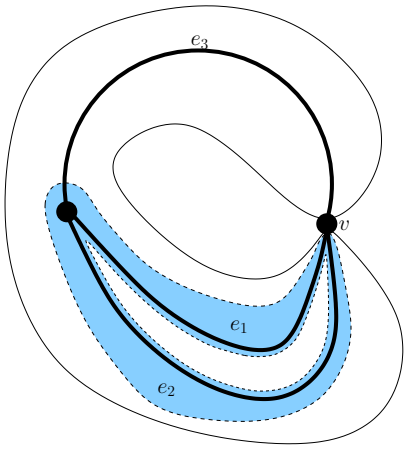


Figure 2.5: A ribbon neighborhood of $e_1 + e_2$.

We now introduce an operation that will help us to compute $\dim_{\mathbb{Z}_2}\langle [z_1]_2, \ldots, [z_n]_2\rangle$ for specific $z_1, \ldots z_n \in Z(G)$. Define $S \triangle z$ to be the result after topologically ungluing S along $G{:}z$. Consider $z \in Z(G)$ such that $G{:}z$ is an orientation-preserving circle whose edges intersect at most one umbrella of each vertex of $G{:}z$. We define $S \triangle z$ to be the cellular 2-complex with two boundary components $(S \triangle z){:}z'$ and $(S \triangle z){:}z''$ that satisfy:

1. each boundary component is isomorphic to $G{:}z$ when considered as a graph, and
2. the quotient space
$$(S \triangle z)/\left((S \triangle z){:}z' \sim (S \triangle z){:}z''\right)$$
is homeomorphic to S.

We define a *set of 1-chains having property* Δ to be a set of 1-chains in $Z(G)$ or $\mathcal{C}(G)$ inducing orientation-preserving circles with ribbon neighborhoods such no two distinct $G{:}z_j$ have any vertices in common. For a set of 1-chains having property Δ, we let $S \triangle \sum_{j\in[m]} z_j$ denote the result after topologically ungluing S along each of the $G{:}z_j$.

2.6 Remark. Note that only the edges (and their darts) appearing in z' and z'' appear as boundary edges of $S \triangle \sum_{j\in[m]} z_j$.

Let $Q\colon S \triangle \sum_{j\in[m]} z_j \to S$ denote the gluing (quotient) map that undoes the $\triangle$ operation. The map Q can be considered a map on topological spaces or a chain map on cellular

chain complexes. Clearly, Q is a bijection between the faces of $S \triangle z$ and the faces of S, is two-to-one onto the vertices and edges of G:$\sum_{j\in[m]} z_j$, and is one-to-one on all other vertices and edges of S. We will let $G \triangle \sum_{j\in[m]} z_j$ denote the 1-skeleton of $S \triangle z$.

2.7 Lemma. Consider $G \to S$ and let $\{z_1, z_2, \ldots, z_m\}$ be a set of 1-chains having property **Δ**. The connected components of $S \triangle \sum_{j\in[m]} z_j$ are the face-connected components of $S \triangle \sum_{j\in[m]} z_j$.

Proof. Let $I \subset F(G \to S)$ be such that S:I is connected and not face connected. Thus, there are face-connected components of $S \triangle \sum_{j\in[m]} z_j$ having only vertices and no edges in common. It follows that the boundary components of $S \triangle \sum_{j\in[m]} z_j$ are not disjoint circles, contradicting Remark 2.6. □

Figure 2.6 illustrates the fact that the conclusion of Lemma 2.7 is not necessarily true if we try to apply the operation of topologically ungluing a pseudosurface P along an orientation-preserving circle. The connected components of the resulting 2-complex are not the face connected components.

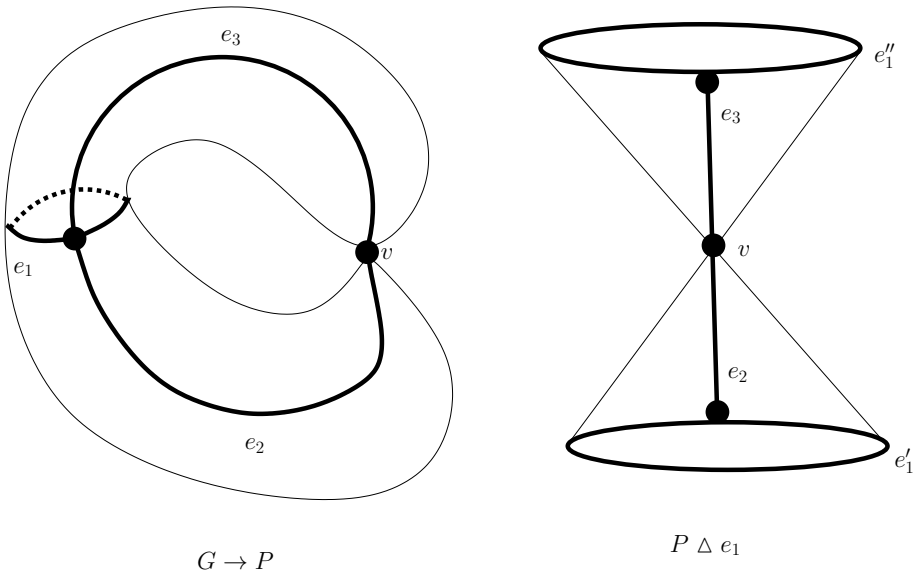


Figure 2.6: A proper graph embedding in the spindle surface and the result after ungluing along an orientation preserving circle with a ribbon neighborhood.

2.8 Lemma. Consider $G \to S$. If $\{z_1, z_2 \ldots, z_m\}$ is a set of 1-chains having property Δ, then

$$\dim_{\mathbb{Z}_2}\langle [z_1]_2, \ldots, [z_m]_2\rangle = m$$

if and only if

$$S \vartriangle \sum_{j\in[m]} z_j$$

is face connected.

Proof. " $\Rightarrow$ " Assume for the sake of contradiction that $S \vartriangle \sum_{i\in[m]} z_i$ is not face connected. Let $I^\vartriangle$ denote the set of faces of a face-connected component $S_I^\vartriangle$ of $S \vartriangle \sum_{i\in[m]} z_i$. Since the boundary components of $S \vartriangle \sum_{j\in[m]} z_j$ are precisely the preimage of the z_i under Q, there is at least one boundary component of $S_I^\vartriangle$ that has the same image under Q as another face-connected component of $S \vartriangle \sum_{i\in[m]} z_i$. By Lemma 2.7, it follows that

$$\partial I^\vartriangle \in \langle z_1, \ldots, z_m\rangle$$

and $\partial I^\vartriangle \neq 0$. Thus, $\dim_{\mathbb{Z}_2}\langle [z_1]_2, \ldots, [z_m]_2\rangle < m$.

" $\Leftarrow$ " Assume for the sake of contradiction that $\dim_{\mathbb{Z}_2}\langle [z_1]_2, [z_2]_2, \ldots, [z_m]_2\rangle < m$. Thus, there exists $I \subset F(G \to S)$ and $z \in Z(G)$ such that

$$z = \partial I \in \langle z_1, \ldots, z_m\rangle$$

and $z \neq 0$. By Theorem 2.5 we know that $z = \partial I^c$ and that I and I^c are nonempty. It follows that $S \vartriangle \sum_{i\in[m]} z_i$ is not face connected. □

For $G \to S$ and a set of (oriented) 1-chains $\{z_1, \ldots z_m\}$ over having property Δ, a *z-region* of S relative to $\{z_1, \ldots z_m\}$ is an induced subcomplex $S{:}I$ for some $I \subset F(G \to S)$ such that $(S \vartriangle \sum_{j\in[m]} z_j){:}I$ is a face-connected component of $S \vartriangle \sum_{j\in[m]} z_j$. We define the associated *z-graph* $\Gamma_z(z_1, \ldots z_m)$ to be the graph whose vertices are the z-regions of S relative to $\{z_1, \ldots, z_m\}$, and whose edges are the z_j. An edge z_j is incident to a vertex $S{:}K$ if $G{:}z_j$ is contained in the boundary of $S{:}K$. Since S is face connected, $\Gamma_z(z_1, \ldots z_m)$ is a

connected graph.

2.9 Theorem. Consider $G \to S$ and let $\{z_1, \ldots, z_m\}$ denote a set of 1-chains with $\mathbb{Z}_2$ coeffeceients having property Δ. We have

$$\dim_{\mathbb{Z}_2}\langle [z_1]_2, \ldots, [z_m]_2\rangle = \beta_1(\Gamma_z(z_1, \ldots, z_m)).$$

Proof. Consider a spanning tree T of $\Gamma_z(z_1, \ldots, z_m)$. Let $K \subset [m]$ be such that $\{z_k : k \in K\}$ is the set of fundamental edges of $\Gamma_z(z_1, \ldots z_m)$ with respect to T. Since

$$\Gamma(z_1, \ldots, z_m) \setminus \{z_k : k \in K\}$$

is connected, $S \vartriangle \sum_{k\in K} z_j$ is face connected. Lemma 2.8 guarantees that the elements of $\{z_k : k \in K\}$ are independent in $H_1(S)$.

We now show that $\{z_k : k \in K\}$ is maximally independent in $\{z_1, z_2, \ldots, z_m\}$. Let j denote an element of $[m] \setminus K$ and let T_1, T_2 denote the two connected components of $T \setminus z_j$. The sum

$$\sum_{v\in V(T_1)} star(v)$$

is nonzero since z_j is an edge joining T_1 and T_2. Since the vertices of $\Gamma_z(z_1, \ldots, z_m)$ correspond bijectively with the z-regions of S, and the z-regions of S are face connected per Lemma 2.7, we have that T_1 and T_2 correspond to face connected components of $S \vartriangle \sum_{i\in K\cup\{j\}} z_i$. Therefore $\{[z_i]_2 : i \in K \cup \{j\}\}$ is dependent. □

2.10 Remark. Note that the proof of Theorem 2.9 is constructive in that it provides a basis for the span of the set $\{z_1, z_2, \ldots, z_m\}$.

Recall that to each proper embedding of a graph in a pseudosurface is associated an embedding $\hat{G} \to \hat{P}$ of a graph $\hat{G}$ in a surface $\hat{P}$ that is created by releasing the pinchpoints of P.

2.11 Corollary. Consider $G \to P$ and the associated embedding $\hat{G} \to \hat{P}$. Let $\{z_1, \ldots, z_m\} \subset Z(G)$ denote a set of 1-chains having property Δ. If $\{\hat{z}_1, \ldots, \hat{z}_m\}$ are the corresponding 1-

chains of $Z(\hat{G})$, then

$$dim_{\mathbb{Z}_2} = \langle [z_1]_2, \ldots, [z_m]_2 \rangle = \beta_1(\Gamma_z(\hat{z}_1, \ldots, \hat{z}_m)),$$

where $\Gamma_z(\hat{z}_1, \ldots, \hat{z}_m)$ is possibly a disjoint union of graphs.

Proof. Since the z_i have ribbon neighborhoods, each of which is homeomorphic to an annulus, the $\hat{z}_i$ form a set of 1-chains having property Δ, which are possibly contained in different components of $\hat{P}$. The result follows from Theorem 2.9. □

2.2 Intersection theory

Here we show how the use of intersection theory can enhance our understanding of a cellular graph embedding. For a more general formulation of this theory see [6] and [7].

Two simple closed curves γ and γ' in a surface S are said to be in *general position* if all intersections are *transversal*, which means that for each point x of $\gamma \cap \gamma'$ the lines that are tangent to γ and γ' span all of the tangent space to S at x. Figure 2.7 makes it clear that if each of γ, γ' passes through two different umbrellas of a given pinchpoint π in a pseudosurface P, then they do not necessarily cross transversely at π; there is not necessarily a unique tangent plane.

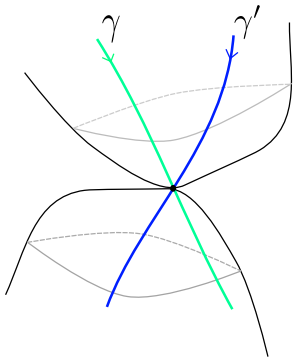


Figure 2.7: Two oriented simple curves that pass through two distinct umbrellas of a pinchpoint.

We will develop the appropriate intersection theory in the setting of graph embeddings

in a surface S, and then extend it to graph embeddings in pseudosurfaces. Consider two oriented simple closed curves γ, γ' in general position. Assume for now that S is oriented. Consider a point $x \in \gamma \cap \gamma'$ and lines ℓ_γ, $\ell_{\gamma'}$ that are tangent to γ and γ' at x, respectively, and have orientations induced by the orientations on γ and γ'. We define the function $sgn(\gamma, \gamma')$ by the following rules

1. $sgn_x(\gamma, \gamma') = 1$ if rotating ℓ_γ about x_j in the direction of $\ell_{\gamma'}$ coincides with the orientation on S, and
2. $sgn_x(\gamma, \gamma') = -1$ otherwise.

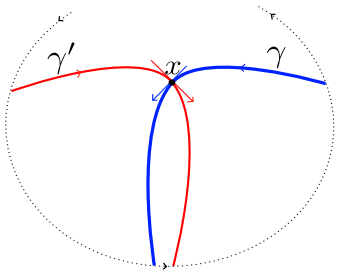


Figure 2.8: An example of a transverse crossing of two curves. Observe that $sgn_x(\gamma, \gamma') = 1$, and that $sgn_x(\gamma', \gamma) = -1$.

2.12 Definition. Let γ and γ' denote two oriented simple closed curves in general position and let $\{x_j : j \in J\} = \gamma \cap \gamma'$. We define the *intersection index of* γ *and* γ' to be the integer

$$(\gamma, \gamma') = \sum_j sgn_{x_j}(\gamma, \gamma').$$

2.13 Definition. If S is nonorientable or not oriented, then for two simple closed curves γ, γ' in general position we define (γ, γ') to be the residue modulo 2 of the number m of points in $\gamma \cap \gamma'$.

Note that the definition of the intersection index of two simple closed curves in a nonorientable surface also applies to two simple closed curves in an oriented surface, and that the intersection index is antisymmetric in the oriented case. In the nonoriented case the intersection index is symmetric. A more general version of Theorem 2.14 appears as Theorem 15.1.3 in [6].

2.14 Theorem. The intersection index of two simple closed curves is a homotopy invariant of γ and γ'.

Proof. Omitted. □

2.15 Remark. For an orientation-reversing simple closed curve γ, $(\gamma, \gamma) = 1$. For an orientation-preserving simple-closed curve γ', $(\gamma', \gamma') = 0$ for γ' in an oriented or nonoriented surface. See Figure 2.9.

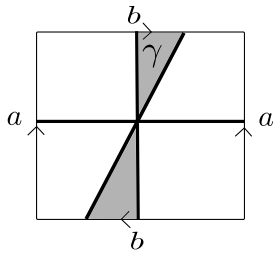


Figure 2.9: A depiction illustrating the fact that for an orientation-reversing simple closed curve γ, $(\gamma, \gamma) = 1$.

As discussed in [6, p. 85], any two simple closed curves that are embedded in a surface can be brought into general position with each other by using a homotopy of one of the embeddings, and by Theorem 2.14, this does not affect the intersection index of the two curves. The intersection product can therefore be defined for any two simple closed curves in a surface. Since $H_1(S)$ is isomorphic to the abelianization of the fundamental group $\pi_1(S)$, we see that the intersection index of two oriented simple closed curves can also be extended to 1-chains inducing oriented circles having ribbon neighborhoods, treating them as representatives of classes of $H_1(S)$. Per [7, Theorem 18.1] and the discussions that follow on [7, pp.220-221], this intersection index is a well defined bilinear map giving a nondegenerate, skew-symmetric, bilinear form on $H_1(S)$. Note that skew-symmetricity of the bilinear form follows from the fact that the intersection index of oriented simple closed curves is antisymmetric.

2.16 Definition. For an oriented surface S, we let

$$\langle \cdot, \cdot \rangle \colon H_1(S) \times H_1(S) \to \mathbb{Z}$$

denote the intersection index of homology classs. For a nonoriented surface S, we let

$$\langle \cdot, \cdot \rangle_2 \colon H_1(S) \times H_1(S) \to \mathbb{Z}_2$$

denote the intersection index of homology classes.

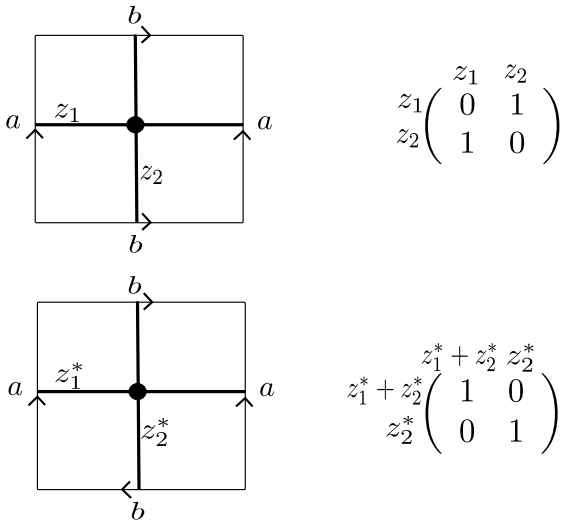
Note that it is possible to ignore the orientation on an oriented surface and use the intersection product $\langle \cdot, \cdot \rangle_2$. We infer from Figures 2.9 and 2.10 that $\langle [z]_2, [z]_2 \rangle_2 = 1$ if $G{:}z$ is an orientation-reversing circle.

2.17 Lemma. Consider $G \to S$. If $z' \in Z(G)$ is homologically trivial, then $\langle [z']_2, [z]_2 \rangle_2 = 0$ for all $z \in Z(G)$. If $z' \in \mathcal{C}(G)$ is homologically trivial, then $\langle [z'], [z] \rangle = 0$ for all $z \in \mathcal{C}(G)$.

Proof. This follows immediately from the bilinearity of the intersection indices since $[z] = 0$. □

Also per [7, pp. 220-221, paragraphs (b) and (c)], if we chose a basis of $H_1(S)$ for which the associated intersection matrix has a canonical form J_{2h} or I_c (see Theorem 1.16), the matrix can be said to algebraically associate the elements of the the basis to the handles or crosscaps of S. Note, as Example 2.18 points out, that the existence of such a basis does not imply that the basis elements themselves induce circles in G.

2.18 Example. Observe that in the case of the embedding in the torus in Figure 2.10 there is a basis whose elements are represented by 1-chains inducing circles, and with respect to these classes the matrix representation of the bilinear form is in a canonical form. In the case of the embedding in Klein bottle no such basis exists, but a different kind of basis can be used to obtain a matrix representation of the bilinear form that is in a canonical form.



$$\begin{array}{c} \\ z_1 \\ z_2 \end{array}\begin{array}{c} \begin{array}{cc} z_1 & z_2 \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ z_1^*+z_2^* \\ z_2^* \end{array}\begin{array}{c} \begin{array}{cc} z_1^*+z_2^* & z_2^* \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

Figure 2.10: A pair of graph embeddings with associated canonical intersection forms.

For $G \to S$ in an oriented surface S let $\{x_1, \ldots, x_l\} \subset \mathcal{C}(G)$ denote an ordered set of 1-chains, with each element $x_i \in X$ inducing a circle in G. Let M_X denote the skew-symmetric $l \times l$ matrix over $\mathbb{Z}$ whose entry m_{ij} is $\langle [x_i], [x_j] \rangle$.

2.19 Theorem. If the rows of M_X are linearly independent, then the elements of X are homologically independent.

Proof. Assume for the sake of contradiction that the nontrivial $\mathbb{Z}$-sum $\sigma = \sum a_i x_i$ is homologically trivial. Write $\sigma = (a_1, a_2, \ldots, a_l) \in \mathbb{Z}^l$. Since the rows of M_x are linearly independent, then $M_X \sigma^T$ is a nonzero vector in $\mathbb{Z}^l$. Thus, there exists a nonzero $y = (y_1, y_2, \ldots, y_l) \in \mathbb{Z}^l$ such that

$$y M_X \sigma^T \neq 0.$$

Let $\sigma_y = \sum y_i x_i$. It follows that

$$\begin{aligned} y(M_X \sigma) &= \sum_{i,j} a_i y_j (G{:}x_i, G{:}x_j) \\ &= \langle [\sigma_y], [\sigma] \rangle \\ &\neq 0. \end{aligned}$$

This contradicts Lemma 2.17. □

For a nonoriented pseudosurface S_2, similar conclusions follow for a similarly-constructed (symmetric) matrix M_{X_2} and an ordered set of 1-chains X_2 summed over $\mathbb{Z}_2$, where the entry m_{ij} of M_{X_2} is $\langle [x_i]_2, [x_j]_2 \rangle_2$.

2.20 Theorem. If the rows of M_{X_2} are linearly independent, then the elements of X_2 are homologically independent.

Proof. The proof of this result is a routine reformulation of the proof of Theorem 2.19. This proof is omitted. □

The implication of Theorems 2.19 and 2.20 is that we can determine that a set of 1-chains is homologically independent by evaluating the rank of the associated matrices M_X or M_{X_2}, which can be said to be interpreted as bilinear intersection forms determined with respect to a chosen set of 1-chains. While it may seem that M_{X_2} is a homological invariant that is too coarse to be useful, it will be repeatedly used later on. Note that even if M_X or M_{X_2} is not of full rank, it is not necessarily the case that the elements of X or X_2 are homologically dependent. This is clear when considering a 2-holed torus and letting X denote a set of 1-chains whose elements induce the longitudinal circles associated to each handle.

Recall Remark 2.10, which states that the proof of Theorem 2.9 is constructive in that it provides for a basis for the span of a set of 1-chains having property Δ. A consequence of Theorem 2.19 is that a homologically independent set that is constructed in this way can be extended to a homologically independent set of 1-chains that is twice as large.

2.21 Theorem. Consider $G \to S$ for a surface S. Let $\{z_1, z_2, \ldots, z_m\}$ denote a set of 1-chains having property Δ. Let $B = \{z_i : i \in I\}$ for some $I \subset [m]$ denote a basis for the span of $\{z_1, z_2, \ldots, z_m\}$ in the $\mathbb{Z}_2$ vector space $H_1(S)$. There is a another set of 1-chains $B' = \{b_i, i \in I\}$ such that the following are true:

$$\langle [z_i]_2, [b_i]_2 \rangle_2 = 1, \langle [z_i], [z_i] \rangle_2 = 0$$

and

$$\langle [z_i]_2, [b_j]_2 \rangle_2 = 0 \text{ for } i \neq j.$$

Proof. Consider the z-regions of S relative to $\{z_1, z_2, \ldots, z_m\}$ and the associated z-graph $\Gamma_z(z_1, \ldots, z_m)$. Per the proof of Theorem 2.9, B is the set of 1-chains of fundamental edges with respect to some spanning tree T of $\Gamma_z(z_1, \ldots, z_m)$. Consider the associated set of fundamental cycles $\{C_{z_i}\}$ of G and a fixed $C_{z_{i'}}$. Let $v_1e_1v_2e_2 \ldots v_le_lv_1$ denote an enumeration of the vertices and edges of $\Gamma{:}C_{z_{i'}}$ such that e_j joins v_j and v_{j+1}, where the subscripts are taken modulo l. For the remainder of this proof, abusing some notation, we let $G{:}e_i$ denote the subgraph induced by the edges of e_i. Since $G \to S$ is a cellular embedding, for any z-region v_k there is a $G{:}e_{k-1}$-$G{:}e_k$ path H_k contained in v_k, as in Figure 2.11

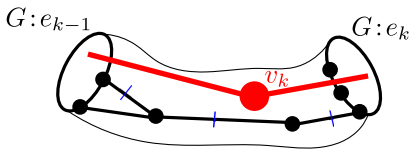


Figure 2.11: A z-region of a graph embedding with the corresponding component of the z-graph colored red and the edges of the path H_k colored blue.

Note that for any j, the paths H_j and H_{j+1} can be joined in $G{:}e_j$ by another path H'_j. Thus, the subgraph

$$C'_i = \bigcup_j H_j \cup H'_j$$

is therefore a circle that transversely crosses only $G : C_{z_{i'}}$ among all $G : G_{z_i}$. Let $b_i \in Z(G)$ denote the 1-chain inducing C'_i and observe that the C_{z_i} and b_i satisfy the desired conditions. □

2.22 Example. Consider Figure 2.12. The red circles of the embedded graph are induced by a set of 1-chains having property Δ, and they correspond to the edges of the z-graph drawn to the right. Observe that the dimenson in $H_1(S)$ of the homology classes represented by the 1-chains inducing the red circles is 3. Note also the existence of homology classes represented by 1-chains inducing circles that have $\mathbb{Z}_2$-intersection product 1 with a unique homology

class among those represented by 1-chains inducing the red circles, and $\mathbb{Z}_2$ intersection product 0 with all other such homology classes. Thus we see that we can double the size of a homologically independent set of 1-chains corresponding to edges not in a spanning tree of the corresponding z-graph.

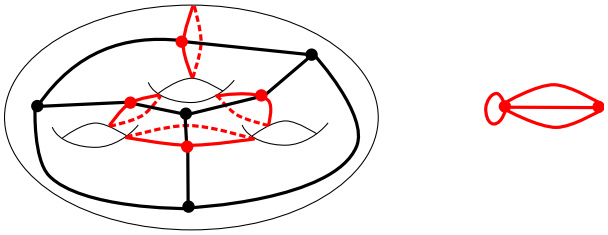


Figure 2.12: A set of 1-chains having property Δ and the corresponding z-graph.

2.23 Corollary. The set of 1-chains $B \cup B'$ is homologically independent in $H_1(S)$.

Proof. Let $X_2 = \{z_1, b_1, z_2, b_2, \ldots, z_n, b_n\}$. Construct the matrix M_{X_2}. For the case that $|B| = 1$ the result is apparent from Theorem 2.20 since

$$M_{X_2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.24}$$

We now consider the case that B is larger. Given the unknown values for $\langle [b_i]_2, [b_j]_2 \rangle_2$ for any i, j, we see that for the case $|B| = 2$, M_{X_2} is a symmetric matrix of the form

$$\left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & * \\ \hline 0 & 0 & 0 & 1 \\ 0 & * & 1 & 0 \end{array}\right),$$

whose rows are easily seen to be linearly independent, and so, by Theorem 2.20, the result follows in this case.

For other cases, note that M_{X_2} is a similar $2|B| \times 2|B|$ matrix with 2×2 blocks of the

form appearing in Equation 2.24 on the diagonal and blocks of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$$

appearing everywhere else. It is apparent that the rows of M_{X_2} are linearly independent. as there is no way to eliminate the 1 appearing in any of the entries corresponding to $\langle z_i, b_i \rangle_2$ using a row sum. The result follows by Theorem 2.20. □

2.25 Remark. Consider $G \to P$. For a set of 1-chains $\{z_1, \ldots z_n\}$ (with $\mathbb{Z}$ or $\mathbb{Z}_2$ coefficients) that induce circles with ribbon neighborhoods, we may still use the intersection indices in exactly the same way as we do for circles of graphs embedded in surfaces. To do this, one must consider the cellular embedding $\hat{G} \to \hat{P}$ that is created by releasing the pinchpoints of P, and apply the theorems in this section to $\hat{G} \to \hat{P}$. The conclusions of the results of this section would still be valid since releasing pinchpoints does not change the incidence of faces and edges.

Figure 2.6 shows why it is not correct to directly use the intersection index of circes to determine whether or not certain cycles in a pseudosurface are homologically independent or homologically nontrivial. Consider that the two curves $G{:}e_1$ and $G{:}(e_2 + e_3)$ have exactly one point in common even though e_1 is homologically trivial and $e_2 + e_3$ is not. The circle $G{:}e_1$ can be said to correspond to a face walk whose first and last face (the same face in this case) are incident to the two sides of the 1-chain $e_2 + e_3$. Since $G{:}e_2 + e_3$ contains a pinchpoint vertex, we are not assured of a similar face walk that witnesses both sides of e_1, which would indicate that e_1 is homologically nontrivial.

Two applications of the theory developed in this subsection are Lemmas 2.26 and 2.27. which build on the aforementioned [7, p 222 paragraph (c)]. We let P^2 denote the projective plane and KB denote the Klein bottle.

2.26 Lemma. [7, p 222, paragraph (c)] Consider $G \to P^2$ and $z_1, z_2 \in Z(G)$. If z_1 and z_2 both induce orientation-reversing circles, then $\langle [z_1]_2, [z_2]_2 \rangle_2 = 1$.

Proof. Omitted. □

2.27 Lemma. Consider $G \to \mathrm{KB}$ and $z_1, z_2 \in Z(G)$. If z_1 induces an orientation-reversing circle and z_2 is homologically nontrivial and induces an orientation-preserving circle, then $\langle [z_1]_2, [z_2]_2 \rangle_2 = 1$.

Proof. Assume for the sake of contradiction that $\langle [z_1]_2, [z_2]_2 \rangle_2 = 0$. By assumption, $\langle [z_1]_2, [z_1]_2 \rangle_2 = 1$ and $\langle [z_2]_2, [z_2]_2 \rangle_2 = 0$. Since z_2 is homologicaly nontrivial, $G{:}z_2$ is a nonseparating curve and so there exists $z_3 \in Z(G)$ such that $G{:}z_3$ is a circle and $\langle [z_2]_2, [z_3]_2 \rangle_2 = 1$, yet we can assume nothing about the values of $\langle [z_1]_2, [z_3]_2 \rangle_2$ and $\langle [z_3]_2, [z_3]_2 \rangle_2$. Let $X_2 = \{z_1, z_2, z_3\}$ and note that M_{X_2} is a symmetric matrix over $\mathbb{Z}_2$ of the form

$$\begin{pmatrix} 1 & 0 & * \\ 0 & 0 & 1 \\ * & 1 & * \end{pmatrix},$$

where the elements of M_{X_2} denoted by $*$ are elements of $\mathbb{Z}_2$.

Since the rows of this matrix are linearly independent, Theorem 2.20 implies that $\dim_{\mathbb{Z}_2}(KB) \geq 3$, which contradicts the fact that $\dim_{\mathbb{Z}_2}(KB) = 2$. □

Chapter 3

Ordinary Voltage Graphs

Here we give the definitions and associated constructions of ordinary voltage graph theory as well as a few more preliminary results.

3.1 Definitions and basic constructions

Recall the set $D(G)$ of all darts on the edges of G, and that for each edge e of G we designate one dart the *positively directed edge* and the other dart the *negatively directed edge*. Following the notation of [11], the positively directed edge on an edge e is also denoted e and the negatively directed edge is denoted e^-.

Let A denote a finite group and 1_A denote the identity element of A. An *ordinary voltage graph* is an ordered pair $\langle G, \alpha \to A \rangle$ such that $\alpha\colon D(G) \to A$ satisfies $\alpha(e^-) = \alpha(e)^{-1}$. The reader may notice the absence of the domain of α, which is the notation used in [11]. The group element $\alpha(e)$ is called the *voltage* of e. Associated to each ordinary voltage graph is a *derived graph* $G^\alpha = (V \times A, E \times A)$. The darts on the edges of G^α obey the following rule: if e is a positively directed edge from u to v and $\alpha(e) = c$, then the dart (e, a) is a dart from (u, a) to (v, ac). As a consequence of this rule and the condition imposed on α, the dart (e^-, ac^{-1}) is the dart opposite (e, a). From now on, we use these abbreviations: v^a for (v, a), e^a for (e, a), and e^{a^-} for (e^-, ac). Define $p\colon G^\alpha \to G$ to be the projection (covering) map satisfying $p(e^a) = e$, $p(e^{a^-}) = e^-$, and $p(v^a) = v$.

3.1 Theorem. [11, Theorem 2.1.1] Let W be a walk in an ordinary voltage graph $\langle G, \alpha \to A\rangle$ such that the initial vertex of W is u. Then for each vertex u^a in the fiber over u, there is a unique lift of W that starts at u^a.

For a walk $W = d_1 d_2 \dots d_m$ let $\omega(W) = \alpha(d_1) \cdots \alpha(d_m)$ denote the *net voltage of* W. If $c \in C_1(G)$ is such that $G{:}c$ is connected and W is a closed walk of $G{:}c$ based at $v \in V(G{:}c)$, then, per Theorem 3.1, each lift of W is uniquely identifiable by the vertex v^a at which it begins. For each $a \in A$, let W_v^a be the lift of W beginning at v^a. Note that W_v^a ends at the vertex at which $W_v^{a\omega(W)}$ begins. We say that lifts W_v^a and W_v^b are *consecutive* if $a\omega(W) = b$ or $b\omega(W) = a$. We call a set of lifts of W of the form

$$\left\{W_v^a, W_v^{a\omega(W)}, W_v^{a(\omega(W))^2}, \dots\right\}$$

a *set of consecutive lifts of* W. Observe that W_v^a and W_v^b are lifts of W contained in the same set of consecutive lifts of W if and only if $a = b(\omega(W))^m$ for some nonnegative integer m. Let $\hat{W}_v^a$ denote the set of consecutive lifts of W containing W_v^a. Note that the finiteness of A implies that each set of consecutive lifts of W contains a finite number of lifts of W. Unless $\omega(W) = 1_A$, there is more than one group-element superscript that can be used to identify $\hat{W}_v^a$. This conclusion also holds for all other sets of consecutive lifts of W.

An ordinary voltage graph embedding (as formulated in [11, §4.1]) in S is an ordered pair $\langle G \to S, \alpha \to A\rangle$. Note again the absence of the domain of α in this definition, which is how an ordinary voltage graph embedding is denoted in [11]. This ordered pair is an encoding of a *derived embedding* of G^α in a derived surface S^α that is constructed according to the rules:

1. if $v : d_1 d_2 \dots d_n$ is a rotation on v, then the rotation on v^a is $d_1^a, d_2^a \dots d_n^a$, and
2. if d_i is a dart on an orientation-reversing edge, then all darts in the fiber over d_i are orientation reversing darts in $G^\alpha \to S^\alpha$.

Garman in [10] observed that since it is the lifts of facial boundaries that form facial boundaries in S^α, S^α can be formed by "identifying each component of a lifted region

boundary with sides of a 2-cell (unique to that component) and then performing the standard identification of edges from surface topology". Therefore a proper embedding of an ordinary voltage graph in a pseudosurface can be used to encode a proper embedding of the derived graph in another pseudosurface. Observe that if a pseudosurface P_2 is allowed to have boundary edges, then for each (directed) edge e in P_2 that is bounded on only one side by a face of $G \to P_2$, each (directed) edge e^a is bounded on only one side by a face of $G^\alpha \to P_2^\alpha$.

If $I \subset F(G \to P)$ is such that $P{:}I$ is connected, then it makes sense to let

$$\langle (G{:}I) \to (P{:}I)\ , \alpha \to A \rangle$$

denote the restriction of $\langle G \to P, \alpha \to A \rangle$ to $P{:}I$. We let $(G{:}I)^\alpha$ and $(P{:}I)^\alpha$ denote the corresponding restricted derived graph and derived pseudosurface. If $c \in C_1(G)$ is such that $G{:}c$ connected, we let $\langle G{:}c, \alpha \to A \rangle$ be the restriction of $\langle G, \alpha \to A \rangle$ to the edges of $G{:}c$. For a fixed $v \in V(G{:}I)$, we let $(P{:}I)_v^a$ and $(G{:}I)_v^a$ denote the connected components of $(P{:}I)^\alpha$ and $(G{:}I)_v^\alpha$, respectively, containing the vertex v^a. Similarly, we let $(G{:}c)_v^a$ denote the connected component of $(G{:}c)^\alpha$ containing the vertex v^a. We let $c_v^a \in C_1(G^\alpha)$ denote the 1-chain inducing the subgraph $(G{:}c)_v^a$.

The graph covering map $p\colon G^\alpha \to G$ can be extended to a (branched) covering map $p\colon P^\alpha \to P$. By Lemma 3.2, p can be considered a chain map on cellular complexes or as a map on topological spaces. Lemma 3.2 follows immediately from the fact that it is the sets of consecutive lifts of facial boundaries that single-cover the edges of the facial boundaries of P^α.

3.2 Lemma. For each k, the map $p\colon C_k(P^\alpha) \to C_k(P^\alpha)$ satisfies $p \circ \partial^\alpha = \partial \circ p$.

Proof. Omitted. □

Following [11, §2.1.1], the voltage group A acts by left multiplication on G^α; for $c \in A$, let $c \cdot v^a = v^{ca}$, $c \cdot e^a = e^{ca}$, and $c \cdot e^{a^-} = e^{ca-}$. This left group action is regular on the fibers over vertices and (directed) edges of G^α. Following [11, §4.4], the action of A on G^α extends

to a transitive action on the faces in each fiber over a face of $\langle G \to P, \alpha \to A \rangle$, and is free if and only of the net voltage of every facial boundary walk is 1_A. This action also extends to subcomplexes of P^α: for a subcomplex X of P^α and $c \in A$, $c \cdot X$ is the subcomplex consisting of the images of the vertices, edges, and faces of X under the left-action of c. It is a consequence of the theory that G^α is a regular covering space of G, and P^α is a (possibly branched) covering space of P and that the action of A on the faces of $G^\alpha \to P^\alpha$ is transitive on the faces of the fiber over any face of $G \to P$ but not necessarily free.

Recall that if W is a closed walk in G based at some vertex v, the lifts of W are uniquely identifiable by their group-label superscripts. For the rest of this paragraph, let ω stand in the place of $\omega(W)$ and $|\omega|$ stand in the place of $|\langle \omega(W) \rangle|$. Consider the set of consecutive lifts

$$\hat{W}_v^{1_A} = \left\{ W_v^{1_A}, W_v^{\omega}, W_v^{\omega^2}, \ldots, W_v^{\omega^{|\omega|-1}} \right\},$$

and a particular lift W_v^a. By definition $c \cdot W_v^a = W_v^{ca}$. Observe that W_v^{ca} ends at the vertex at which $W_v^{ca\omega}$ begins. We can now extend the action of A to sets of consecutive lifts of W. Let

$$c \cdot \hat{W}_v^a = \left\{ W_v^{ca}, W_v^{ca\omega}, W^{ca\omega^2}, \ldots, W_v^{ca\omega^{|\omega|-1}} \right\}.$$

Though this action of A on the set of sets of consecutive lifts is clearly transitive, it is free if and only if $\omega(W) = 1_A$.

Since the action of A on P^α is regular, the connected components of G^α are all isomorphic, as are their embeddings in the connected components P^α. Thus, the connected components of P^α are isomorphic as 2-complexes and homeomorphic as topological spaces. Similar consequences follow for induced subcoplexes of P and G.

The significance of Proposition 3.3 below is that the intersection of the fiber over a homologically trivial 1-chain and a connected component of P^α is homologically trivial, and that this conclusion has no dependence on A nor α.

3.3 Proposition. Consider $\langle G \to P, \alpha \to A \rangle$ and $z \in Z(G)$ such that z is trivial in the $\mathbb{Z}_2$-vector space $H_1(P)$. Let $v \in V(G{:}z)$. The $\mathbb{Z}_2$-sum of edges appearing in $p^{-1}(z) \cap G_v^a$ is trivial in $H_1(P_v^a)$ for all $a \in A$.

Proof. Since all of the embeddings of the connected components of G^α in the connected components of P^α are isomorphic, it suffices to verify the conclusion for $P_v^{1_A}$. By assumption, there is some $I \subset F(G \to P)$ such that $z = \partial I$. For each $f \in I$, there is a finite set J_f of faces of $P_v^{1_A}$ that is the intersection of $p^{-1}(f)$ and $P_v^{1_A}$. Thus, by linearity of p^{-1},

$$p^{-1}(\partial I) \cap G_v^{1_a} = \partial^\alpha(p^{-1}(I)) \cap P_v^{1_A} = \sum_{f \in I} \partial^\alpha J_f.$$

Therefore, $p^{-1}(z) \cap G_v^a$ is homologically trivial in $H_1(P_v^a)$ for all $a \in A$. □

The proof of Theorem 3.5, which can be extracted from the proof of Theorem 3.4, generalizes from ordinary voltage graphs to ordinary voltage graph embeddings since the latter proof does not depend on the existence of an embedding.

3.4 Theorem. [11, Theorem 2.5.3] Consider $\langle G, \alpha \to A \rangle$ and a spanning tree T of G. The derived graph is isomorphic to a derived graph of an ordinary voltage graph $\langle G, \alpha_T \to A \rangle$ such that $\alpha_T(d) = 1_A$ for each directed edge d of T.

Proof. Omitted. □

3.5 Theorem. Consider $\langle G \to P, \alpha \to A \rangle$ and a spanning tree T of G. The derived graph embedding is isomorphic to a derived graph embedding of an ordinary voltage graph embedding $\langle G, \alpha_T \to A \rangle$ such that $\alpha_T(d) = 1_A$ for each directed edge d of T.

Proof. Omitted. □

We will call the spanning tree T appearing in the proof of Theorems 3.4 and 3.5 a *zero voltage spanning tree* and the corresponding voltage assignment α_T the T-voltage assignment.

3.2 Archdeacon's medial voltage graph enhancement

In this section we state some of the definitions and theorems associated Archdeacon's transfer [3] of the voltage assignment of an ordinary voltage graph embedding in a surface

to a subdivided medial graph of an ordinary voltage graph embedding. We state only those that we will need for future use. This section's material is borrowed from Archdeacon's 1992 paper [3] and is included for the sake of completeness.

For $v \in V(G)$ and $c \in A$, define a *local voltage modification* at v to be the result of replacing $\alpha(e)$ with $c\alpha(e)$ for all darts directed away from v. Each $\alpha(e^-)$ is replaced with $\alpha(e)^{-1}c^{-1}$. We will say that *two voltage graphs are equivalent* if one can be obtained from the other by a sequence of local voltage modifications.

3.6 Lemma. Consider $\langle G \to S, \alpha \to A\rangle$, and let c be an element of A and v be a vertex of G. Define $\overset{1}{\alpha}$ to be the result after performing a local voltage modification at v. The derived embeddings $G^{\alpha} \to S^{\alpha}$ and $G^{\overset{1}{\alpha}} \to S^{\overset{1}{\alpha}}$ are isomorphic.

Proof. Omitted. □

Construction 3.7 introduces a tool that we will use in later chapters.

3.7 Construction. Recall that for an edge e of a graph G, the operation of subdividing e is the result of replacing e with a path e_1ve_2 of length two. The resulting *subdivided voltage assignment* has voltage $\alpha(e_1) = \alpha(e)$, $\alpha(e_1^-) = \alpha(e_1)^{-1}$ and $\alpha(e_2) = \alpha(e_2^-) = 1_A$.

3.8 Remark. Note that by Lemma 3.6, the assignment of voltages to e_1 and e_2 can be switched.

3.9 Lemma. Let $\langle G', \alpha' \to' A\rangle$ be the ordinary voltage graph resulting from subdividing every edge of $\langle G, \alpha \to A\rangle$ and accordingly subdividing the voltage assignment α. The resulting graph $G'^{\alpha'}$ is isomorphic to the result of subdividing every edge of G^{α}.

Proof. Omitted. □

3.10 Construction. Consider the subdivided graph G' of G consider the medial graph M of $G \to S$. Subdivide every edge of M to obtain the subdivided medial graph M'. We choose a preferred direction e^+ (recall that this is denoted e) on a fixed edge e of G'. Let v_e denote the corresponding vertex of the subdivided medial graph. There are two corners

of $G \to P$ at the initial vertex of e and containing e that correspond to two edges e_1, e_2 incident to v_e in the subdivided medial graph. Direct these two edges e_1^+ and e_2^+ so that they have terminal vertex v_e, and assign the voltage $\alpha(e^+)$ to both e_1^+ and e_2^+ and $\alpha(e^{-1})$ to e_1^- and e_2^-. The other darts of M' whose head or tail vertex is v_e are assigned voltage 1_A. See Figure 3.1.

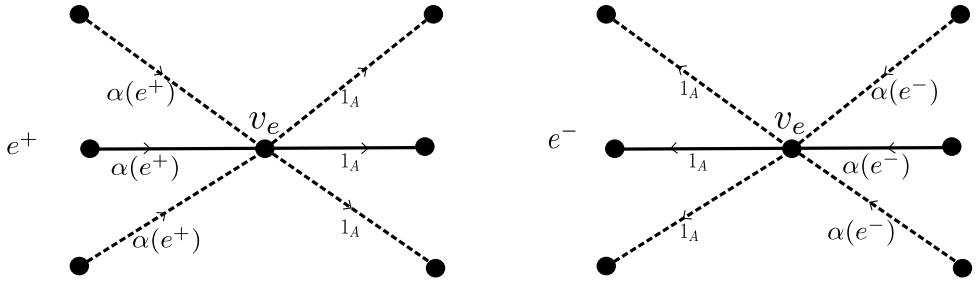


Figure 3.1: The transferred medial voltage assignment with different choices of preferred direction. The edges of M' are the dashed edges.

We define a *voltage assignment transferred to the subdivided medial graph* M' of $G' \to S$ to be the result after performing Construction 3.10.

Note that each edge of M' is incident to exactly two vertices, one of which corresponds to an edge of G. Each edge is therefore assigned a voltage that is transferred from only one edge of G', and so the transferred voltage assignment is well defined. Thus, we can transfer all of the voltages simultaneously without getting conflicting voltage assignments on an edge of M'. Switching to e^- as the preferred direction corresponds to multiplying all voltages of darts with tail vertex v_e by $\alpha(e)^{-1}$, as illustrated by Figure 3.1. By Lemma 3.6 this does not effect the isomorphism class of $(M')^\alpha$.

For a corner d_1d_2 of a cellular graph embedding, we define the operation of *drawing an edge across the corner* to be the result of adding an edge joining $t(d_1)$ and $h(d_2)$. Note that if d_1 and d_2 are opposites of eachother, the edge drawn is a loop. Define the *total graph* $T(G)$ of $G \to S$ to be the graph formed by subdividing each edge of G and drawing in subdivided edges across each corner. Thus, the total graph has both the medial graph and G as minors. We define an *extended voltage assignment to the total graph* $T(G)$ to be

an assignment that is created by subdividing each edge of G per Construction 3.7, adding edges across each corner, and transferring the voltage assignment from G' to M', as in Construction 3.10. We denote th extended voltage assignment $\alpha_E \to A$. Note the existence of a function $\psi\colon T(G) \to T(G)$, which is the identity map on G' and maps a corner of M', and the vertex of M' that it induces, onto the corner of G' to which it corresponds, as in Figure 3.2. Note that for each dart d of $T(G)$, the extended voltage assignment α_E satisfies $\alpha(\psi(d)) = \alpha_E(d)$.

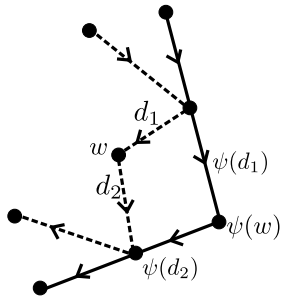


Figure 3.2: An elucidation of the definition of the map $\psi\colon M' \to G'$ for one corner of $G \to P$. The dashed edges are the edges of M'.

3.11 Theorem. [3]**(Archdeacon, 1992)**Consider $\langle G \to S, \alpha \to A\rangle$ and let M' be the subdivided medial graph with the transferred voltage assignment. The derived graph $(M')^\alpha$ is the medial graph of the derived graph of the subdivided G'.

Proof. Omitted. □

Chapter 4

Coset Lemmas

Given $\langle G \to P, \alpha \to A \rangle$, for a pseudosurface P, we examine carefully-constructed subgroups of A corresponding to special induced subcomplexes of P. Using these subgroups, we establish relationships between certain cosets of these subgroups and induced subcomplexes of P^α.

4.1 Cosets not requiring Archdeacon's medial-graph enhancement

Following [11, §2.5.1], the net voltages of the set of closed walks of G based at a vertex v form a group, which we denote $A(v)$. We first state a known result.

4.1 Theorem. (Alpert and Gross, 1976)[11, Theorem 2.5.1] Let v be a vertex of $\langle G, \alpha \to A \rangle$. The vertices v^a and v^b are in the same component of G^α if and only $a^{-1}b$ is an element of $A(v)$.

Proof. Omitted. □

Lemma 4.2 establishes a relationship between cosets of $A(v)$ in A and the connected components of P^α. Lemma 4.2 is a reformulation and strengthening of Theorem 4.1.

4.2 Lemma. Given $\langle G \to P, \alpha \to A \rangle$ and a fixed vertex v of G, the map

$$\phi_v \colon \{aA(v) : a \in A\} \to \{P_v^a : a \in A\}$$

defined by $\phi_v(aA(v)) = P_v^a$ is a bijection.

Proof. Since G_v^a is the 1-skeleton of P_v^a, it suffices to verify that ϕ_v is a bijection between the left cosets of $A(v)$ in A and the connected components of G^α. Since G is connected, for any vertex $u \in G$ there is a u-v-path H that lifts to a total of $|A|$ distinct $p^{-1}(u)$-$p^{-1}(v)$-paths, each one joining a vertex in the fiber over u to a distinct vertex in the fiber over v. Thus, it suffices to show that $G_v^a = G_v^b$ if and only if $a^{-1}b \in A(v)$, which follows from Theorem 4.1. □

Before we proceed any further, we present two technical lemmas.

4.3 Lemma. Let X, Y, and Z denote groups such that $X \leq Y \leq Z$. For each $z \in Z$, the left coset zY can be partitioned into left cosets xX, $x \in zY$.

Proof. Treating Y as a left coset in Z, observe that Y can be partitioned into left cosets

$$\{y_1 X, \ldots, y_n X\}$$

where the y_i are representatives of the left cosets of X in Y. Thus

$$Y = \dot{\bigcup}_{i \in [n]} y_i X.$$

Fix $z \in Z$. Consider the left coset zY and note that for $i \neq j$, if $y^* \in zy_iX \cap zy_jX$, then there exist $x_1,\ x_2 \in X$ such that $y^* = zy_ix_1 = zy_jx_2$, and so $z^{-1}y^* = y_ix_1 = y_jx_2$, which is an element of $y_iX \cap y_jX$. This contradicts the earlier hypothesis that y_iX and y_jX are distinct cosets of X in Y.

Thus,

$$zY = \dot{\bigcup}_{i \in [n]} zy_i X.$$

□

We need to introduce some more notation before stating our next result. Let P_B denote a pseudosurface, possibly with boundary, and let G_B denote a connected graph properly embedded in P_B. Let A_B denote a finite group, and let $A_B(v)$ denote the local voltage group of closed walks based at $v \in V(G_B)$. Let X denote a connected sub 2-complex of P_B containing v. Consider $\langle G_B \to P_B, \alpha \to A_B \rangle$. Let $A_B(v, X)$ denote the group of net voltages of closed walks in X based at v, and fix $a \in A_B$. Lemma 4.4 establishes a relationship between the left cosets of $A_B(v, X)$ contained in $aA_B(v)$ and the connected components of X^α contained in $(P_B)^a_v$. The general formulation of Lemma 4.4 allows for easier application to a wide range of cases.

4.4 Lemma. For a fixed $a \in A_B$, the map

$$\phi_X \colon \{bA(v, X) : b \in aA_B(v)\} \to \left\{X^b_v : b \in aA_B(v)\right\}$$

defined by

$$\phi_X(bA_B(v, X)) = X^b_v$$

is a bijection.

Proof. We consider the case for which the domain of ϕ_X is the set of left cosets of $A_B(v, X)$ in $A_B(v)$, which is the case in which $a = 1_A$. By Lemma 4.3, and the transitive left action of A_B on the fiber over X, this case will suffice. Let $G(X)$ denote the 1-skeleton of X, and note that $G(X)^b_v$ is the 1-skeleton of X^b_v. Since $G(X)^b_v$ is the 1-skeleton of X^b_v, it suffices to show that ϕ_X is a bijection between the $bA_B(v, I)$ and the $G(X)^b_v$. Since $G(X)$ is connected, for any other vertex u of $G(X)$, there is a v-u path in $G(X)$ that lifts to $|A_B|$ distinct $p^{-1}(u)$-$p^{-1}(v)$ paths in $G(X)^\alpha$, each one joining a vertex in the fiber over u to a distinct vertex in the fiber over v. Thus, if we substutute $G(X)$ and $A_B(v, X)$ in the statement of Theorem 4.1, we see that $G(X)^b_v = G(X)^c_v$ if and only if $b^{-1}c$ is an element of $A_B(v, x)$. Moreover, per Theorem 4.1, after substituting G_B, A_B and $A_B(v)$ in the place of G, A, and $A(v)$, respectively, we see that $G(X)^b_v$ is contained in the same connected component

of P_B^α if and only if $b^{-1}c$ is an element of $A_B(v)$. The result follows. □

Let I denote a subset of $F(G \to P)$ such that $G{:}I$ is connected and assume that v is a vertex of $G{:}I$. Let $A(v, I)$ be the subgroup of $A(v)$ consisting of net voltages of closed walks in $G{:}I$ based at v. Observe that $A(v, I) \leq A(v) \leq A$. By an application of Lemma 4.3 to the groups $A(v, I), A(v)$, and A, we see that for each $a \in A$, the left coset $aA(v)$ can be partitioned into left cosets of the form $bA(v, I)$ for $b \in aA(v)$. Lemma 4.5 establishes a relationship between cosets of $A(v, I)$ contained in a coset $aA(v)$ and components of $(P{:}I)^\alpha$ contained in P_v^a.

4.5 Lemma. Given $\langle G \to P, \alpha \to A\rangle$ and a fixed $a \in A$. The map

$$\phi_I\colon \{bA(v, I)\colon b \in aA(v)\} \to \left\{(P{:}I)_v^b : b \in aA(v)\right\}$$

defined by $\phi_I(bA(v, I)) = (P{:}I)_v^b$ is a bijection.

Proof. This follows from Lemma 4.4 after letting P take the place of P_B, A take the place of A_B, $A(v)$ take the place of $A_B(v)$ $P{:}I$ take the place of X, and $A(v, I)$ take the place of $A_B(v, X)$. □

Let $y \in C_1(G{:}I)$ denote a 1-chain having the property that $G{:}y$ is connected and assume that v is a vertex of $G{:}y$. Let $A(v, G{:}y)$ denote the subgroup of $A(v, I)$ consisting of net voltages of closed walks in $G{:}y$ based at v. Observe that $A(v, y) \leq A(v, I) \leq A(v)$. By an application of Lemma 4.3 to the groups $A(v, y)$, $A(v, I)$, and $A(v)$, we see that for a fixed $a \in A$, each of the left cosets $bA(v, I)$ contained in $aA(v)$ can be partitioned into left cosets $cA(v, y)$ for $c \in bA(v, I)$. Lemma 4.6 establishes a relationship between the cosets of $A(v, y)$ contained in $bA(v, I)$ and the connected components of $(G{:}y)^\alpha$ contained in $(P{:}I)_v^b$.

4.6 Lemma. Given $\langle G \to P, \alpha \to A\rangle$ with fixed $a \in A$, $b \in aA(v)$, the map

$$\phi_y\colon \{cA(v, y) : c \in bA(v, I)\} \to \{(G{:}y)_v^c : c \in bA(v, I)\}$$

defined by $\phi_y(cA(v, y)) = (G{:}y)_v^c$ is a bijection.

Proof. This follows from Lemma 4.4 after letting $P{:}I$ take the place of P_B, A take the place of A_B, $A(v, I)$ take the place of $A_B(v)$ $G{:}y$ take the place of X, and $A(v, G{:}y)$ take the place of $A_B(v, X)$. □

Now let W denote a closed walk in $G{:}x$ based at v. Clearly

$$\langle \omega(W) \rangle \leq A(v, z) \leq A(v, I) \leq A(v) \leq A.$$

By an application of Lemma 4.3 to the groups $\langle \omega(W) \rangle$, $A(v, z)$, $A(v, I)$ we see that for a fixed $a \in A$, $b \in A(v)$, each of the left cosets $cA(v, z)$ contained in $bA(v, I)$ can be partitioned into left cosets of the form $d\langle \omega(W) \rangle$ for $d \in cA(v, x)$. Lemma 4.7 establishes a relationship between the cosets of $\langle \omega(W) \rangle$ contained in the same coset of $A(v, x)$ in $A(v, I)$ and sets of consecutive lifts of W of the same component of $(G{:}x)^c_v$.

4.7 Lemma. Fix $a \in A$, $b \in aA(v)$, and $c \in baA(v, I)$. The function

$$\phi_{\hat{W}} \colon \{d\langle \omega(W) \rangle : d \in cA(v, x)\} \to \left\{ \hat{W}^d_v : d \in cA(v, x) \right\}$$

defined by $\phi_{\hat{W}}(\langle d\omega(W) \rangle) = \hat{W}^d_v$ is a bijection whose image is the set of consecutive lifts of W contained in $(G{:}x)^c_v$.

Proof. We treat the case for which $c = 1_A$. By Lemma 4.3 and the transitive action of A on the components of $(G{:}X)^\alpha$, this case will suffice. The map $\phi_{\hat{W}}$ is well defined and injective since $\hat{W}^{d_1}_v = \hat{W}^{d_2}_v$ if and only if $d_1 = d_2 x$ for some $x \in \langle \omega(W) \rangle$. Surjectivity follows immediately from the definition of $\phi_{\hat{W}}$.

Note that $\hat{W}^{d_1}_v$ and $\hat{W}^{d_2}_v$ are sets of consecutive lifts of W in $(G{:}x)^{1_A}_v$ if and only if v^{d_1} and v^{d_2} are contained in $(G{:}x)^{1_A}_v$, which by letting $G{:}x$ and $A(v, x)$ take the places of G and $A(v)$ in Theorem 4.1, respectively, is true if and only if $d_1^{-1} d_2 \in A(v, x)$. The result follows. □

The results in Theorem 4.8 follow readily from Lagrange's theorem for cosets and the above lemmas. The proofs are omitted.

4.8 Theorem. Consider $\langle G \to P, \alpha \to A\rangle$, and let I be a subset of $F(G \to P)$ such that $P{:}I$ is connected. Let $x \in C_1(P{:}I)$ be conected, v denote a vertex of $G{:}x$, and W be a closed walk of $G{:}x$ based at v.

1. There are $\frac{|A|}{|A(v)|}$ components of P^α.
2. There are $\frac{|A|}{|A(v,I)|}$ components of $(P{:}I)^\alpha$ contained in each component of P^α.
3. There are $\frac{|A(v,I)|}{|A(v,x)|}$ components of $(G{:}x)^\alpha$ contained in each component of $(P{:}I)^\alpha$.
4. There are $\frac{|A(v,x)|}{|\langle\omega(W)\rangle|}$ sets of consecutive lifts of W covering the edges of each component of $(G{:}x)^\alpha$.

4.9 Corollary. Assume that $x \in C_1(G)$ is an element of $Z(G)$ and that W is an Eulerian walk of $G{:}x$ based at $v \in V(G{:}x)$. There are $\frac{|A(v,x)|}{|\langle\omega(W)\rangle|}$ sets of consecutive lifts forming closed walks in each component of $(G{:}x)^\alpha$, and each edge in each component of $(G{:}x)^\alpha$ is singly covered by one of these closed walks.

Proof. The first conclusion follows by Part 4 of Theorem 4.8.

To prove the second conclusion let $W = d_1, \ldots d_l$ and assume without loss of generality that d_i is the positively-directed edge e_i. An Eulerian walk singly covers each edge of $G{:}x$. Therefore, each lift of W singly covers the edges of $(G{:}x)^\alpha$ that it covers. Note that each (directed) edge in the fiber over e_i is covered by a lift of W; for each $a \in A$, e_i^a appears in $W_v^{a\omega(d_1,\ldots d_{i-1})^{-1}}$. To see that no edge of $(G{:}x)^\alpha$ is covered by two closed walks formed by a set of consecutive lifts of W, note that with $|A|$ lifts of W contained in G^α, $|A|$ edges forming the fiber over each edge in G^α, and each edge of $(G{:}x)^\alpha$ being single covered by at least one lift of W, it must be that each edge of $(G{:}x)^\alpha$ is single covered by exactly one lift of W. □

4.10 Example. Consider Figure 4.1. Let $W = e_1e_2$ and $z = e_1 + e_2$. Observe that $A(v,z) \cong \mathbb{Z}_2$ and $\langle\omega(W)\rangle = \{1_A\}$. By Corollary 4.9, there are two sets of consecutive lifts of W singly covering the single connected component of $(G{:}z)^\alpha$.

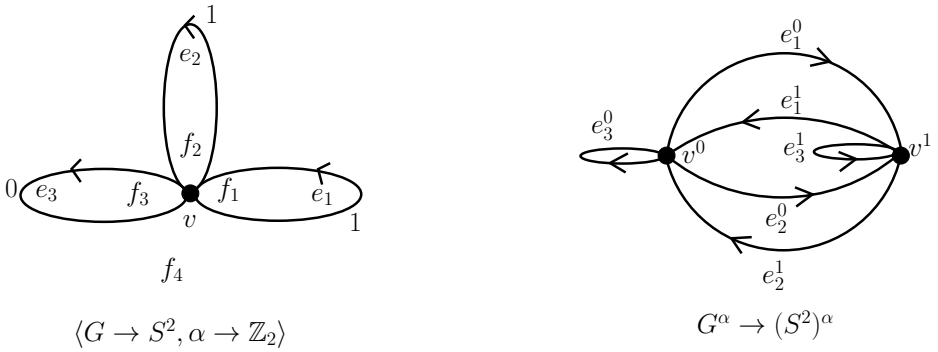


Figure 4.1: An ordinary voltage graph embedding and its corresponding derived embedding. Both embeddings are in the sphere.

4.11 Example. Consider Figure 4.2. Let $I = \{f_1, f_2\}$, $z = e_2$. Observe that $A(v) \cong \mathbb{Z}_8$, $A(v, I) \cong \langle 2 \rangle \leq \mathbb{Z}_8$, and $A(v, I^c) \cong \mathbb{Z}_8$. By Part 1 of Theorem 4.8, there is one connected component of $(S^2)^\alpha$. By Part 2 of Theorem 4.8, there are two connected components of $(S^2{:}I)^\alpha$ and one connected component of $(S{:}I^c)^\alpha$ that form the connected component of $(S^2)^\alpha$. Figures 4.3 and 4.4 show how the derived embedding can be expressed in terms of the connected components of $(S^2{:}I)^\alpha$ and $(S^2{:}I^c)^\alpha$.

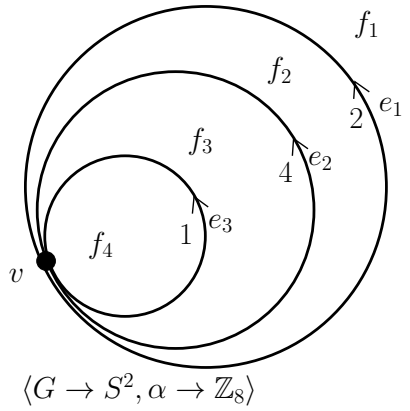
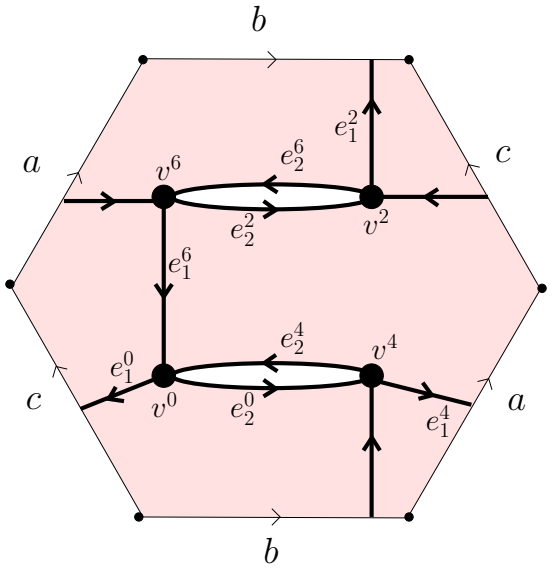
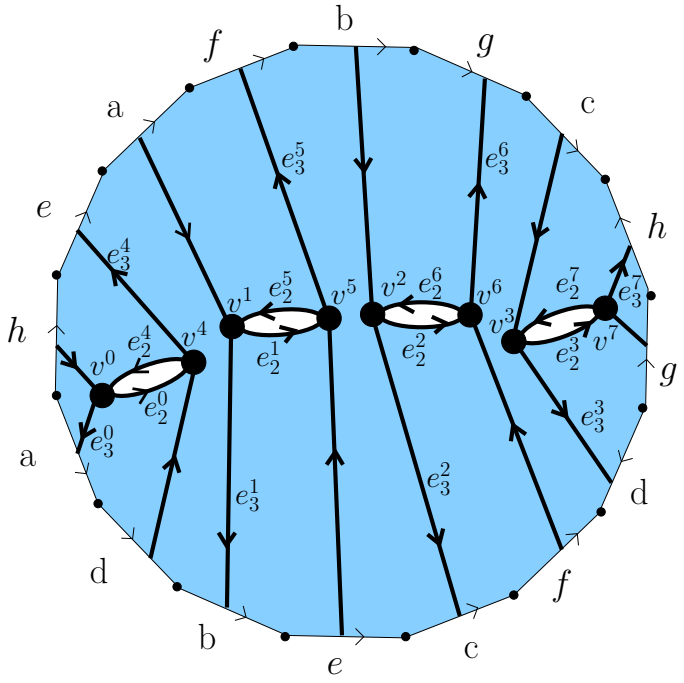


Figure 4.2: An ordinary voltage graph embedding.



$(S^2{:}I)_v^0$

Figure 4.3: The induced region $(S^2{:}I)_v^0$ is drawn with boundary components bordering the non-shaded regions in the interior of the polygon. The induced region $(S^2{:}I)_v^1$ (not shown) is isomorphic to $(S^2:I)_v^0$ per the transitive action of A on the connected components of $(S^2{:}I)^\alpha$.



$(S^2{:}I^c)^\alpha$

Figure 4.4: The induced region $(S^2{:}I^c)^\alpha$ is drawn with boundary components bordering the non-shaded regions in the interior of the polygon.

4.2 Subgroups of A requiring Archdeacon's medial-graph enhancement and the associated coset lemmas

In this section we show how the definitions and results of Sections 2.2 and 3.2 can be used to further enhance our understanding of the derived embedding of an ordinary voltage graph embedding.

Recall from Section 3.2 the subdivided medial graph M' and the total graph $T(G)$ associated to a cellular graph embedding. Consider $\langle G \to S, \alpha \to A \rangle$, a transferred voltage assignment to the subdivided medial graph M', and an extended voltage assignment to the total graph $T(G)$. Since M'^{α} captures the incidence of faces and edges of $G^\alpha \to S^\alpha$, we may utilize M'^{α} to determine homological information about $G^\alpha \to S^\alpha$. We first examine some more carefully constructed subgroups of A corresponding to special induced complexes of S.

Definition 4.12 introduces a bookkeeping tool for computation that will allow for easier proofs of results.

4.12 Definition. Consider $\langle G \to S, \alpha \to A \rangle$, two adjacent vertices $u, v \in V(G)$, and a v-u positively-directed edge e. Let v_e denote the vertex of the subdivided medial graph corresponding to e, and consider Figure 4.5. The subgraph $T(G)$:$\{(v, v_e), (w, v_e), (v_e, y)\}$ is called the *special claw corresponding to* e.

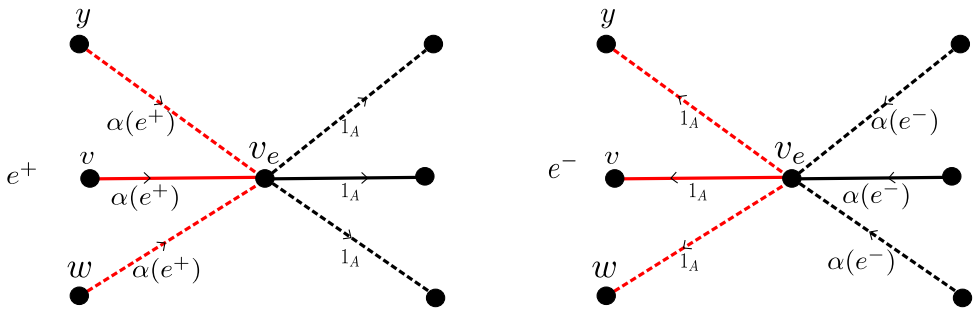


Figure 4.5: The edges of the special claw corresponding to e appear in red on the left hand side of both graphics, each corresponding to a choice of preferred direction on e.

4.13 Remark. No matter the choice of preferred direction of e, observe that any walk joining any two of the vertices w, v, and y in the special claw corresponding to w has net voltage 1_A. Therefore, the vertices in each component in the fiber over the special claw corresponding to e will have vertices v^a, w^a, y^a with identical group-element superscripts.

Let $A'(w)$ denote the group of net voltages of closed walks in M' based at $w \in V(M')$.

4.14 Lemma. Consider $\langle G \to S, \alpha \to A \rangle$. For a fixed vertex v of G, an edge e incident to v, the special claw corresponding to e, and a vertex w' of M' that takes the place of either of the vertices w or y in Definition 4.12, $A'(w') = A(v)$.

Proof. Any closed walk W' in M' based at w' maps under ψ to a closed walk in G' based at v. So $A'(w) \leq A(v)$. Given a closed walk W in G', it is easy to construct a closed walk W_2' in M' based at w' such that $\psi(w') = W$. It follows that $A(v) \leq A'(w)$. □

Now, consider $\langle G \to S, \alpha \to A\rangle$ and a connected $c_1 \in C_1(G)$. We introduce Definitions 4.15 and 4.16 here for the sake of presenting them in the most general context, and alert the reader to the fact that Definitions 4.15 and 4.16 will be used in distinctly refined settings later in this chapter and Chapter 5, where we will use the cosets of specially constructed subgroups of A to understand the z-regions of S^α with respect to the 1-chains forming the fiber over z, with an eye toward using the z-regions to get a homological understanding of the derived embedding. Recall that for a dart d, $h(d)$ is the head vertex of d.

4.15 Definition. Let W' be a walk in M'. We say that W' *transversely crosses* $G{:}c_1$ if W' contains a subwalk $d_i d_{i+1}$ such that d_i intersects one component of $U^*(h(d_i)) \setminus G{:}c_1$ and d_{i+1} intersects the other.

4.16 Definition. Let $A^{\veebar}(w, c_1)$ denote the subgroup of $A'(w)$ of net voltages of walks in M' that do not transversely cross $G{:}c_1$. (We use the superscript $\veebar$ in order to convey the nontransversality of intersections of two embedded subgraphs).

For $z \in Z(G)$ inducing a circle such that the circles of $(G{:}z)^\alpha$ are orientation-preserving circles, we will let $\bar{S}^b_w(z)$ denote the z-region of S^α, taken with respect to the set $\{z^a_v : a \in A\}$ of 1 chains with property Δ, that contains w^b.

4.2.1 Circles over orientation-reversing circles

Consider $\langle G \to P, \alpha \to A\rangle$ and $z \in Z(G)$ such that $G{:}z$ is an orientation-reversing circle with a ribbon neighborhood. Let W denote an Eulerian walk of $G{:}z$ based at $v \in V(G{:}z)$, and let $v_1, \ldots, v_k$ denote an enumeration of the vertices of $G : z$ satisfying $v_1 = v$ and $v_i = t(d_i)$ for each i. For the remainder of this subsection, we let ω denote $\omega(W)$ and $|\omega|$ denote $|\langle\omega(W)\rangle|$. We also assume that $|\omega|$ is even, and so the circles forming the fiber over $G{:}z$ are orientation preserving, per Lemma 4.17.

4.17 Lemma. Let $\langle G \to P, \alpha \to A\rangle$ denote an ordinary voltage-graph embedding in a nonorientable P. Let $z \in Z(G)$ denote a 1-chain inducing an orientation-reversing circle with a ribbon neighborhood, and let W be an Eulerian walk of $G{:}z$. Each circle of $(G{:}z)^\alpha$

is orientation-reversing if and only if $|\omega|$ is odd and orientation preserving if and only if $|\omega|$ is even.

Proof. Recall that a circle is orientation-reversing if and only if it has a ribbon neighborhood and an odd number of orientation-reversing edges. Following the proof of Theorem 3.1, there are $|\omega|$ consecutive lifts of W that form each set of consecutive lifts of W that single cover the edges of each component $(G{:}z)^{\alpha}$. The result follows. □

We return to ordinary voltage graph embeddings in (nonorientable) *surfaces*. For the remainder of this subsection, we consider $\langle G \to S, \alpha \to A\rangle$, and we retain the definitions of z and W. Even though we do not need the assumption that z has a ribbon neighborhood, we will still use a ribbon neighborhood.

Consider a vertex-star neighborhood, $U^*(v)$, of v and consider an orientation of $U^*(v)$ induced by the rotation on v and the two components of $U(v) \setminus (G{:}z)$. Considering as well the vertices and edges of G as subsets of S and the orientation induced by the rotation map ρ, it is thus reasonable to denote one component of $U(v) \setminus (G{:}z)$ the *clockwise side of* v and the other open set the *counter-clockwise side of* v according to the orientation on $U^*(v)$ induced by ρ and the direction of the darts d_1 and d_k, as in Figure 4.6.

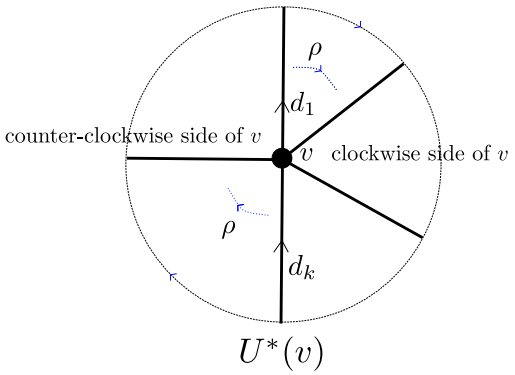


Figure 4.6: The clockwise and counter-clockwise sides of v.

Let $U^a(v)$ denote the component of $p^{-1}(U^*(v))$ containing v^a. We lift the distinction of clockwise and counterclockwise sets to the preimages of $p^{-1}(U^*(v))$; let the *clockwise side*

of v^a refer to the component of $U^a(v) \setminus (G{:}z)$ that p maps to the clockwise side of v and the *counterclockwise side of* v^a refer to the other component.

We now consider an extended voltage assignment to the total graph $T(G)$ associated to $G \to S$. For clarity's sake, we require that $U^*(v)$ contains the vertices v, x, w, y of the special claw corresponding to $E(d_1)$ (d_1 is the first dart in W). Assume without loss of generality that the clockwise side of v contains w and the counterclockwise side of v contains y.

Consider a ribbon neighborhood $R(z_v^{1_A})$ of $(G{:}z)_v^{1_A}$ containing $U^{1_A}(v)$. Let the *east side of* $z_v^{1_A}$ refer to the component of $R(z_v^{1_A}) \setminus (G{:}z)_v^{1_A}$ containing the clockwise side of v^{1_A}, and the *west side of* $z_v^{1_A}$ refer to the other component; the east wide of $z_v^{1_A}$ contains w^{1_A} and the west side contains y^{1_A}. Since each lift of W single covers an odd number of orientation-reversing edges in G^α, we see that the east side of $z_v^{1_A}$ contains the counterclockwise side of v^ω and the west side of $z_v^{1_A}$ contains the clockwise side of v^ω. Continuing in this way we see that for l even, the east side of $z_v^{1_a}$ contains the clockwise side of w^{ω^l} and the west side of $z_v^{1_A}$ contains the counterclockwise side of $z_v^{\omega^l}$. We also see that for l odd, the west side of $z_v^{1_A}$ contains the clockwise side of v^{ω^l}, and the east side of $z_v^{1_A}$ contains the counterclockwise side of v^{ω^l}. Similar conclusions also follow for w^{ω^l} and y^{ω^l}. Figure 4.7 illuminates this discussion. Observe the alternation of the rotations of the vertices in the fiber over v following each lift of W. For the sake of a more intelligible picture, we show neither the total graph embedding nor the derived total graph embedding.

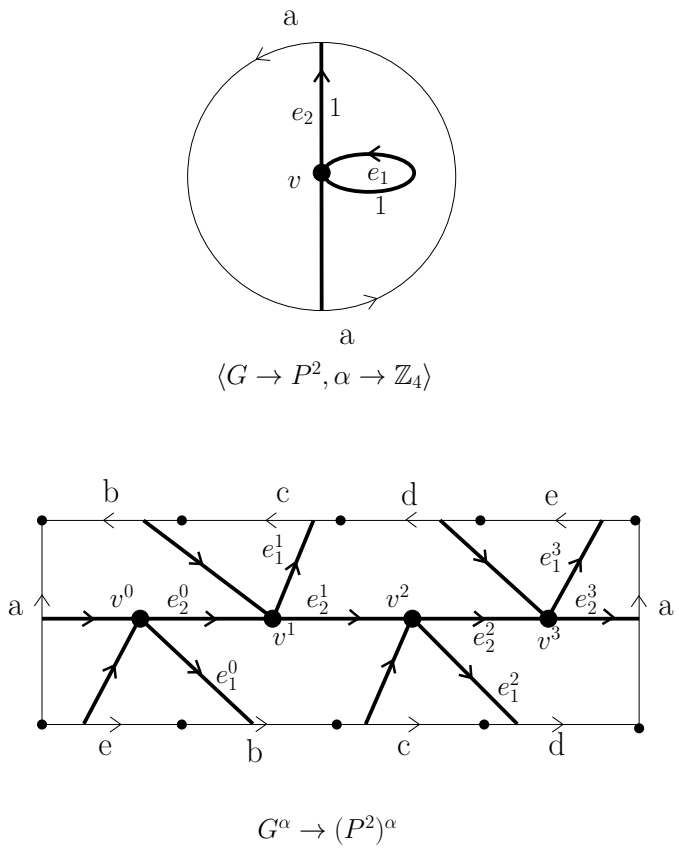


Figure 4.7: A nonorientable ordinary voltage graph embedding and its derived embedding.

4.18 Remark. Since $|\omega|$ must be even and nonzero for the circles of $(G{:}z)^\alpha$ to be orientation-preserving, we may conclude that both the east and west side of $z_v^{1_A}$ contain the same nonzero number of connected components of the lifts of the clockwise and counterclockwise sides of v. Since S^α is a regular covering of S, it follows that both of the connected components of $R(z_v^a) \setminus (G{:}z)_v^a$ intersect the same nonzero number of connected components of the lifts of the clockwise and counterclockwise sides of v. We also see that the action of ω on S^α is a rotation and reflection along the connected components of $(G{:}z)^\alpha$. The 1-chains in the fiber over z are a set of 1-chains having property Δ. If there exist two z-regions of S^α relative to the 1-chains in the fiber over z that share a common boundary component, then the action of ω on S^α switches those z-regions, and so we see that any two of these z-regions are homeomorphic. If there are two or more components of $(G{:}z)^\alpha$ bounding a z-region F, then it necessarily follows that each connected component bounds only two z-regions

and that these z-regions are homeomorphic; it is impossible to switch F with two distinct z-regions simultaneously. It follows that there are at most two z-regions of S^α relative to the 1-chains over z contained in any component of S^α. Figures 4.8 and 4.9 illustrate this discussion. We show the total graph embedding with the extended voltage assignment in Figure 4.8 for future use.

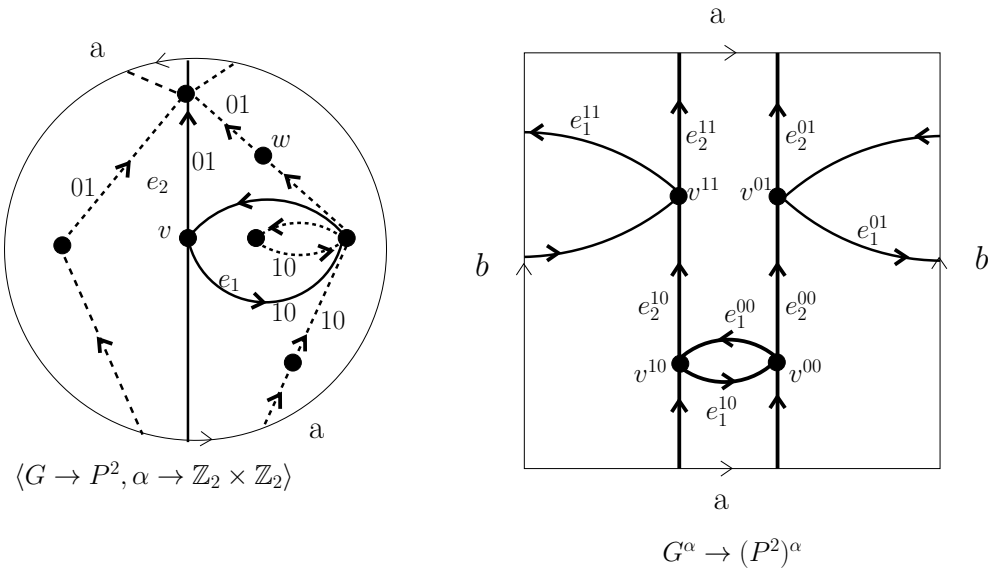
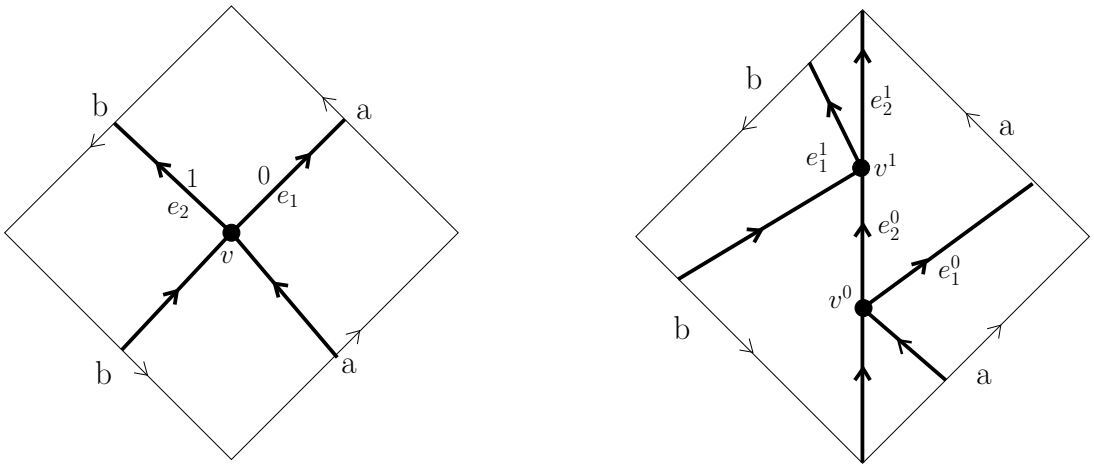


Figure 4.8: An ordinary voltage graph embedding in the projective plane and its derived embedding in the torus. The total graph embedding is shown only in the base embedding, where the edges of M' are dashed. The only voltages shown are nonidentity voltages. We let ab stand in the place of (a, b).



$\langle G \to KB, \alpha \to \mathbb{Z}_2 \rangle$ $\quad$ $G^\alpha \to (KB)^\alpha$

Figure 4.9: An ordinary voltage graph embedding and its derived embedding, both of which are in the Klein bottle KB.

Per the discussion between following 4.17, we see for the vertices in the list w^{1_A}, w^ω, ..., $w^{\omega^{|\omega|-1}}$, that every other vertex beginning with w^{1_A} is contained in the east side of $z_v^{1_A}$ and every other vertex beginning with w^ω is contained in the west side of $z_v^{1_A}$.

4.19 Remark. Since $|\omega|$ is even and nonzero, we see that each z-region contains at least one vertex of the fiber over w.

4.2.2 Circles over nonseparating orientation-preserving circles

Consider $\langle G \to S, \alpha \to A \rangle$, and let $z \in Z(G)$ be homologically nontrivial and induce an orientation-preserving circle. Let $W = d_1, \ldots d_n$ denote an Eulerian walk of $G{:}z$ based at a vertex v of $G{:}z$. Let $e = E(d_1)$. Since $G{:}z$ is orientation preserving, all circles forming the fiber over $G{:}z$ are orientation preserving. Since $G{:}z$ is orientation-preserving and z is homologically nontrivial, there is a $G{:}z$-path H nontrivially intersecting both components of $R(z) \setminus (G{:}z)$. Consider now an extended voltage assignment to the embedded total graph $T(G)$, and the special claw corresponding to e. Because of the existence of H, there is a w-y walk W' in the subdivided medial graph M' that does not transversely cross $G{:}z$. For each w^a, there is a lift W'^a_w joining w^a to $x^{a\omega(W')}$ that does not transversely cross any of the circles forming $(G{:}z)^\alpha$. Remark 4.20 follows from this discussion.

4.20 Remark. Each z-region of S^α relative to the set of 1-chains $\{z_v^a : a \in A\}$ having property Δ contains at least one vertex of $p^{-1}(w)$ and at least one vertex of $p^{-1}(y)$.

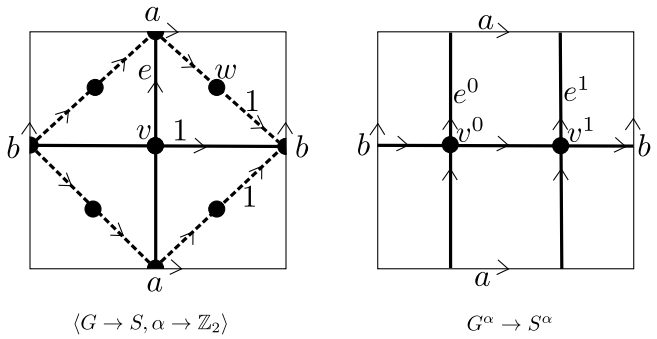


Figure 4.10: An ordinary voltage graph embedding and its corresponding derived embedding, both of which are in the torus. The total graph embedding is shown only in the base embedding, where the edges of M' are dashed. The only voltages drawn are nonzero (nonidentity) voltages.

4.2.3 z-regions

Fix $a \in A$, and $z \in Z(G)$ such that z induces a homologically nontrivial circle. Fix $v \in V(G)$ and let $W = d_1 \dots d_n$ be an Eulerian walk in $G{:}z$ based at v. Let $e = E(d_1)$. Consider the special claw corresponding to e. Implicitly using the equality $A'(w) = A(v)$ occurring in Lemma 4.14, we are able to establish a bijection between cosets of $A^{\underline{\vee}}(w, z)$ contained in $aA(v)$ and the z-regions of S_v^a with respect to $\{z_v^b : b \in aA(v)\}$.

4.21 Lemma. The function $\phi_{\underline{\vee}}\colon \{bA^{\underline{\vee}}(w, z) : b \in aA'(w)\} \to \{\bar{S}_w^b(z) : b \in aA'(w)\}$ defined by

$$\phi_{\underline{\vee}}(bA^{\underline{\vee}}(w, z)) = \bar{S}_w^b(z)$$

is a bijection.

Proof. By Remarks 4.19 and 4.20, we have that every z-region of S_v^a relative to $\{z_v^b : b \in A'(w)\}$ contains at least one vertex in the fiber over w. By Lemma 4.3 and the regular A-action on S^α, it suffices to consider the case $a = 1_A$. It is apparent that $\bar{S}_w^c = \bar{S}_w^d$ if and only if there is w^c-w^d walk W' in M'^α that does not transversely cross any component of $(G{:}z)^\alpha$.

Given such a w^c-w^d walk W', it follows that $p(W^c)$ is a w-path in M' of net voltage $c^{-1}d$ that does not transversely cross $G{:}z$. The result follows. □

4.22 Example. Consider Figure 4.8. Let $z = e_2$. Observe that $A^{\underline{\vee}}(w, z) = \langle(1, 0)\rangle$, which has two left cosets in $\mathbb{Z}_2 \times \mathbb{Z}_2$. Thus, there are two z-regions of $(P^2)^\alpha$ with respect to $\{z_v^a : a \in \mathbb{Z}_2 \times \mathbb{Z}_2\}$.

4.23 Example. Consider Figure 4.10. Observe that $A(v) = \mathbb{Z}_2$ and $A^{\underline{\vee}}(w, e) = \{0\}$. Thus, there are two z-regions of S^α relative to $\{e^0, e^1\}$.

Chapter 5

Homological Applications of the Coset Lemmas

In this chapter we show how the intersection theory developed in Section 2.2, and the results of Chapter 3 and Chapter 4 can be utilized to derive homological data concerning derived embeddings. Some of these results will not make use of intersection theory, thus demonstrating the utility of the results of Chapter 4 just by themselves. Proposition 5.1 is a result concerning ordinary voltage graphs embedded in pseudosurfaces.

5.1 Lifts of homologically trivial 1-chains

Consider $\langle G \to P, \alpha \to A \rangle$ and let $z \in Z(G)$ be a connected 1-chain such that $z = \partial I$ for some $I \in C_2(G \to S)$. Let $v \in V(G{:}z)$. We first prove a simple result that is an application of earlier theorems. Fix $a \in A$. Proposition 5.1 establishes a bound for the $\mathbb{Z}_2$-dimension of $\langle [z_v^b]_2 : b \in aA(v) \rangle$.

5.1 Proposition. For a fixed $a \in A$, in the connected component P_v^a of P^α,

$$dim_{\mathbb{Z}_2} \left(\langle [z_v^b]_2 : b \in aA(v) \rangle \right) \leq \frac{|A(v)|}{|A(v,z)|} - 1.$$

Proof. By Proposition 3.3, the intersection of $p^{-1}(z)$ with G_v^a is homologically trivial. By

part 3 of Theorem 4.8, there are $\frac{|A(v)|}{|A(v,z)|}$ components of $(G{:}z)^\alpha_v$ contained in G^a_v. The result follows. □

We move to consider ordinary voltage graph embeddings in surfaces. Now assume that z is still homologically trivial but induces a circle, so $A(v,z) = \langle\omega(W)\rangle$ for any Eulerian walk W of $G{:}z$. Theorem 2.5 guarantees that $z = \partial I^c$. The 1-chains z^a_v are obviously a set of 1-chains having property Δ. By Lemma 2.7, we see that the connected components of $(S{:}I)^\alpha$ are the face-connected components of $(S{:}I)^\alpha$, and the same conclusion follows for connected components of $(S{:}I^c)^\alpha$.

5.2 Construction. Let $b_1, \ldots b_j$, $j = \frac{|A|}{|\langle\omega(W)\rangle|}$ denote representatives of the cosets of $\langle\omega(W)\rangle$ contained in $aA(v)$. By Lemmas 4.5 and 4.6, we may conclude that for the 1-chains z^b_v contained in S^a_v, which are a set of 1-chains having property Δ, the vertices of the corresponding z-graph $\Gamma(z^{b_1}_v, \ldots, z^{b_j}_v)$ correspond bijectively with the union of the cosets of $A(v,I)$ contained in $aA(v)$ and the cosets of $A(v,I^c)$ contained in $aA(v)$. Similarly, the cosets of $\langle\omega(W)\rangle$ contained in $aA(v)$ correspond bijectively to the 1-chains z^b_v contained in S^a_v, which are the edges of $\Gamma(z^{b_1}_v, \ldots, z^{b_j}_v)$. Thus, implicitly using the aforementioned bijections, we may say that edges of $\Gamma(z^{b_1}_v, \ldots, z^{b_j}_v)$ are incident to the vertices of $\Gamma(z^{b_1}_v, \ldots, z^{b_j}_v)$ that contain them.

5.3 Proposition. Consider $\langle G \to S, \alpha \to A\rangle$. If $z \in Z(G)$ induces a circle, W is an Eulerian walk of $G{:}z$, $z = \partial I$ for some $I \in C_2(G \to S)$ and $v \in V(G{:}z)$, then

$$dim_{\mathbb{Z}_2}\left(\langle [z^b_v]_2 : b \in aA(v)\rangle\right) = \frac{|A(v)|}{|\langle\omega(W)\rangle|} - \left(\frac{|A(v|}{|A(v,I)|} + \frac{|A(v)|}{|A(v,I^c)|} - 1\right).$$

Proof. Following Construction 5.2, the above equation represents the dimension of the cycle space of $\Gamma(z^{b_1}_v, \ldots, z^{b_j}_v)$. The result follows by Theorem 2.9. □

5.4 Remark. Note that the application of Theorem 2.9 implies that Proposition 5.3 is constructive in that we may use its proof to produce a basis for the span of $\left\{[z^b_v]_2 : b \in aA(v)\right\}$ in $H_1(S^a_v)$, where we treat $H_1(S^a_v)$ as a $\mathbb{Z}_2$-vector space. By Theorem 2.21 and Corollary

2.23, we may extend a basis for the span of the $[z_v^b]_2$ to an independent set that is twice as large.

5.5 Example. Consider Example 4.11. Let $z = e_2$ in Figure 4.2. Following the proof of Proposition 5.3, the set of homology classes $\left\{[z_v^0]_2, [z_v^1]_2\right\}$ is a basis for the span of $\left\{[z_v^0]_2, [z_v^1]_2,\ [z_v^2]_2, [z_v^3]_3\right\}$. By Corollary 2.23, the independent set $\left\{[z_v^0]_2, [z_v^1]_2\right\}$ can be augmented by other homology classes $[z_v'^0]_2,\ [z_v'^1]_2$, represented by 1-chains inducing circles, such that the set $\left\{[z_v^0]_2, [z_v^1]_2, [z_v'^0]_2,\ [z_v'^1]_2\right\}$ is homologically independent.

5.2 Circles over orientation-reversing circles

5.2.1 Results not utilizing Archdeacon's medial graph enhancement

Here we will utilize the results of Chapter 2 and Section 4.1 to prove some basic results about the derived surface of an ordinary voltage graph embedding. Proposition 5.6 is surprising in that the crux of its proof lies only in intersection theory; there is no explicit reliance on the incidence of faces and edges.

5.6 Proposition. Consider $\langle G \to S, \alpha \to A\rangle$. Let $\{z_1, \ldots, z_n\}$ denote a set of 1-chains inducing orientation-reversing circles satisfying $\langle [z_i]_2, [z_j]_2\rangle_2 = 0$ for all $i \neq j$. For each i, let v_i be a vertex of $G{:}z_i$ and let W_i denote an Eulerian walk of $G{:}z_i$. If, for all i, $|\langle\omega(W_i)\rangle|$ is odd, then the set of 1-chains

$$\left\{(z_i)_{v_i}^a : i \in [n],\ a \in A\right\}$$

is independent in $H_1(S^\alpha)$, and each circle induced by the 1-chains therein is orientation-reversing.

Proof. By Lemma 4.17 each circle in the fiber over $G{:}z_i$ is orientation reversing, and so each $(z_i)_{v_i}^a$ satisfies

$$\langle [(z_i)_{v_i}^a]_2, [(z_i)_{v_i}^a]_2\rangle_2 = 1.$$

The circles forming the fiber over $G{:}z_i$ have no vertices in common, so any two distinct

1-chains of the form $(z_i)^a_{v_i}, (z_i)^b_{v_i}$ inducing circles of $(G{:}z_i)^\alpha$ satisfy

$$\langle [(z_i)^a_{v_i}]_2, [(z_i)^b_{v_i}]_2 \rangle_2 = 0.$$

Now, consider two 1-chains $(z_i)^a_{v_i}$, $(z_j)^b_{v_j}$ for $i \neq j$. If $G{:}z_i$ and $G{:}z_j$ have no vertices in common, then neither do $(G{:}z_i)^a_{v_i}$ and $(G{:}z_j)^b_{v_j}$, and

$$\langle [(z_i)^a_{v_i}]_2, [(z_j)^b_{v_j}]_2 \rangle_2 = 0$$

follows. If $G{:}z_i$ and $G{:}z_j$ do have a vertex u in common, let $\dot{W}$ denote an Eulerian walk of $G{:}z_i$ based at u and consider the lift $\dot{W}^a_u$ of $\dot{W}$. There are an odd number of subwalks $d_k, \ldots d_{k+l}$ contained in the interior of $\dot{W}^a_u$ having the properties:

1. $h(d_k) \in V(G{:}z_j)^a_u$,
2. $t(d_k) \in V(G{:}z_j)^a_u$,
3. $E(d_k) \cup E(d_{k+l})$ intersects both components of $R(z^a_u) \setminus (G{:}z)^a_u$, and
4. the interior darts of $d_{j+1} \ldots d_{j+k-1}$ comprise a walk in $(G{:}z_j)^a_u$.

The enumerated properties mean that for each $\dot{W}^a_u$, there are an odd number of instances of $\dot{W}^a_u$ entering and leaving $R(z^a_u)$ through different components of $R(z^a_u) \setminus (G{:}z)^a_u$ contained in the interior of $\dot{W}^a_u$.

Also, observe that the first dart of $\dot{W}^a_u$ is part of another subwalk of $\hat{\dot{W}}^a_u$ that satisfies properties 1 through 4 above. It follows that to each lift $\dot{W}^a_u$ of $\dot{W}$ we can associate an even number of subwalks satisfying properties 1 through 4 above. So, in general position, the circles $(G{:}z_i)^a_u$ and $(G{:}z_j)^a_u$ have an even number of vertices in common. Therefore

$$\langle [(z_i)^a_{v_i}]_2, [(z_j)^b_{v_j}]_2 \rangle_2 = 0$$

for all i, j, a, b.

Let

$$X_2 = \left\{ (z_i)^a_{v_i} : i \in [n],\ a \in A \right\}.$$

We deduce that M_{X_2} is an identity matrix. By Theorem 2.20, the elements of X_2 are independent in $H_1(S^\alpha)$. □

5.7 Corollary. Consider $\langle G \to S, \alpha \to A\rangle$. Let $\{z_1, \ldots, z_n\}$ denote a set of 1-chains inducing orientation-reversing circles and satisfying $\langle [z_i]_2, [z_j]_2\rangle_2 = 0$ for all $i \neq j$. Let each z_i have the property that for each Eulerian walk W_i of $G{:}z_i$, the order $|\langle\omega(W_i)\rangle|$ is odd. Let v_i be a vertex of $G{:}z_i$. The set of homology classes

$$\left\{[(z_i)^b_{v_i}]_2 : i \in [n],\ b \in aA(v)\right\}$$

has dimension

$$|A(v_1)|\sum_{i=1}^{n}\frac{1}{|\langle\omega(W_i)\rangle|}$$

over $\mathbb{Z}_2$.

Proof. If we let

$$X_2 = \left\{(z_i)^b_{v_i} : i \in [n],\ b \in aA(v)\right\},$$

then we have by Proposition 5.6 that the 1-chains contained in X_2 are independent in $H_1(S^a_v)$. Note that for any vertex v^* in G, $A(v)$ is conjugate to $A(v^*)$ and so $|A(v)| = |A(v^*)|$. Therefore

$$\frac{|A(v_1)|}{|\langle\omega(W_i)\rangle|} = \frac{|A(v_i)|}{|\langle\omega(W_i)\rangle|} \text{ for all i.}$$

This result now follows from Part 3 of Theorem 4.8. □

We now transition to the case in which the circles forming the fiber over $G{:}z$ are orientation preserving. Thus, per Lemma 4.17, for the remainder of this section, any Eulerian walk W of $G{:}z$ based at $v \in V(G{:}z)$ (or any other vertex) has the property that $|\langle\omega(W)\rangle|$ is even. We consider first a very simple case. Let P^2 denote the projective plane and assume for now that $dim_{\mathbb{Z}_2}(Z(G)) = 1$. For $G \to P^2$, the one circle contained in G must be orientation-reversing, otherwise the embedding $G \to P^2$ wouldn't be cellular. Let $z \in Z(G)$ be the 1-chain inducing this one circle. Proposition 5.12 will establish necessary and sufficient conditions for each of the z^a_v to be homologically trivial. We will need a few technical

lemmas.

5.8 Lemma. Let T be a tree and consider $\langle T, \alpha \to A\rangle$. For all $v \in V(T)$, $A(v) = 1_A$.

Proof. For any $v \in V(T)$ and any closed walk W in T based at v, the fact that T is acyclic implies that $\omega(W) = 1_A$. The result follows by induction on the number of edges of T. □

5.9 Lemma. Consider $\langle G, \alpha \to A\rangle$ and let $v \in V(G)$. If $A(v) = \{1_A\}$, then each component of G^α is isomorphic to G, and there are $|A|$ such components of G^α.

Proof. By Part 1 of Theorem 4.8, there are $|A|$ components of G^α, each one containing at least one edge in the fiber over any (directed) edge e of G, and at least one vertex in the fiber over any vertex u of G. Note that for any other vertex u of G, the group $A(u)$ is conjugate to $A(v)$. Since the trivial subgroup $\{1_A\}$ of A is invariant under conjugation, it follows that each component of G^α contains exactly one vertex in the fiber over u. Moreover, since directed edges in the fiber over a directed edge of G are uniquely identifiable by their tail vertices, it follows that each component of G^α contains exactly one directed edge in the fiber over each edge of G.

Because A acts transitively on the components of G^α, it suffices to prove the result for a single component $G_v^{1_A}$ of G. Let $p_{1_A}\colon G_v^{1_A} \to G$ denote the restriction of the covering map $p\colon G^\alpha \to G$ to $G_v^{1_A}$. The map p_{1_A} is necessarily bijective. Moreover, the directed edge e^a beginning at u^a and ending at a vertex w^b maps under p_{1_A} to the directed edge e beginning at u and ending at w. So, p_{1_A} is indeed a graph isomorphism between $G_v^{1_A}$ and G_v. □

5.10 Lemma. Let $\langle G, \alpha \to A\rangle$ denote an ordinary voltage graph such that $\dim_{\mathbb{Z}_2}(Z(G)) = 1$. Each component G_v^c of G^α satisfies $\dim_{\mathbb{Z}_2}(G_v^c) = 1$.

Proof. Since the action of G on G^α by left multiplication is regular, it suffices to prove the lemma for a specific connected component, $G_v^{1_A}$, of G^α. Let $z \in Z(G)$ be the only nonzero element of $Z(G)$ and let $v \in V(G{:}z)$.

Per Theorem 3.4, it suffices to consider a voltage assignment for which $\alpha(e) = 1_A$ for each directed edge e of a spanning tree T of G. Since $\dim_{\mathbb{Z}_2}(Z(G))$=1, there is only one

edge of $G \setminus T$. Let e_1 denote the positively directed edge appearing in $G \setminus T$. If $\alpha(e_1) = 1_A$, then Lemma 5.9 completes the proof.

Now assume that $\alpha(e_1) = c \neq 1_A$. For the sake of cleaner notation, let $|c|$ denote $|\langle c \rangle|$. Let $E(T)$ denote the edge set of T. By Lemmas 5.8 and 5.9, there are $|A|$ components of $(G{:}E(T))^\alpha$, each isomorphic to T. let T_v^a denote the unique connected component of T^α containing v^a; each edge e_1^a connects T_v^a to T_v^{ac}. Since the set

$$\left\{T_v^a, T_v^{ac}, \ldots T_v^{ac^{c-1}}\right\}$$

is a set of trees, the graph

$$H = T_v^a \cup e_1^a \cup T_v^{ac} \cup e_1^{ac} \cup \ldots \cup e^{ac^{|c|-2}} \cup T_v^{ac^{|c|-1}}$$

is also a tree.

Let

$$H' = H \cup e_1^{ac^{|c|-1}}.$$

The edge $e_1^{ac^{|c|-1}}$ joins the trees $T^{ac^{|c|-1}}$ and T^a. Since H was already connected and a tree, H' contains exactly one circle. Further, since H' is the result of removing the fiber over e_1 from $G_v^{1_A}$ and adding it back again, $H' = G_v^{1_A}$. □

5.11 Remark. If $z \in Z(G)$ is the only nonzero element of $Z(G)$, then it is apparent that $(G{:}z)_v^a$ is the only circle contained in G_v^a.

Recall that we use P^2 to denote the projective plane.

5.12 Proposition. Let $\langle G \to P^2, \alpha \to A \rangle$ be such that $\dim_{|mathbbZ_2}(Z(G)) = 1$. Let $z \in Z(G)$ be the only nonzero element of $Z(G)$ and let W be an Eulerian walk on $G{:}z$ based at $v \in V(G{:}z)$. For each $a \in A$, $[z_v^a]_2 = [0]_2$ if and only if $|\omega(W)|$ is even.

Proof. Per Lemma 5.10, the fact that derived embeddings are cellular implies that each connected component of $(P^2)^\alpha$ satisfies $\dim_{\mathbb{Z}_2}(H_1(P^2))^\alpha \leq 1$. So, $(P^2)^\alpha$ is a disjoint union

of spheres or projective planes. Per Lemma 4.17, the single circle in each component of G^α is orientation preserving if and only if $|\langle\omega(W)\rangle|$ is even. It follows that $\dim_{\mathbb{Z}_2}(H_1((P^2)^\alpha) = 0$ if and only if $|\langle\omega(W)\rangle|$ is even. □

We now transition to more complicated graphs; we allow for $\dim_{\mathbb{Z}_2}(Z(G)) > 1$.

5.13 Proposition. Consider $\langle G \to S, \alpha \to A\rangle$ and let $z \in Z(G)$ be a 1-chain inducing an orientation-reversing circle containing a vertex v. If an Eulerian walk W of $G{:}z$ has the property that $|\langle\omega(W)\rangle|$ is even, then the dimension of $\langle [z_v^b]_2 : b \in aA(v)\rangle$ is $\frac{|A(v)|}{|\omega(W)|} - 1$ or $\frac{|A(v)|}{|\omega(W)|}$.

Proof. By Part 3 of Theorem 4.8, the set $\left\{z_v^b : \ b \in aA(v)\right\}$ is the set 1-chains in the fiber over z contained in S_v^a. If there is only one z-region of S_v^a relative to the 1-chains over z, then $\dim_{\mathbb{Z}_2}(H_1(S_v^a)) = \frac{|A(v)|}{|\omega(W)|}$ by Theorem 2.9. If there are more than one such z-region, then by Remark 4.18, there are two z-regions of S_v^a relative to the z_v^b, and so the corresponding z-graph is a graph consisting of two vertices and $\frac{|A(v)|}{|\omega(W)|}$ parallel edges. The result follows. □

Corollary 5.14 follows easily in light of Proposition 5.13.

5.14 Corollary. If the set $\left\{z_v^b : b \in aA(v)\right\}$ contains more than one element, that is, $A(v)$ is not generated by $\omega(W)$, then each $z_v^{a_i}$ is homologically nontrivial.

Proof. Omitted. □

5.2.2 Results utilizing Archdeacon's medial graph enhancement

Referring to Remark 2.10, we see that we would be able to determine a basis for the $\mathbb{Z}_2$-span of homology classes represented by the elements of $\left\{z_v^b : b \in aA(v)\right\}$, a set having property Δ, in $H_1(S_v^a)$ if we had a method to construct the appropriate z-graph. Since $G{:}z$ is a circle, we remind the reader that $A(v, z) = \langle\omega(W)\rangle$ for an Eulerian walk W of $G{:}z$ based at v.

5.15 Construction. Let $b_1, \ldots, b_j$ denote representatives of the cosets of $\langle\omega(W)\rangle$ contained in $aA(v)$, $W = d_1 \ldots d_k$, and consider a special claw corresponding to $E(d_1)$. Let w' denote

one of the vertices w or y appearing in Definition 4.12. Per Lemma 4.14, $A'(w') = A(v)$. Using this equality implicitly, we see that per Lemma 4.21, the z-regions of S_v^a relative to the z_v^b for $b \in aA(v)$ correspond bijectively to the left cosets of $A^{\vee}(w', z)$ contained in $aA(v)$. Per Lemma 4.6, the cosets of $\langle \omega(W) \rangle$ contained in $aA(v)$ correspond bijectively to edges of $\Gamma(z_v^{b_1}, \ldots, z_v^{b_j})$. Thus, using the aforementioned bijections implicitly, we may say that the left cosets of $A^{\vee}(w, z)$ contained in $aA(v)$ are the vertices of the z-graph $\Gamma(z_v^{b_1}, \ldots, z_v^{b_j})$ and the left cosets of $\langle \omega(W) \rangle$ contained in $aA(v)$ are the edges.

5.16 Proposition. Consider $\langle G \to S, \alpha \to A \rangle$ for a nonorientable surface S. Let $z \in Z(G)$ induce an orientation-reversing circle such that an Eulerian walk $W = d_1, \ldots d_k$ of $G{:}z$ based at $v \in V(G{:}z)$ has the property that $|\langle \omega(W) \rangle|$ is even. Consider a special claw corresponding to $E(d_1)$, and let w' take the place of the vertex w appearing in Definition 4.12. For any $a \in A$,

$$dim_{\mathbb{Z}_2} \left(\langle [z_v^b]_2 : b \in aA(v) \rangle \right) = \frac{|A(v)|}{|\langle \omega(W) \rangle|} - \left(\frac{|A(v)|}{|A^{\vee}(w', z)|} - 1 \right).$$

Proof. Per Construction 5.15, we see that right hand side of the above equation is the dimension of the cycle space of the z-graph $\Gamma(z_v^{b_1}, \ldots, z_v^{b_j})$. The result follows by Theorem 2.9. □

5.17 Remark. Following the proof of Theorem 2.9, in order to find a basis for span of $\left\{ [z_v^b]_2 : b \in aA(v) \right\}$ in $H_1(S_v^a)$, we could describe the incidence of vertices and edges of the associated z-graph $\Gamma(z_v^{b_1}, \ldots, z_v^{b_j})$. Recall that the vertex w' takes the place of w in Definition 4.12. We let y' take the place of y. Recall that in the derived embedding of the total graph, the vertices w'^a and y'^a are in the same component of the fiber over the special claw corresponding to e as v^a. Recall also that the east and west sides of $z_v^{1_A}$ contain alternate vertices in the list

$$w^{1_A}, w^{\omega(W)}, w^{\omega(W)^2}, w^{\omega(W)^3}, \ldots.$$

Thus, we see that the z-regions containing $(G{:}z)_v^{1_A}$ in their boundaries contain all of the

vertices in one (or both) of the lists

$$w^{1_A}, w^{\omega(W)^2}, \ldots,$$

$$w^{\omega(W)}, w^{\omega(W)^3}, \ldots.$$

In the case of there being only one z-region of S_v^a with respect to $\{z_v^b : b \in aA(v)\}$, we see that that z-region will contain all of the vertices of $w^{1_A}, w^{\omega(W)}, w^{\omega(W)^2}, w^{\omega(W^3)}, \ldots$. And so, we may characterize the incidence of the vertices and edges of $\Gamma(z_v^{b_1}, \ldots, z_v^{b_j})$ by saying that an edge $c\langle\omega(W)\rangle$ is incident to a vertex $baA^{\underline{\vee}}(w, z)$ that intersects that edge nontrivially.

5.18 Remark. Note that the application of Theorem 2.9 implies that Proposition 5.16 is constructive in that its proof can be used to produce a basis for the span of $\{[z_v^b]_2 : b \in aA(v)\}$. By Theorem 2.21 and Corollary 2.23, we may extend a basis for the span of the $[z_v^b]_2$ to an independent set that is twice as large.

5.19 Example. Consider Example 4.22. For the moment, let cd stand in the place of (c, d). In light of the construction of $\Gamma(z_v^{00}, z_v^{10})$, we may deduce that $\{[z_v^{00}]_2\}$ is a basis for the span of $\{[z_v^a]_2 : a \in \mathbb{Z}_2 \times \mathbb{Z}_2\}$, and by Remark 5.18, we may find another 1-chain z^1 of G^α such that $\{[z_v^{00}]_2, [z_1]_2\}$ is an independent set.

5.3 Circles over orientation-preserving circles

Consider $\langle G \to S, \alpha \to A\rangle$ and let $z \in Z(G)$ be homologically nontrivial and induce an orientation-preserving circle. We begin with a simple result. Note that the proof of Proposition 5.20 uses homology with $\mathbb{Z}$ coefficients.

5.20 Proposition. Let $\{z_1, \ldots, z_n\}$ be a set of homologically independent 1-chains inducing orientation-preserving circles in $\langle G \to S, \alpha \to A\rangle$. Fix $a \in A$. For all $i \in [n]$, let v_i be a vertex of $G{:}z_i$, and let $Y_i = \langle[(z_i)_{v_i}^b] : \ b \in aA(v)\rangle$. Let $I^\alpha \subset F(G_v^a \to S_v^a)$ be such that $\partial^\alpha I^\alpha$ is a nonzero element in $\cup_i Y_i$. If, for any specific i, there is an element of Y_i with a nonzero coefficient in $\partial^\alpha I^\alpha$, then there must be at least two such elements of Y_i.

Proof. Assume without loss of generality and for the sake of contradiction that

$$\partial I^{\alpha} = a^*(z_1)_{v_1}^{b^*} + \Sigma$$

for an $a^* \in \mathbb{Z}$, $(z_i)_{v_1}^{b*} \in Y_1$ and Σ in the $\mathbb{Z}$-span of $\cup_{j\geq 2} Y_j$. Let W_1 be an Eulerian walk of $G{:}z_1$. Since p is a linear chain map and $[\partial I^{\alpha}] = [0]$, we have

$$[0] = [p(\partial I^{\alpha})] = [p(a^*(z_1)_{v_1}^{b*} + \Sigma)] = [a^*|\langle\omega(W_1)\rangle|z_1] + [p(\Sigma')]$$

where $p(\Sigma)$ is in the span of $\cup_{j\geq 2} Y_j$. Since $\mathbb{Z}$ is an integral domain, $a^* = 0$, which is a contradiction. □

Corollary 5.21 follows immediately from Theorem 5.20.

5.21 Corollary. For any $i \in [n]$ and $b \in aA(v)$, $(z_i)_{v_i}^{b}$ is nontrivial in $H_1(S_v^a)$, where $H_1(S_v^a)$ is taken to be a $\mathbb{Z}$-module.

Proof. Omitted. □

5.3.1 Results utilizing Archdeacon's medial graph enhancement

Consider $\langle G \to S, \alpha \to A\rangle$ and let $z \in Z(G)$ be homologically nontrivial and induce an orientation-preserving circle. Let v be a vetex of $G{:}z$. Consider as well the subdivided medial graph M' with a transferred voltage assignment and the total graph $T(G)$ with an extended voltage assignment. Let W be an Eulerian walk of $G{:}z$ based at $v \in V(G{:}z)$, and consider a special claw corresponding to the edge e of $G{:}z$ containing the first dart of W. Let w' and y' denote vertices of the zero-voltage claw corresponding to e that take the place of w and y, respectively, in Definition 4.12.

5.22 Construction. Let $b_1, \ldots, b_j$ denote representatives of the cosets of $\langle\omega(W)\rangle$ contained in $aA(v)$. Per Lemma 4.14 $A'(w') = A(v)$. Using this equality implicitly, we see per Lemma 4.21 that the the left cosets of $A^{\vee}(w', z)$ contained in $aA(v)$ correspond bijectively to the vertices of the z-graph $\Gamma(z_v^{b_1}, \ldots, z_v^{b_j})$. Per Part 3 of Theorem 4.8, the cosets of $\langle\omega(W)\rangle$ contained in $aA(v)$ correspond bijectively to the edges of $\Gamma(z_v^{b_1}, \ldots, z_v^{b_j})$. Thus, using the

aforementoned bijections implicitly, we may say that the left cosets of $A^{\veebar}(w,z)$ contained in $aA(v)$ are the vertices of the z-graph $\Gamma(z_v^{b_1},\ldots,z_v^{b_j})$ and the left cosets of $\langle\omega(W)\rangle$ contained in $aA(v)$ are the edges.

Proposition 5.23 follows immediately in light of Construction 5.22.

5.23 Proposition. Consider $\langle G\to S,\alpha\to A\rangle$ and let $z\in Z(G)$ induce an orientation-preserving circle. For any $a\in A$,

$$\dim_{\mathbb{Z}_2}\left(\langle[z_v^b]_2 : b\in aA(v)\rangle_2\right)=\frac{|A(v)|}{|\langle\omega(W)\rangle|}-\left(\frac{|A(v)|}{|A^{\veebar}(w,z)|}-1\right).$$

Proof. Omitted.

5.24 Remark. Following the proof of Theorem 2.21, we could construct a basis for the span of $\left\{[z_v^b]_2 :\ b\in aA(v)\right\}$ by considering the incidences of the edges and vertices of the z-graph associated to the z_v^b. We define the set $A^{\veebar}(w',y',z)$ to be the set (not necessarily a group) of net voltages of walks in M' beginning at w', ending at w' or y', and not transversely crossing $G{:}z$. Observe that two vertices w'^a and y'^b are in the same z-region of S_v^a with respect to $\left\{z_v^b : b\in aA(v)\right\}$ if and only if there is a walk W' in the embedded M'^{α} beginning at w'^a, ending at y'^b, and not transversely crossing any of the components of $(G{:}z)^{\alpha}$. The last statement is true if and only if $\omega(p(W'))=a^{-1}b$ is an element of $A^{\veebar}(w',y',z)$. For similar reasons, w'^a and w'^b are contained in the same z-region if and only if $a^{-1}b$ is an element of $A^{\veebar}(w,y,z)$. For $a_1\in A$, let $a_1\cdot A^{\veebar}(w,y,z)=\left\{a_1r : r\in A^{\veebar}(w',y',z)\right\}$. Recall that the vertices w'^c, y'^c, and v^c are in the same lift of the special claw corresponding to e, and each lift of W is an orientation-preserving walk. For the sake of cleaner notation, we once again let ω denote $\omega(W)$ and $|\omega|$ denote $|\langle\omega(W)\rangle|$. Observe that if a z-region of S_v^a with respect to $\left\{z_v^b : b\in b\in aA(v)\right\}$ contains a vertex w'^c, then it contains $w'^{c\omega},w'^{c\omega^2},\ldots,w'^{c\omega^{|\omega|-1}}$. Similarly, if a z-region of S_v^a with respect to $\left\{z_v^b : b\in aA(v)\right\}$ contains y'^c, then it contains $y'^{c\omega},y'^{c\omega^2},\ldots,y'^{c\omega^{|\omega|-1}}$. It follows that each left coset of $\langle\omega(W)\rangle$ is contained in one or two sets of the form $a_1\cdot A^{\veebar}(w,y,z)$.

Note that to any left coset $bA^{\veebar}(w,z)$, we may associate a unique set $b\cdot A^{\veebar}(w,y,z)$ that contains the group-element superscripts of the vertices in the fibers over w' and y'

contained in the z-region $\bar{S}_w^b$. Implicitly using this bijection and the bijections appearing in Construction 5.15, we may say that each edge of $\Gamma(z_v^{b_1}, \ldots, z_v^{b_j})$ is incident to the one or two sets of the form $a_1 \cdot A^{\veebar}(w, y, z)$ that contain it.

5.25 Remark. Note that the application of Theorem 2.9 implies that Proposition 5.23 is constructive in that we can use its proof to produce a basis for the span of $\langle [z_v^b]_2 : b \in aA(v) \rangle$. By Theorem 2.21 and Corollary 2.23, we may extend a basis for the $\mathbb{Z}_2$- span of the $[z_v^b]_2$ to an independent set that is twice as large.

5.26 Example. Consider Example 4.23. In light of the construction of $\Gamma(e^0, e^1, e^2, e^3)$, we may deduce that $\{[e^0]_2\}$ is a basis for $\langle [e^0]_2, [e^1]_2, [e^2]_2, [e^3]_2 \rangle$, and by Theorem 2.21 we may find another 1-chain z_1 of G^α such that $\{[e^0]_2, [z_1]_2\}$ is independent.

Chapter 6

Applications to Embeddability of Graphs in Surfaces

Two of the central accomplishments of ordinary voltage graph theory are Theorems 6.1 and Theorem 6.2, below, which state conditions under which a graph or graph embedding may be encoded in the form of an ordinary voltage graph or an ordinary voltage graph embedding, respectively. We need these and a few more results before proving the main results of this chapter.

6.1 Theorem. [11, Theorem 2.2.2]**(Gross and Tucker, 1974)** If A is a group acting freely on a graph $\tilde{G}$, and G is the resulting quotient graph, then there is an assignment α of ordinary voltages in A to the quotient graph G such that G^α is isomorphic to $\tilde{G}$ and that the given action of A on $\tilde{G}$ is the natural left action of A on G^α.

As discussed in [11, §4.3] a finite-sheeted covering space $(\tilde{S}, p)$ of a surface S is said to be regular if S is the quotient of $\tilde{S}$ modulo the action of a finite group A such that the action of A has a finite number of fixed points. Such an action on a surface is called pseudofree. Theorem 6.2 establishes that all such actions of finite groups on surfaces are be encoded in the form of an ordinary voltage graph embedding.

6.2 Theorem. [11, Theorem 4.3.5]**(Gross and Tucker, 1987)** Let $p\colon \tilde{S} \to S$ denote a regular branched covering of surfaces, and let $G \to S$ denote an embedding in S having at

most one branch point in any face and no branch points in G. There there is an ordinary voltage assignment α in the group of (covering) deck transformations of p such that p is equivalent to a natural surface projection $S^\alpha \to S$.

We say that an automorphism of a surface S respects (reverses) the rotation of an embedded graph G if the automorphism preserves the orientation type (orientation-preserving or orientation-reversing) of each edge and preserves (reverses) the rotations at each vertex of G.

6.3 Theorem. [11, Corollary to Theorem 4.3.2.] Consider $G \to S$. Let A be a group of automorphisms acting on the graph G. If every element of A respects or reverses the rotation system, then the action of A can be extended to an action on embedding surface S. Moreover, if A acts freely on the graph G, then the extended action can be chosen to have at most one fixed point inside any one face.

Applying Theorems 6.2 and 6.3, we see that if G is embedded in S in such a way that the free-action of a group A on G extends to a cellular automorphism of S, then the embedding can be encoded using an ordinary voltage graph embedding since the action of A on the derived embedding respects the rotations of the vertices of G. We will investigate the embeddability of two infinite families of graphs in fixed surfaces as derived embeddings; we will show that there are no embeddings of these graphs as derived embeddings despite the fact that these graphs do embed in these fixed surfaces.

For any derived embedding of an ordinary voltage graph embedding $\langle G \to P, \alpha \to A\rangle$, there is at most one branch point contained in each face of the base embedding. Another vital tool is the Riemann-Hurwicz Equation, which is stated here as Theorem 6.4. In order to understand Theorem 6.4, we must understand the deficiency of a branch point y contained in a face f_y, denoted def(y). Let W_{f_y} denote a boundary walk of f_y, and recall that it takes $|\langle\omega(W_{f_y})\rangle|$ lifts of W_{f_y} to form each set of consecutive lifts of W_{f_y}. Let $|\omega_y|$ denote $|\langle\omega(W_{f_y})\rangle|$ for the remainder of this paragraph, and note that in P^α, which is an

$|A|$-fold branched covering of P,

$$|A| - \frac{|A|}{|\omega_{f_y}|} = |p^{-1}(f_y)|.$$

We define $\mathrm{def}(y)$ to be $|A| - |p^{-1}(f_y)|$. So, for an n-fold cover of P,

$$\mathrm{def}(y) = n - |p^{-1}(y)|.$$

6.4 Theorem. (The Riemann-Hurwicz Equation) Let $p : \tilde{P} \to \dot{P}$ denote an n-fold branched covering of pseudosurfaces and let Y denote the set of branch points of $\dot{P}$. Then

$$\chi(\tilde{P}) = n\chi(\dot{P}) - \sum_{y \in Y} \mathrm{def}(y).$$

Proof. The proof, which we omit, is a routine sharpening of the analogous result for surfaces, [11, Theorem 4.2.4]. □

Lemma 6.5 will be used in in the proof of Theorem 6.9.

6.5 Lemma. For manifolds $\tilde{X}$ and X with X compact, let ϕ: $\tilde{X} \to X$ denote a finite-sheeted covering map. If $\tilde{X}$ is nonorientabe, then X is nonorientable.

Proof. Assume for the sake of contradiction that X is orientable. Consider a closed curve c in X that is closed in the topology on X. Since X is compact, c is compact as well. Therefore, we may chose an open cover O of c consisting of a finite number of sets that are small enough for the local homeomorphism property to apply. Since X is assumed to be orientable, each of the open sets in O can be compatibly oriented.

Let $\tilde{O}$ denote the set of open sets lying above O, which, by assumption, contains a finite number of sets. Using the local homeomorphism property, we may extend the orientation of one open set of $\tilde{O}$ to all open sets that nontrivially intersect it, one open set at a time. Continuing in this way, we may extend this orientation to a regular neighborhood of a lift of c.

It follows that no closed curve $\tilde{c}$ in $\tilde{X}$ that has a nonorientable regular neighborhood can be a lift of $\phi(\tilde{c})$, a contradiction. □

Another tool that we will make use of is the well-known Classification of Surfaces, with and without boundary, which can be found in [16, Theorem 5.1]. The following Lemmas, concerning the decomposition of a surface S as the connected sum $S_1 \# S_2$ of two surfaces, S_1 and S_2, are consequences of the Classification of Surfaces, Theorem 1.2, and the (non)orientability of the projective plane, the torus T, and the Klein bottle KB. We let S^2 denote the sphere.

6.6 Lemma. As a connected sum of surfaces, the projective plane P^2 is only decomposable as $P^2 \# S^2$.

6.7 Lemma. As a connected sum of surfaces, the Klein bottle KB has exactly two decompositions: $KB = KB \# S^2$ and $KB = P^2 \# P^2$.

6.8 Lemma. As a connected sum of surfaces, the torus T is only decomposable as $T = T \# S^2$.

6.1 Cartesian products of circles in Dyck's Surface

Following [11, §1.3.3], the Cartesian product of two graphs G and G' is denoted $G \times G'$ and defined to be the graph with vertex set $V_G \times V_{G'}$ and edge set $(E_G \times V_{G'}) \cup (V_G \times E_{G'})$. If the edge (e, v') is an element of $E_G \times V_{G'}$ and the endpoints of the edge e are v_1 and v_2, then the endpoints of the edge (e, v') are the vertices (v_1, v') and (v_2, v'). If $(v, e') \in V_G \times E_{G'}$ and the endpoints of the edge e are v_1' and v_2', then the endpoints of the edge (v, e') are (v, v_1') and (v, v_2').

Recall that a graph is a circle if it is a connected 2-regular graph. Let C_n denote a circle with n edges. Figure 6.1 shows the cartesian product of $C_2 = (E, V)$ and $C_3 = (V', E')$ in terms of the notation described above.

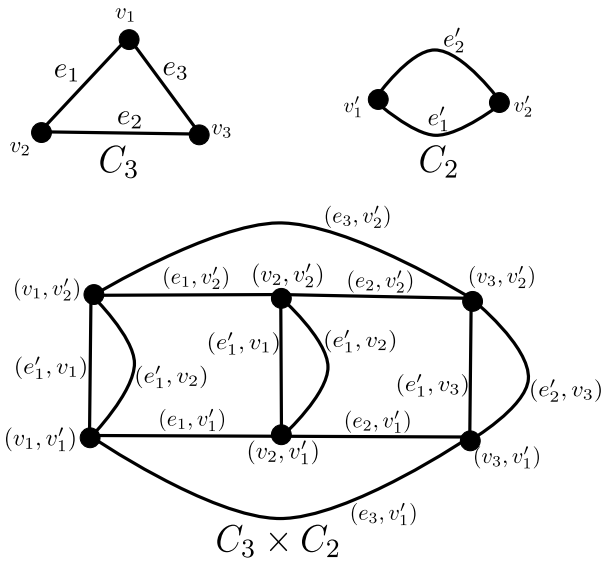


Figure 6.1: The Cartesian product of two circles.

Let p and q denote distinct primes, each at least five, and consider $C_p \times C_q$. Note that each graph of the form $C_p \times C_q$ has a derived embedding in T whose base embedding is a bouquet of two loops with voltage group $\mathbb{Z}_{pq}$; see Figure 6.2. We refer to the edges e, e_p, and e_q later.

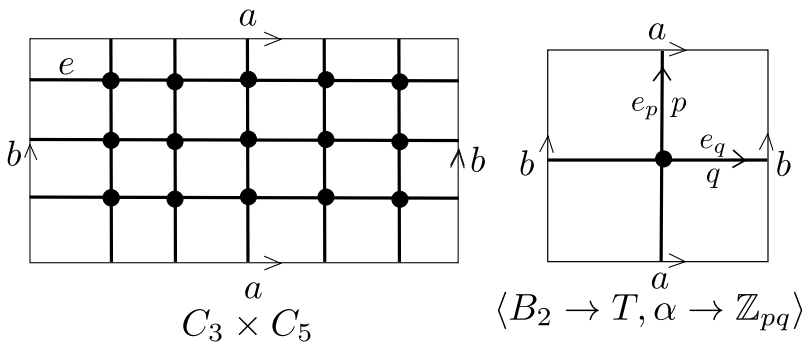


Figure 6.2: The Cartesian product of two circles embedded in the torus as a derived embedding.

By applying edge twisting surgery [11, p. 134] to any single edge of the derived embedding of $C_p \times C_q$ in T described above, we may produce a (cellular) embedding of $C_p \times C_q$ in Dyck's Surface, which is the connected sum of three projective planes. Figure 6.3 shows the result after twisting a specific edge.

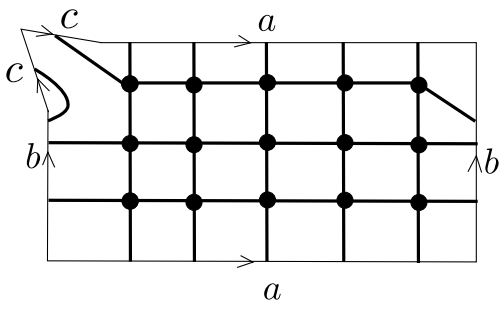


Figure 6.3: The result after twisting the edge e in Figure 6.2

And so we see that any graph of the form $C_p \times C_q$ has a cellular embedding in Dyck's surface. However, as we will show in Theorem 6.9, there is no embedding of $C_p \times C_q$ in Dyck's Surface as a derived embedding of an ordinary voltage graph embedding. By Theorems 6.2 and 6.3, this means that there is no embedding of $C_p \times C_q$ in Dyck's surface such that a free action of a group on $C_p \times C_q$ extends to a cellular automorphism of the surface.

6.9 Theorem. For distinct primes, $p,\ q \geq 5$, there is no embedding of $C_p \times C_q$ in Dyck's Surface that is encoded in the form of an ordinary voltage graph embedding.

Proof. Without loss of generality, assume $p < q$. First, note that, up to isomorphism of groups, the three free actions of groups that generate all such actions on $C_p \times C_q$ are those that are generated by the voltage-group actions of the subgroups isomorphic to $\mathbb{Z}_p$, $\mathbb{Z}_q$, and $\mathbb{Z}_{pq}$ on $C_p \times C_q$. We consider first the $\mathbb{Z}_p$ and $\mathbb{Z}_q$ actions.

The $\mathbb{Z}_p$-action is a free action on the vertices and edges of $C_p \times C_q$, but which fixes each of the p circles of the form C_q whose edges cover the edge e_q and permutes the q circles of the form C_p. The $\mathbb{Z}_q$-action is an action that permutes the q circles of the form C_p whose edges cover e_2 and fixes the p circles of the form C_q. It is a consequence of Theorem 6.1 that $C_p \times C_q$ is the derived graph of $\langle C_p \times C_1, \alpha_q \to \mathbb{Z}_q \rangle$ with α_q assigning voltage 0 to all darts on links and 1 to the positively directed darts on loops. Similarly $C_p \times C_q$ is the derived graph of $\langle C_1 \times C_q, \alpha_p \to \mathbb{Z}_p \rangle$ with α_p assigning voltage 0 to all darts on links and 1 to the positively directed darts on loops.

The full $\mathbb{Z}_{pq}$-action is also a free action on the vertices and edges of $C_p \times C_q$, and fixes none of the circles. The quotient of $C_p \times C_q$ modulo this action is the bouquet of two loops, B_2, and $C_p \times C_q$ can be recovered as the derived graph with voltage assignment described in Figure 6.2.

By Theorem 6.4, we see that the only way to obtain Dyck's surface as the derived surface of any ordinary voltage graph embedding is to consider base embeddings in surfaces of nonnegative Euler Characteristic; if $\chi(P) \leq -1$, then Theorem 6.4 implies that $\chi(\tilde{P}) < -1$. Moreover, by Lemma 6.5, we must consider only base embeddings in P^2 and KB. We first consider base embeddings of $\langle C_p \times C_1, \alpha_q \rightarrow \mathbb{Z}_q \rangle$ and $\langle C_1 \times C_q, \alpha_p \rightarrow \mathbb{Z}_p \rangle$ in P^2, and then KB. We will prove the result in detail for ordinary voltage graphs of the form $\langle C_1 \times C_q, \alpha_p \rightarrow \mathbb{Z}_p \rangle$ and point out how the same arguments work for ordinary voltage graphs of the form $\langle C_1 \times C_q, \alpha_p \rightarrow \mathbb{Z}_p \rangle$. Let $z_1,\ z_2 \in Z(C_1 \times C_q)$ denote a loop and the 1-chain inducing the circle of length q, respectively. By Theorem 6.2, this will suffice.

We first treat the cases of base embeddings in the projective plane, P^2. Per Lemma 2.26, any two 1-chains representing the nontrivial homology class of P^2 must represent homology classes having nonzero intersection product. Since the loops of $C_1 \times C_q$ do not have any vertices in common, it follows that at most one loop of $C_1 \times C_q$ is homologically nontrivial, and since $5 \leq p < q$ there are at least six loops bounding faces. For each loop, the positively directed edge is assigned voltage 1, and so the fiber over any face bounded by a loop contains only one face. It follows that there are at least 6 branch points of deficiency $p - 1$ in P^2. So, by Theorem 6.4, we have that

$$\chi(P^2)^\alpha \leq p \cdot 1 - 6 \cdot (p-1) = 6 - 6p < -1,$$

which means that $(P^2)^\alpha$ is not Dyck's surface since Dyck's surface has Euler Characteristic -1 and $p \geq 5$. A similar analysis shows that for $\langle C_p \times C_1 \rightarrow P^2, \alpha_q \rightarrow \mathbb{Z}_q \rangle$, there are at least four loops bounding faces, and thus

$$\chi((P^2)^\alpha \leq q \cdot 1 - 4(q-1) = 4 - 4q < -1,$$

also excluding $(P^2)^\alpha$ from being Dyck's surface.

Now, we consider embeddings of $C_1 \times C_q$ in the Klein bottle. Again, we will prove the result in detail for ordinary voltage graphs of the form $\langle C_1 \times C_q, \alpha_p \to \mathbb{Z}_p \rangle$ and point out how the same arguments work for ordinary voltage graphs of the form $\langle C_p \times C_1, \alpha_q \to \mathbb{Z}_q \rangle$.

First we assume that z_2 induces an orientation-reversing circle. Since all darts of $G{:}z_2$ are assigned voltage 0, we have by Proposition 5.6 that there are at least p homologically independent 1-chains in KB^α. Since $\dim_{\mathbb{Z}_2}(P^2 \# P^2 \# P^2) = 3$, this excludes the possibility that KB^α from being Dyck's surface. It similarly follows from Proposition 5.6 that under the assumption that the 1-chain inducing the circle of length p in $\langle C_p \times C_1 \to KB, \alpha \to \mathbb{Z}_q \rangle$ induces an orientation reversing circle, there are at least q homologically independent 1-chains in KB^α, so KB^α is not Dyck's surface.

Now assume that z_2 induces an orientation-preserving circle.

If z_2 is homologically trivial, then we will show that there is least one loop bounding a face f. Since the positive-directed edges on all loops are assigned voltage 1, it follows that f contains a branch point y of deficiency $(q-1)$. By Theorem 6.4, it will follow that

$$\chi(KB^\alpha) \leq q \cdot 0 - (q-1) = 1 - q \leq 0,$$

which implies that KB^α is not Dyck's surface. Since $\dim_{\mathbb{Z}_2}(KB) = 2$, under the assumption that z_2 is homologically trivial, there must be two homologically nontrivial, nonhomologous loops that represent the elements a basis for $H_1(KB)$. Note again that no two loops in $C_1 \times C_q$ induce circles with vertices in common. Since the embedding is in the Klein bottle, one of the loops, say z_1, must induce an orientation-reversing circle. If another loop, which we denote z_3, induced an orientation-preserving circle, then, by Lemma 2.27, $\langle [z_1]_2, [z_3]_2 \rangle_2 = 1$, which is impossible. It follows that z_3 also induces an orientation-reversing circle. Therefore

$$\langle [z_1]_2, [z_1]_2 \rangle_2 = \langle [z_3]_2, [z_3]_2 \rangle_2 = 1$$

and

$$\langle [z_1]_2, [z_3]_2 \rangle_2 = 0.$$

Moreover, since none of the loops in $C_1 \times C_q$ induce circles with vertices in common, no other loop, say z_4 can be homologically nontrivial, since $[z_4]_2$ would have a nonzero intersection product with $[z_1]_2$ or $[z_3]_2$. It follows by Lemma 6.7 that the only other way for z_4 to be homologically trivial would be for z_4 to bound a face or for z_4 to induce the boundary of the punctured projective planes forming the connected sum $KB = P^2 \# P^2$. If z_4 is the former, then KB^α is not Dyck's surface. If z_4 is the latter, then there is another loop, say z_5, that is homologically trivial and contained within one of the two projective planes forming the connected sum $P^2 \# P^2$. It follows by Lemma 6.6 that z_5 bounds a face.

Now assume that z_2 is homologically nontrivial and induces an orentation-preserving circle. Without loss of generality, a loop, say z_1, induces an orienatation-reversing circle and by Lemma 2.27 $\langle [z_1]_2, [z_2]_2 \rangle_2 = 1$. Since none of the loops induce circles having vertices in common, it follows that no other loops can be homologous to z_2, and no other loops can be homologous to z_1. Let z_3 denote another loop. We see that z_3 cannot induce the boundary of a punctured projective plane since the circle induced by z_2 passes through both crosscaps in the decomposition $KB = P^2 \# P^2$ and would have to share two vertices with the circle induced by z_3, which is impossible. By Lemma 6.7, the only other way that z_3 could be homologically nontrivial is if z_3 bounds a face, which implies that KB^α is not Dyck's surface. If z_3 is not homologically nontrivial, then the fact that z_3 cannot be homologous to z_1 or z_2 means that z_3 is homologous to $z_1 + z_2$. Consider another loop z_4 and note that the above exclusions force z_4 to be homologically nontrivial, which means that z_4 is homologous to z_1, z_2, or $z_1 + z_2$. If z_4 is homologous to z_1, then $\langle [z_4]_2, [z_1]_2 \rangle_2 = 1$, which is impossible. If z_4 is homologous to z_2, then $\langle [z_4]_2, [z_1]_2 \rangle_2 = 1$, which is also impossible. If z_3 is homologous to $z_1 + z_2$, then the fact that z_3 is homologous to $z_1 + z_2$ means that

$$\langle [z_3]_2, [z_4]_2 \rangle_2 = \langle [z_1 + z_2]_2, [z_1 + z_2]_2 \rangle_2 = \langle [z_1]_2 + [z_2]_2, [z_1]_2 + [z_2]_2 \rangle_2 = 1,$$

which is impossible. It follows that z_4 must bound a face, and so KB^α is not Dyck's surface.

Since $p \geq 5$, $C_p \times C_1$ contains at least five loops. It similarly follows that under the assumption that the circle of length p in $\langle C_p \times C_1 \to KB, \alpha_q \to \mathbb{Z}_q \rangle$ induces an orientation-preserving circle, then there is at least one loop bounding a face.

We move to considering base embeddings of the bouquet of two loops, B_2,

$$\langle B_2 \to S, \alpha \to \mathbb{Z}_{pq} \rangle,$$

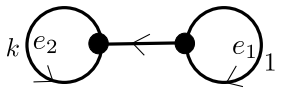
where S is either the projective plane or the the Klein bottle and the voltages p and q are assigned to the positively directed edges on the two loops e_p and e_q, respectively. In either case, one of the loops must induce an orientation reversing circle. Since p and q are at least 5, it follows by Proposition 5.6 that there are at least 5 homologically independent 1-chains in S^α. Therefore, S^α is not Dyck's surface. This completes the proof. □

6.2 Generalized Petersen Graphs in the torus

In this section we show that there is infinite class of graphs with each graph having no embedding in the torus T as a derived embedding. It follows from Theorem 6.2 that no free group actions on these graphs extend to cellular automorphisms of any of the embeddings of these graphs in T.

Given positive integers n, k, the Generalized Petersen Graph $GP(n,k)$ has vertices v_0, v_1, ..., v_{n-1}, u_0, u_1, ..., u_{n-1}. For all $i \in \mathbb{Z}_n$, $GP(n,k)$ has edges (v_i, v_{i+1}), (v_i, u_i), and (u_i, u_{i+k}), where all subscripts are taken modulo n. Note that this action is transitive on the vertices v^i, and it is transitive on the vertices u^i.

6.10 Remark. Note the presence of a free action of $\mathbb{Z}_n$ on $GP(n,k)$. An integer l acts on $GP(n,k)$ by mapping v_i to v_{i+l}, and u_i to u_{i+l}, where the subscripts are reduced modulo n. The quotient of $GP(n,k)$ modulo this $\mathbb{Z}_2$-action the free action is the barbell graph $GP(1,1)$. Each Generalized Petersen Graph can be recovered as the derived graph of the ordinary voltage graph appearing in Figure 6.4.



$\langle GP(1,1), \alpha \to \mathbb{Z}_n \rangle$

Figure 6.4: An ordinary voltage graph whose derived graph is $GP(n,k)$.

6.11 Example. Two well known Generalized Petersen Graphs are the Petersen Graph, $GP(5,2)$, and the Dürer Graph, $GP(6,2)$. Both of these graphs appear as derived graphs in Figure 6.5. The zero voltages are not shown.

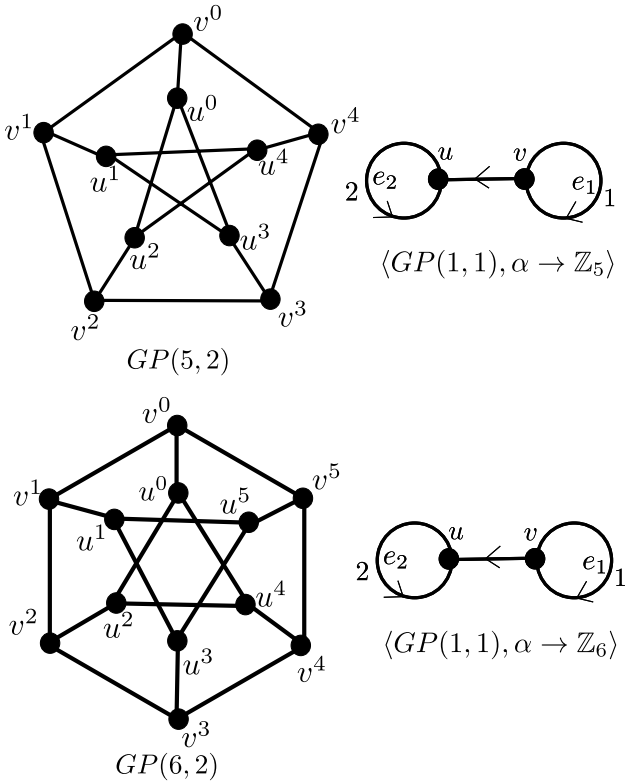
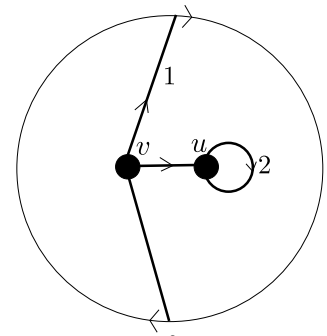
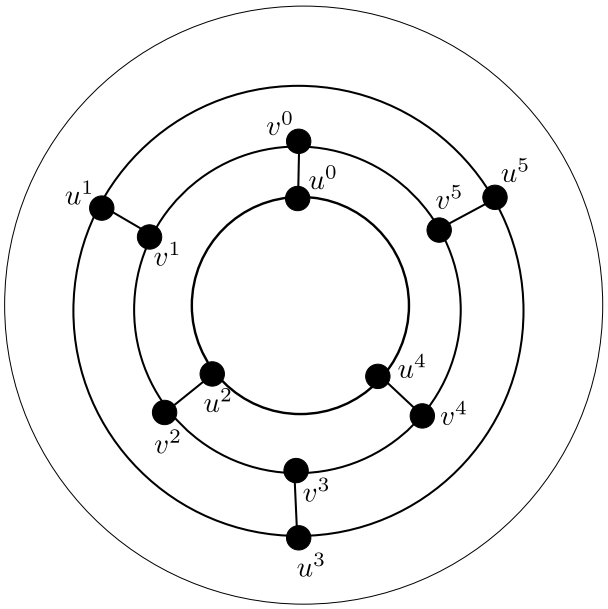


Figure 6.5: The Petersen Graph and the Dürer Graph as derived graphs of ordinary voltage graphs. The darts on the edges of each derived graph are omitted, as are the zero voltages in the base embedding.

First note that $GP(2p,2)$ has a derived embedding in the sphere, as evidenced by Figure 6.6.



$\langle GP(1,1) \to P^2, \alpha \to \mathbb{Z}_{2p} \rangle$



$GP(6,2) \to S^2$

Figure 6.6: $GP(2p, 2)$ embedded in the sphere as a derived embedding of the special case of the base embedding for which $p = 3$. The zero voltages are not shown. The darts on the edges of the derived graph are omitted.

6.12 Remark. To see that, for all p, the derived embedding of the base embedding in Figure 6.6 is in the sphere, note that the boundary walk of one of the faces has net voltage 2, and the other has net voltage 0. It follows that there is a single branch point and it has deficiency $2p - 2$. By Theorem 6.4, we see that in this case, $(P^2)^\alpha$ has Euler characteristic $2p \cdot 1 - (2p - 2) = 2$, and so $(P^2)^\alpha = S^2$ by Theorem 1.2.

6.13 Construction. Given the derived embedding of $GP(2p, 2)$ in the sphere described in Remark 6.12, we construct a cellular embedding of $GP(2p, 2)$ in the torus. Form the

connected sum $S^2\#T$ with the requirement that the circle of attachment is contained in the face f bounded by the circle of $GP(2p,2)$ that contains the vertices u^{2l}. Now, remove the edges (u_0,u_2), (u_2,u_4), as in Figure 6.7. Note that the face f in Figure 6.7 is homeomorphic to a punctured torus.

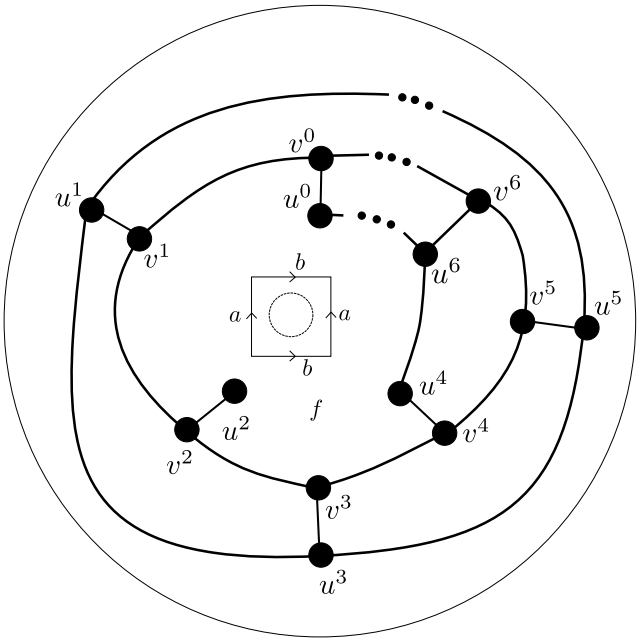


Figure 6.7: A partial and noncelluar embedding of $GP(2p,p)$ in the torus. The dashed circle in the middle of the square represents a boundary component of the interior of the square with edges identified. The face f is homeomorphic to a punctured torus.

Now, redraw the edges (u_0,u_2) and (u_2,u_4) such that they complete longitudinal and meridianal circles of the torus, as in Figure 6.8.

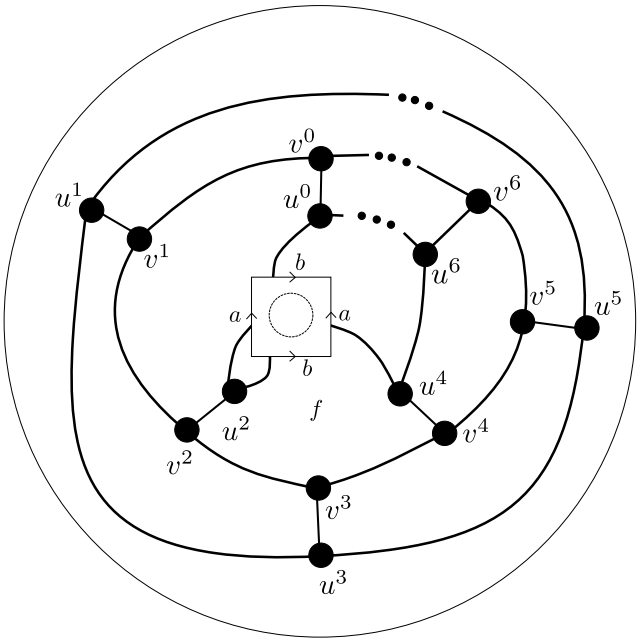


Figure 6.8: $GP(2p, 2)$ cellularly embedded in the torus.

It remains to check that the face f shown in Figure 6.8 is homeomorphic to a disc. To verify this, note that in the case of each embedding of $GP(2p, 2)$ in the torus constructed this way, a facial boundary boundary of f is always the same and can be described as a sequence of vertices,

$$u^0v^0v^1v^2u^2u^4u^6 \dots u^0u^2v^2v^3v^4u^4u^2u^0,$$

where the vertices $\{u^i : i \geq 6\}$ exist for $p > 3$. It is apparent that f is homeomorphic to a disc.

6.14 Remark. The rerouting of the edges in this manner, which ensures a cellular embedding in the torus, is borrowed from the proof of a theorem of Xuong [11, Theorem 3.4.13].

Before we come to our discussion of the main results of this chapter, we need to state a theorem of Frucht, Graver and Watkins.

6.15 Theorem. [9] The graph $GP(n, k)$ is vertex transitive if and only if $k^2 \equiv 1$ (mod n), or $n = 10$ and $k = 2$.

Now assume that p is a prime greater than 5. If there were any automorphism of $GP(2p,2)$ that maps a vertex v_j to a vertex u_k, then the $\mathbb{Z}_n$-action defined in Remark 6.10, which is transitive on the vertices v_i and the vertices u_i, respectively, can be used to construct an automorphism of $GP(2p,2)$ that maps any vertex to any other vertex. This would mean that $GP(2p,2)$ is vertex transitive, which contradicts Theorem 6.15. So, the only free actions of groups on $GP(2p,2)$, for primes $p > 5$ are those which have the vertices v_i and the vertices u_i in distinct orbits. Recall that the vertices v_i are vertices of a circle of length n. Since the automorphism group of a circle of length n is the dihedral group of order $2n$, and reflections of the circle have fixed points, it follows that the only free action of groups on $GP(2p,p)$ are the cyclic actions of the subgroups $\langle 1\rangle$, $\langle 2\rangle$, and $\langle p\rangle$ of $\mathbb{Z}_{2p}$ which act on $GP(2p,p)$ in the manner described in Remark 6.10. The quotient graphs of $GP(2p,2)$ modulo these actions are $GP(1,1)$, $GP(2,1)$ and $GP(p,2)$, respectively. By Theorem 6.1, there exist voltage assignments to these graphs such that the corresponding derived graphs are $GP(2p,2)$. Note by Theorem 3.4 that we can require that α assigns voltage 0 to the darts of a spanning tree of the ordinary voltage graph. The three possibilities for ordinary voltage graphs whose derived graphs are $GP(2p,2)$ are:

1. $\langle GP(1,1), \alpha \to \mathbb{Z}_{2p}\rangle$ such that α assigns voltage 1 and 2 to the positively directed edges on the loops, and 0 to both darts on the link, as in Figure 6.4,

2. $\langle GP(2,1), \alpha \to \mathbb{Z}_p\rangle$, as in Figure 6.9;

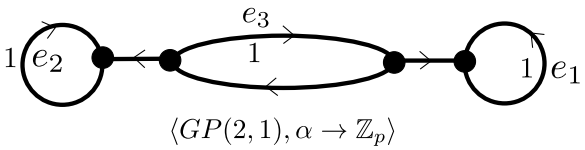


Figure 6.9: An ordinary voltage graph whose derived graph is $GP(2p,2)$. The zero voltages are not shown.

3. $\langle GP(p,2), \alpha \to \mathbb{Z}_2\rangle$ such that α assigns 1 to all darts on the edges joining the vertices v_i, and 0 to all other darts. An example of this appears in Figure 6.10.

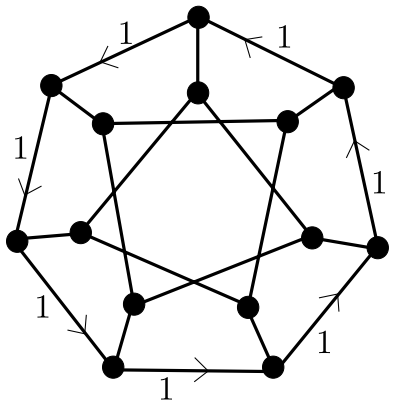


Figure 6.10: An ordinary voltage graph whose derived graph is $GP(14, 2)$. The zero voltages are not shown.

We now state and prove Theorem 6.16, which was alluded to earlier in this dissertation.

6.16 Theorem. For each prime $p > 5$, the Generalized Petersen Graph $GP(2p, 2)$ has no embedding in the torus as a derived embedding.

Proof. Per Theorems 1.2, 3.5, and 6.4 it suffices to consider the three necessary ordinary voltage graphs described in this section and all of their possible embeddings in the sphere, projective plane, torus, and Klein bottle. We organize the proof by each ordinary voltage graph and then by surface of the base embedding.

Case 1: $GP(1, 1)$

Case 1a: Base embeddings of $GP(1, 1)$ *in* S^2

Since $dim_{\mathbb{Z}_2}(Z(GP(1, 1)) = 2$ and the two loops induce circles that have no vertices in common, there is only one embedding with three faces: two faces are bounded by a loop, and the other is bounded by both loops and the link. Figure 6.11 contains the only embedding of $GP(1, 1)$ in the sphere, though we will have to consider different orientations of the positive directed edges.

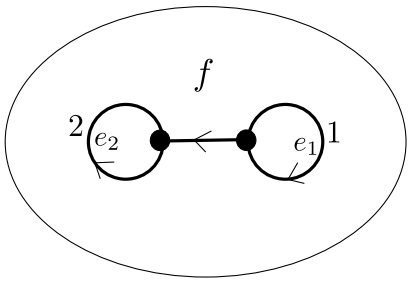


Figure 6.11: An ordinary voltage graph embedding in the sphere. The zero voltages are not shown.

Given the embedding depicted in Figure 6.11, note that the two faces bounded by loops will always have facial boundary walks with net voltages of order $2p$ and p, respectively. It follows that the deficiencies of the branch points contained in these two faces are $2p-1$ and $2p-2$. The face f has a facial boundary walk W_f whose net voltage depends on whether the positively-directed edges e_1 and e_2 (or both of their opposites) appear in W_f. If this is true, which is the case depicted in Figure 6.11, then $\omega(W_f) = 3$. Since $p > 5$, and 3 does not divide $2p$, it follows that the deficiency of the branch point contained in f is $2p-1$. By Theorem 6.4,

$$\chi((S^2)^\alpha) = 2p \cdot 1 - ((2p-1) + (2p-1) + (2p-2)) = 4 - 4p < 0.$$

And so, $(S^2)^\alpha$ can't be the torus. The only other possibility is that W_f contains e_1 and e_2^{-1} (or vice versa). In this case it is apparent that

$$\langle \omega(W_f) \rangle = \mathbb{Z}_{pq}$$

, and so the deficiency of the branch point contained in f is $2p-1$. Just as before, $\chi(S^2)^\alpha < 0$, which precludes $(S^2)^\alpha$ from being the torus.

Case 1b: Base embeddings of $GP(1,1)$ in P^2

Consider the embedding of $GP(1,1)$ appearing in the base embedding in Figure 6.6. First note that at least one of e_1 and e_2 must represent the nontrivial homology class of

P^2. Since e_1 and e_2 induce circles in $GP(1,1)$ which have no vertices in common, Lemma 2.26 implies that that only one of the loops can be homologically nontrivial, and so, the other loop must bound a face. Therefore, there is no other embedding of $GP(1,1)$ in P^2 to consider.

We now move to considering the possible net voltages of facial boundary walks, which will determine the deficiencies of any branch points. By Lemma 4.17, it cannot both be true that the edge e_2 is homologically nontrivial and $(P^2)^\alpha$ is orientable. It follows that the face bounded by a loop must contain a branch point of deficiency $2p-2$. The other face has a facial boundary walk whose net voltage depends on whether the positively directed edges e_1 and e_2 (or both of their opposites) appear in a facial boundary walk W_f. If they do, then $\omega(W_f)$ generates the subgroup $\langle 4 \rangle$ of Z_{2p}. In $\mathbb{Z}_{2p}$, $\langle 2 \rangle = \langle 4 \rangle$ since 2 and 4 have the same greatest common divisor with $2p$. It follows that both faces contain branch points of deficiency $2p-2$, and by Theorem 6.4, we have that

$$\chi((P^2)^\alpha) = 2p \cdot 1 - ((2p-2) + (2p-2)) = 4 - 2p,$$

which precludes $(P^2)^\alpha$ from being the Klein bottle. The other case, where e_1 and e_2^-, or their respective opposites, appear in W_f, was treated in Figure 6.6 and Remark 6.12 and always results in the sphere as the derived surface.

Case 1c: Base embeddings of $GP(1,1)$ in the Klein bottle

Recall that $\dim_{\mathbb{Z}_2}(H_1(KB)) = 2$ and note that $\dim(Z(GP(1,1)) = 2$. So, e_1 and e_2 must be homologically nontrivial and homologically independent. We will show that e_2 induces an orientation-reversing circle, which, by Lemma 4.17, will preclude KB^α from being the torus. Since the Klein bottle is nonorientable, there must at least one orientation-reversing circle; assume that it is induced by e_1. Since e_2 must be homologically nontrivial, if it does not induce an orientation-reversing circle, it must induce a nonseparating orientation-preserving circle. If e_2 induces a nonseparating orientation-preserving circle, Lemma 2.27 implies that the circles induced by e_1 and e_2 must transversely cross each other, which is impossible since they have no vertices in common. It follows that e_2 must induce an

orientation-reversing circle for any cellular embedding of $GP(1,1)$ in the Klein bottle.

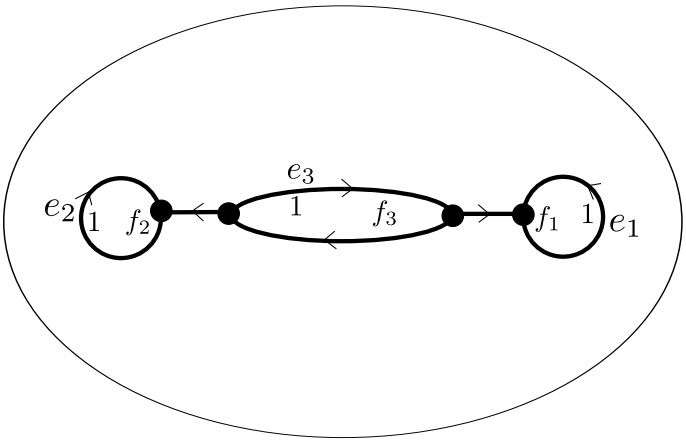
Case 1d: Base embeddings of $GP(1,1)$ in the torus

Recall that $\dim_{\mathbb{Z}_2}(H_1(T)) = 2$ and note again that $\dim_{\mathbb{Z}_2}(Z(GP(1,1)) = 2$. So, e_1 and e_2 must be homologically nontrivial and homologically independent. Note that there is an automorphism of $GP(1,1)$ that permutes the loops and fixes the link. Since T is orientable, without loss of generality, there must be a 1-chain, say e_1 inducing a nonseparating and orientation-preserving circle. Since there must be another circle of $GP(1,1)$ which crosses this circle transversely, we conclude that the circles induced by e_1 and e_2 must have at least one vertex in common, which is not true. It follows that $GP(1,1)$ has no cellular embeddings in the torus.

Case 2: $GP(2,1)$

Case 2a: Base embeddings of $GP(2,1)$ in the sphere

First note that since the three circles of $GP(2,1)$ have no vertices in common, there can be no crossings of the circles induced by any of the nonzero elements of $Z(GP(2,1))$. Since all circles induced by all graphs embedded in the sphere are separating, as $H_1(S^2) \cong \{0\}$, it follows that there are only two embeddings of $GP(2,1)$ in the sphere: one of the embeddings features both loops contained in a single component of the complement of the circle of length 2, and the other has one loop in each of those components. Consider Figure 6.12.



$\langle GP(2,1) \to S^2, \alpha \to \mathbb{Z}_p \rangle$

Figure 6.12: An ordinary voltage graph embedding featuring one of the two possible embeddings of $GP(2,1)$ in the sphere. The zero voltages are not shown.

Note that no matter how the positively directed edges e_1 and e_2 are drawn in this embedding, each of the three faces f_1, f_2, and f_3 will have boundary walks whose net voltage is 1, which generates $\mathbb{Z}_p$. It follows that f_1, f_2 and f_3 contain branch points of deficiency $p-1$, and so, by Theorem 6.4

$$\chi((S^2)^\alpha) \leq p \cdot 2 - ((p-1) + (p-1) + (p-1)) = 3 - p.$$

Since $p > 5$, this precludes $(S^2)^\alpha$ from being the torus. Now consider Figure 6.13.

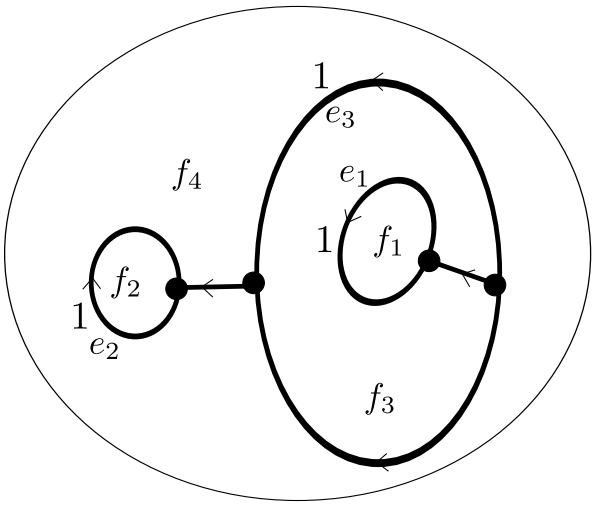


Figure 6.13: An ordinary voltage graph embedding featuring one of two possible embeddings of $GP(2,1)$ in the sphere.

Note that no matter how the positively-directed edges e_1 and e_2 are drawn in this embedding, each of the faces f_1 and f_2 will have facial boundary walks whose net voltage is 1, which generates $\mathbb{Z}_p$. In the depicted case, the facial boundary walks of f_3 and f_4 have net voltage 0, and so f_3 and f_4 contain no branch points. It follows that $(S^2)^\alpha = S^2$ since Theorem 6.4 implies that

$$\chi(S^2)^\alpha = p \cdot 2 - ((p-1) + (p-1)) = 2.$$

However, if either e_1, e_2 or e_3 were to be drawn in such a way that at least one of the faces f_3 or f_4 would have a facial boundary walk with net voltage 2, which generates $\mathbb{Z}_p$, then Theorem 6.4 implies

$$\chi((S^2)^\alpha \leq p \cdot 2((p-1) + (p-1) + (p-1) = 3 - p.$$

Since $p \geq 5$, this precludes $(S^2)^\alpha$ from being the torus.

We have therefore excluded any ordinary voltage graph embedding of $GP(2,1)$ with the voltage assignment depicted in Figure 6.9 from having derived embeddings in the torus.

Case 2b: Base embeddings of $GP(2,1)$ in the projective plane and the Klein bottle

In either of these surfaces, there must be an orientation-reversing circle. Note that there are only three circles in $GP(2,1)$, and each of them is traversed by an Eulerian walk having net voltage 1, which has odd order in $\mathbb{Z}_p$. It follows by Lemma 4.17 that any base embedding of $GP(2,1)$ in the projective plane or the Klein bottle with the voltage assignment described in Figure 6.9 results in a nonorientable derived surface.

Case 2c: Base embeddings of $GP(2,1)$ in the torus

Since $dim_{\mathbb{Z}_2}(H_1(T)) = 2$, there must be at least two homologically nontrivial circles in $GP(2,1)$. Moreover, there must be a 1-chain inducing an orientation-preserving and nonseparating circle to which is associated another circle which transversely crosses it at least once. This is impossible since none of the three circles in $GP(2,1)$ have any vertices in common.

Case 3: $GP(p,2)$

Let $C_v, C_u \in Z(GP(p,2))$ denote the 1-chains inducing the circles whose vertex sets are $\{v_i\}$ and $\{u_i\}$, respectively. Since 2 does not divide p, there is a single circle connecting the vertices u_i. It follows that $GP(p,2) \setminus (GP(p,2){:}C_v)$ is connected. Also, for future use we define a set of fundamental cycles with respect to a spanning tree of $GP(p,2)$. We chose the spanning tree

$$GP(p,2){:}\Big(\{(v_i, v_{i+1}) : i \in \mathbb{Z}_{p-1}\} \bigcup \{(v_i, u_i) : i \in \mathbb{Z}_p\}\Big),$$

which leaves us with the fundamental edges

$$\{(v_{p-1}, v_0)\} \bigcup \{(u_i, u_{i+2}) : i \in \mathbb{Z}_p\}.$$

We note that the fundamental cycles of $GP(p,2)$ with respect to this spanning tree are C_v and the cycles

$$\{C_i = (u_i, u_{i+2}) + (v_{i+2}, u_{i+2}) + (v_{i+1}, v_{i+2}) + (v_i, v_{i+1}) + (v_i, u_i) : \ i \in \mathbb{Z}_n\}.$$

Note that for any i, an Eulerian walk W_i of $GP(p,2){:}C_i$ satisfies $\omega(W_i) = 0$ since $A = \mathbb{Z}_2$ and we are considering a voltage assignment that assigns voltage 1 to exactly two darts in W_i. It follows by Part 3 of Theorem 4.8 that there are exactly two circles forming the fiber over each circle $GP(p,2){:}C_i$. Making shorthand of our previous notation, we will let C_i^0 and C_i^1 denote the 1-chains forming the fiber over C_i, according to whether their induced circles contain v_i^0 or v_i^1, respectively.

Case 3a: $GP(p,2)$ in the sphere

We will show that $GP(p,2)$ has a $K_{3,3}$ minor which, by Kuratowski's theorem, precludes $GP(p,2)$ from being planar. Let $X = \{v_0, v_2, u_1\}$ and $Y = \{v_1, v_3, u_2\}$. We will identify 9 edge sets inducing paths that have no interior vertices in common and join each of the three vertices in Y with each of the three vertices in X. It follows that $K_{3,3}$ is a minor of $GP(p,2)$.

1. Edge sets inducing paths joining v_1 to all vertices in X are

   (a) $\{(v_1, v_0)\}$ inducing a path joining v_1 and v_0, and

   (b) $\{(v_1, u_1)\}$ inducing a path joining v_1 and u_1, and

   (c) $\{(v_1, v_2)\}$ inducing a path joining v_1 and v_2.

2. Edge sets inducing paths joining v_3 to all vertices in X are:

   (a) $\{(v_3, v_4),\ (v_4, v_5) \ldots (v_{p-1}, v_0)\}$ inducing a path joining v_3 and v_0.

   (b) $\{(v_3, u_3),\ (u_3, u_1)\}$ inducing a path joining v_3 and u_1, and

   (c) $\{(v_3, v_2)\}$ inducing a path joining v_3 and v_2.

3. Edge sets inducing paths joining u_2 to all vertices in X are:

   (a) $\{(u_2, u_4), (u_4, u_6), \ldots, (u_{p-1}, u_1)\}$ inducing a path joining u_2 and u_1, and

   (b) $\{(u_2, u_0),\ (u_0, v_0)\}$ inducing a path joining u_2 and v_0, and

   (c) $\{(u_2, v_2)\}$ inducing a path joining u_2 and v_2.

It follows that $K_{3,3}$ is a minor of $GP(p,2)$. Therefore, there is no ordinary voltage graph embedding of $GP(p,2)$ in the sphere.

Case 3b: $GP(p,2)$ in the projective plane

In this case, since each dart on an edge appearing in C_u has voltage 0, it follows by Lemma 4.17, that for $(P^2)^\alpha$ to be the torus, C_u must be homologically trivial. Lemma 6.6 implies that C_u induces a contractible circle. Or course, either C_v is homologically trivial or it is homologically nontrivial.

Subcase 1: C_v is homologically nontrivial

First, consider a cellularly embedded subgraph of a cellular embedding of $GP(p,2)$ having the property that C_v is homologically nontrivial. If the circle $GP(p,2){:}C_1$ is orientation preserving, then, since $GP(p,2)\setminus(GP(p,2){:}C_i)$ is connected, we must have the edge (v_{i+1},u_{i+1}) appearing as it does in the disc U as drawn in Figure 6.14.

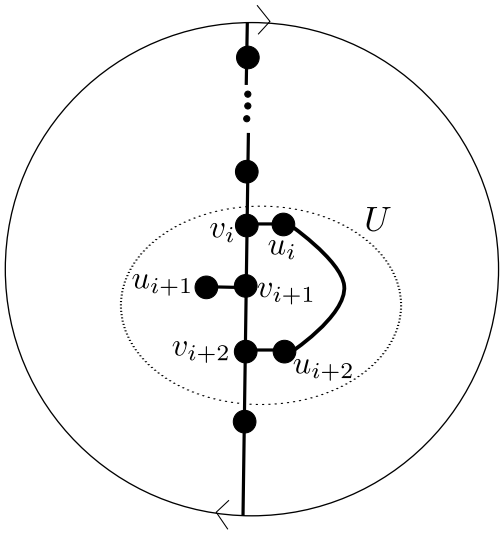


Figure 6.14: A partial embedding of $GP(p,2)$ in P^2.

If the circle $GP(p,2){:}C_i$ is orientation reversing, then Lemma 4.17 implies that $(P^2)^\alpha$ is nonorientable. So, by Lemma 4.17, all of the circles $GP(p,2){:}C_i$ must be contractible circles. It follows from this discussion that for $\langle GP(p,2),\alpha\to\mathbb{Z}_2\rangle$ to be embedded in the projective plane in such a way that the derived surface is orientable, the embedding must be the one depicted in Figure 6.15.

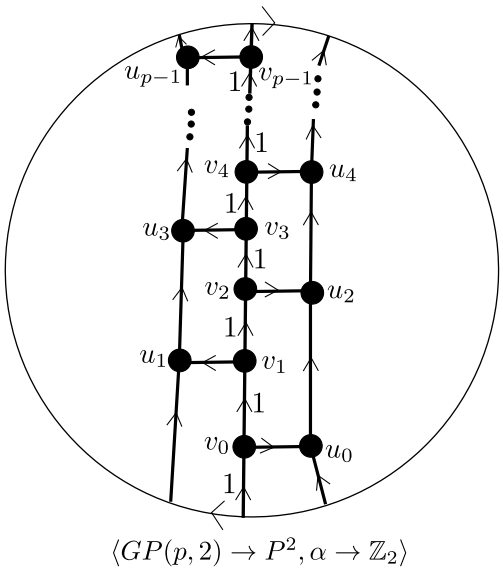


Figure 6.15: An ordinary voltage graph embedding in the projective plane. The zero voltages are not shown.

Since each face of the embedding in Figure 6.15 is bounded by an even number of edges contained in C_v, it follows that there are no branch points. It follows from Theorem 6.4 that $(P^2)^\alpha$ satisfies

$$\chi((P^2)^\alpha) = 2 \cdot 1 = 2.$$

And so, $(P^2)^\alpha$ is not the torus.

Subcase 2: C_v is homologically trivial

In this case, by Lemma 6.6, $GP(p,2)$:C_v is a contractible circle. Observe that $GP(p,2) \setminus GP(p,2)$:$C_v$ is connected. Since $GP(p,2)$ was already determined to be nonplanar, it follows that $GP(p,2) \setminus E(C_v)$ must not be contained in the disc bounded by $GP(p,2)$:C_v.

In Figure 6.16, a subgraph of $GP(p,2)$ is drawn in a manner that reflects the above constraints. If C_0 is homologically trivial then $GP(p,2)$:C_0 bounds a disc containing the vertex u_1 as in Figure 6.16. It follows that if C_0 is homologically trivial under these constraints, then $GP(p,2)$ cannot be embedded in the projective plane.

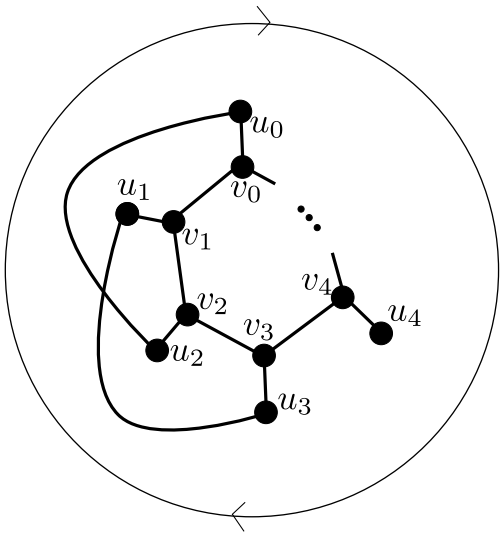


Figure 6.16: A partial drawing of $GP(p, 2)$ in the projective plane depicting a forced immersion of $GP(p, 2)$.

It follows that C_0 must be homologically nontrivial in this case. Since this implies that $GP(p, 2){:}C_0$ is orientation-reversing, it follows by Lemma 4.17 that $(P^2)^\alpha$ is nonorientable, which precludes $(P^2)^\alpha$ from being the torus.

Case 3c: $GP(p, 2)$ in the Klein bottle

Subcase 1: C_v is homologically trivial

If $GP(p, 2){:}C_v$ is a separating curve, then the fact that $GP(p, 2) \setminus E(C_v)$ is connected means that $GP(p, 2) \setminus GP(p, 2) : C_v$ is contained within one of the two regions of $KB \setminus (GP(p, 2){:}C_v)$. By Lemma 6.7, $GP(p, 2){:}C_v$ either induces a contractible circle forming the boundary of the punctured Klein bottle and the punctured sphere in the connected sum $KB = KB\#S^2$, or it induces a circle forming the boundary of the two punctured projective planes forming the connected sum $KB = P^2\#P^2$.

If $GP(p, 2){:}C_v$ is contractible, the fact that $GP(p, 2)$ is nonplanar means that $GP(p, 2) \setminus E(C_v)$ must be contained within the closure of the punctured Klein bottle, in which case $GP(p, 2) : C_v$ bounds a face f_v. Since p is odd, and each of the darts on the edges of $GP(p, 2) : C_v$ is assigned voltage 1, it follows that a facial boundary walk W_f of f has nonzero voltage. Since $A = \mathbb{Z}_2$ in this case, we have that f contains a branch point y of

deficiency 1. Therefore, by Theorem 6.4,

$$\chi(KB^{\alpha}) \leq 2 \cdot 0 - 1 = -1,$$

which precludes KB^{α} from being the torus.

If $GP(p,2)$:C_v is the boundary of the the two punctured projective planes forming the connected sum $KB = P^2 \# P^2$, then the fact that $GP(p,2) \setminus E(C_v)$ is connected means that $GP(p,2) \setminus (GP(p,2) : C_v)$ must be contained within one of the two punctured projective planes. It follows that the other punctured projective plane is a face of the embedding, which means that in this instance, the embedding is not cellular, a contradiction.

Subcase 2: C_v is homologically nontrivial

Either $GP(p,2)$:C_v is an orientation-preserving circle, or $GP(p,2)$:C_v is an orientation-reversing circle.

If $GP(p,2) : C_v$ is an orientation-preserving, circle, then by a sequence of local sign switches on the vertices of $GP(p,2) : C_v$, we produce an equivalent base embedding of $GP(p,2)$ in the Klein bottle with no orientation-reversing edges in $GP(p,2) : C_v$. Since $GP(p,2)$ is assumed to be cellularly embedded in a nonorientable surface, there must be an orientation-reversing circle of $GP(p,2)$. Since $Z(GP(p,2))$ is generated by C_v and the $\{C_i\}$, it follows that there must be at least one orientation-reversing circle induced by one of the C_i, say C_0. By Lemma 4.17, it follows that KB^{α} is nonorientable, so it is not the torus.

Now, suppose that $GP(p,2)$:C_v is an orientation-reversing circle. Since $\dim_{\mathbb{Z}_2}(H_1(KB)) = 2$ and every element of $Z(G)$ is generated by C_v and the C_i, it follows that at least one of the C_i, say C_2 is homologically nontrivial. If $GP(p,2)$:C_2 is an orientation-reversing circle, then it follows by Lemma 4.17 that KB^{α} is nonorientable, so KB^{α} is not the torus. Therefore, $GP(p,2)$:C_2 must be orientation preserving. By Lemma 2.27, $GP(p,2)$:C_2 may be drawn as it is shown in Figure 6.17. Note that without loss of generality, (v_3, u_3) may be drawn as it is shown in Figure 6.17, above $GP(p,2)$:C_v. Also note the two possible placements for the edge (v_1, u_1) are also shown.

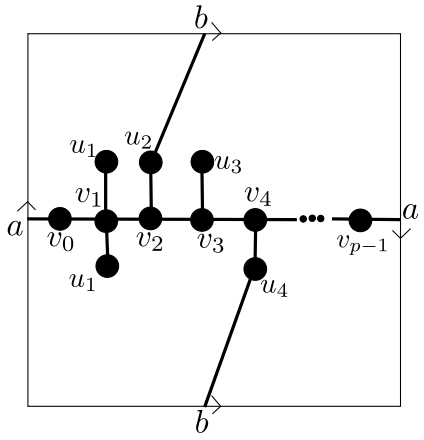


Figure 6.17: A partial embedding of $GP(p,2)$ in the Klein bottle, with two possible choices for how to draw (v_1, u_1).

Note that no matter how we draw the edge (v_1, u_1), the fundamental cycles C_1 and C_2 must satisfy $\langle [C_1]_2, [C_2]_2 \rangle_2 = 1$. Since $[C_v]$ and $[C_2]$ span the $\mathbb{Z}_2$-vector space $H_1(KB)$, C_1 is homologous to C_v, C_2, or $C_v + C_2$. Since $\langle [C_v]_2, [C_2]_2 \rangle_2 = 1$ it follows that C_1 and C_2 are not homologous since $GP(p,2){:}C_2$ is an orientation-preserving circle, which implies that $\langle [C_2]_2, [C_2]_2 \rangle_2 = 0$. If C_1 is homologous to C_v then C_1 is an orientation-preserving circle, and by Lemma 4.17, KB^α is nonorientable and KB^α is not the torus. If C_1 is homologous to $C_v + C_2$, then it follows that

$$\langle [C_1]_2, [C_1]_2 \rangle_2 = \langle [C_v + C_2]_2, [C_v + C_2]_2 \rangle_2 = \langle [C_v]_2 + [C_2]_2, [C_v]_2 + [C_2]_2 \rangle_2 = 1.$$

Therefore, $GP(p,2) : C_1$ is an orientation-reversing circle, and by Lemma 4.17, KB^α is nonorientable, and so KB^α is not the torus.

Case 3d: $GP(p,2)$ in the torus

Subcase 1: C_v is homologically trivial

First, note that if C_v is homologically trivial, then, by Lemma 6.8, $GP(p,2){:}C_v$ is the boundary of the punctured torus and punctured sphere forming $T = T \# S^2$. Therefore, $GP(p,2){:}C_v$ is contractible. Since $GP(p,2) \setminus E(C_v)$ is connected, it follows that $GP(p,2) \setminus GP(p,2) : C_v$ is contained in either the punctured torus or the punctured sphere. Since $GP(p,2)$ is nonplanar, it follows that $GP(p,2){:}C_v$ bounds a face f, and so by Theorem 6.4,

we have that

$$\chi(T^\alpha) \leq 2 \cdot 0 - 1 = -1,$$

which precludes the possibility of T^α being the torus.

Subcase 1: C_v is homologically nontrivial

Since the torus is orientable and $G : C_v$ is an orientation-preserving circle, a ribbon neighborhood $R(C_v)$ is homeomorphic to an annulus, and at least one of our chosen basis elements of $Z(GP(p,2))$, say C_2, satisfies $\langle [C_2]_2, [C_v]_2 \rangle_2 = 1$. Now, consider Figure 6.18 and note that without loss of generality, (v_3, u_3) can be drawn as it is, above $GP(p,2){:}C_v$. Also note the two possible choices for drawing the edge (v_1, u_1) are also drawn.

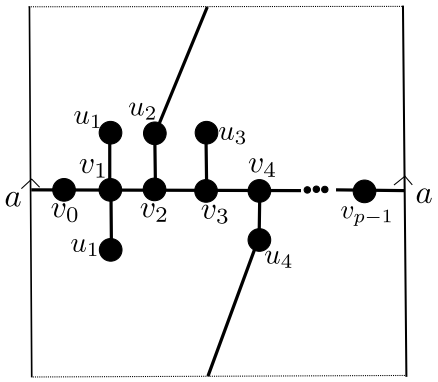


Figure 6.18: A ribbon neighborhood of $GP(p,2){:}C_v$ partially embedded in the torus, with two possible choices for how to draw (v_1, u_1).

Note that no matter how we draw the edge (v_1, u_1), the fundamental cycles C_1 and C_2 must satisfy $\langle [C_1]_2, [C_2]_2 \rangle_2 = 1$. So, C_1 and C_2 are homologically nontrivial. Since the torus is orientable, $GP(p,2){:}C_1$ is also orientation preserving. Therefore,

$$\langle [C_1]_2, [C_1]_2 \rangle_0 = \langle [C_2]_2, [C_2]_2 \rangle_2 = 0.$$

Since there are two circles in $GP(p,2)^\alpha$ forming each the fibers over $GP(p,2) : C_1$ and $GP(p,2){:}C_2$ the fact that $p\colon T^\alpha \to T$ is a covering map implies the following:

$$\langle [C_1^0]_2, [C_2^1]_2 \rangle_2 = 1, \ \langle [C_1^0]_2, [C_2^0]_2 \rangle_2 = 0,$$

and

$$\langle [C_1^1]_2, [C_2^1]_2 \rangle_2 = 1, \ \langle [C_1^1]_2, [C_2^0]_2 \rangle_2 = 0.$$

We conclude that if we let $X_2 = \{C_1^0, C_2^1, C_1^1, C_2^0\}$, then

$$M_{X_2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Since the rows of M_{X_2} are linearly independent, Theorem 2.20 implies that $dim_{\mathbb{Z}_2}(H_1(T^\alpha)) \geq 4$, which precludes the possibility that KB^α is the torus since $dim_{\mathbb{Z}_2}(H_1(T)) = 2$.

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