Spectrum and Factors of Substitution Dynamical Systems

by Joseph L. Herning

B.A. in Mathematics and Philosophy, May 2006, Catholic University of America
M.A. in Mathematics, January 2010, George Washington University

A Dissertation submitted to

The Faculty of
The Columbian College of Arts and Sciences
of The George Washington University
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

August 31, 2013

Dissertation directed by

E. Arthur Robinson, Jr.
Professor of Mathematics

The Columbian College of Arts and Sciences of The George Washington University certifies that Joseph L. Herning has passed the Final Examination for the degree of Doctor of Philosophy as of June 10, 2013. This is the final and approved form of the dissertation.

Spectrum and Factors of Substitution
Dynamical Systems

Joseph L. Herning

Dissertation Research Committee:

E. Arthur Robinson, Jr., Professor of Mathematics, Dissertation Director

John B. Conway, Professor of Mathematics, Committee Member

Yongwu Rong, Professor of Mathematics, Committee Member

© Copyright 2013 Joseph L. Herning
All Rights Reserved

Acknowledgements

This dissertation was supported in part by a 2012 Summer Dissertation Fellowship from the George Washington University. My dissertation director, Professor Robinson, has provided much guidance and insight without which this work would not have been possible. I would also like to thank my other committee members for their time and assistance. I am very grateful to have studied under many wonderful professors at both the George Washington University and the Catholic University of America.

I am indebted to my classmates, particularly Erblin Mehmetaj and Tyler White, for many insightful discussions about mathematics. I also thank my friends and family for their unfailing support. My parents, Ronald and Dorothy Herning, deserve special recognition. Many others not mentioned by name have helped me reach this point, but properly thanking them all here would excessively delay the completion of this document.

Finally, I wish to acknowledge the role of Pope John Paul II's encyclical letter *Fides et Ratio*; I may never have reached this point in my academic career without reading these words as a young student: "Faith and reason are like two wings on which the human spirit rises to the contemplation of truth; and God has placed in the human heart a desire to know the truth – in a word, to know himself – so that, by knowing and loving God, men and women may also come to the fullness of truth about themselves."

Joseph L. Herning

Abstract

Spectrum and Factors of Substitution
Dynamical Systems

This dissertation studies topological factors substitutions, especially substitutions with constant length p. If p is prime we show that any factor is topologically conjugate to a substitution of length p^s. This extends a previous result on the relationship between substitutions of constant length and automatic sequences, and it also complements a result of Mentzen about metric factors of constant length substitutions.

We also prove a topological version of a result by Host and Parreau; we show that factors of constant length substitutions onto substitutions can usually be reduced to 2-block codes.

Our main result is a counterexample to a conjecture of Baake that any substitution has a subshift factor that is metrically isomorphic to its maximal equicontinuous factor. If true, the conjecture would imply that there is always a continuous complex function depending on finitely-many coordinates such that the corresponding diffraction measure contains the entire discrete spectrum. Unfortunately this turns-out to be false, as the example shows by using the structure of bijective substitutions and the previous results to limit the factors which must be considered.

Finally, we show that a partial result to the contrary holds for non-constant length substitutions having irrational eigenvalues, and we examine visual approximations of the diffraction spectra of our examples and other well-known substitutions.

Contents

List of Figures

List of Symbols

(X, T) The dynamical system with homeomorphism $T : X \to X$.

(X, T, μ) The dynamical system (X, T) with T-invariant measure μ.

$[[x]]$ The subshift of $(\mathcal{A}^{\mathbb{Z}}, T)$ defined by the language of the infinite word x.

$\#\mathcal{A}$ The cardinality of the set $\mathcal{A}$.

$\left|\left|f\right|\right|_{\infty}$ The supremum norm of the function f.

$\left|\gamma\right|$ The length of the substitution γ.

$\left|P\right|$ The order of the permutation P.

$\left|w\right|_{\#}$ The length of the word w.

γ_f The substitution induced by the substitution γ and consistent 1-block code f.

γ_h The (second) higher-order substitution induced by γ.

$\hat{\sigma}(k)$ The k-th Fourier coefficient of the measure σ.

$\overline{B}$ The closure of the set B in some topological space.

$\mathcal{A}^*$ The free semigroup on the alphabet $\mathcal{A}$.

$\mathcal{A}^{\mathbb{Z}}$ The set of bi-infinite sequences on the alphabet $\mathcal{A}$.

$\mathcal{A}^m$ The set of length-m words on the alphabet $\mathcal{A}$. $\mathcal{A}^m \subset \mathcal{A}^*$.

$\sigma_{\max}$ A measure of maximal spectral type.

$\mathcal{L}(A)$ The language of the subshift A or the point A.

$\mathcal{L}_n(A)$ The language of length-n of the subshift A or the point A.

$\mathbb{T}$ The unit circle in the complex plane.

$f_{\max}$ A function which generates the maximal spectral type.

h The height of a constant-length substitution or a coboundary function of an arbitrary substitution.

T The shift homeomorphism.

u^f The sequence induced by the sequence u and function f.

U_T The unitary operator induced by $U_T f = f \circ T$.

$w_{[i,j]}$ The word appearing from position i to position j in the larger word w

$X \equiv Y$ The subshifts X and Y are topologically conjugate.

X^m The canonical m-delay of the subshift X.

X_γ The subshift induced by the substitution γ.

$Z(p)$ The cyclic group of order p.

Z_k The k-adic integers.

Chapter 1

Introduction

The 2011 Nobel Prize in Chemistry was awarded to Dan Schechtman for the discovery of an arrangement of matter previously thought to be impossible. Quasicrystals, as they are now called, are aperiodic structures which act much like their traditional crystal cousins. Schechtman observed an X-ray diffraction pattern having the sharpness of a crystalline structure but indicating a ten-fold rotational symmetry – something which is impossible for a crystal. Unbeknownst to him at the time, the new material shared many properties with a newly discovered class of mathematical structures: aperiodic tilings. In particular, they are aperiodic but mimic periodicity enough to produce sharp diffraction patterns.

Such aperiodic structures can be generated in many ways. This work focuses on one-dimensional substitutions. A substitution is simply a function, φ, which replaces (or "substitutes") letters on a finite alphabet, $\mathcal{A}$, with words from that alphabet. The substitution is applied repeatedly in order to generate long strings and, finally, a set of infinite sequences on the alphabet. When certain non-degeneracy conditions are met, the resulting sequences are used to generate a minimal, aperiodic topological dynamical system with translation, or shifting, as the action on the space. In particular, if n is the size of the alphabet, the systems are subshifts of the full shift on n letters. Unlike some other subshifts such as subshifts of finite type, substitution subshifts are particularly bad at holding information: from an information-theory perspective they have entropy zero.

The dynamical system arising from a non-degenerate substitution φ is the couple (X_φ, T), where X_φ is the substitution subshift considered as a topological space and T is the left-shift action. The system (X_φ, T) is uniquely ergodic which means it admits a unique translation-invariant Borel probability measure. This measure is used to construct a unitary operator on the space $L^2(X_\varphi)$ by $U_T \circ f = f \circ T$. It turns out that the spectral study of this unitary operator can say much about the dynamical system. Unitary operators may be characterized by their maximal spectral type, a corresponding measure on the unit circle $\mathbb{T}$, and spectral multiplicity function.

The spectral type of the unitary operator U_T can be used to infer many properties of the dynamical system (X_φ, T). In particular, if U_T has pure-discrete spectrum, i.e., the maximal spectral type is a discrete measure, then (X_φ, T) is isomorphic, in the correct category, to a rotation on the group of characters defined by its eigenvalues – the point masses in the spectral measure. It is conjectured that all substitution dynamical systems with abelianizations characterized by irreducible Pisot matrices are of this type. This is the so-called Pisot conjecture (c.f. [12]) which has been proved in the case of two letters ([4], [14]). On the other hand, a maximal spectral type which is continuous except at 1, where a point mass corresponds to the constant eigenfunctions, indicates that the system is weak-mixing.

Given a sequence from a finite alphabet and an integer length n, a block-code of the sequence is defined by a map from the set of possible finite sequences of length n into some new alphabet. A new sequence is generated by "sliding" the length-n windows along the original sequence and concatenating the results. The theorem of Curtis, Hedlund, and Lyndon demonstrates that between two subshifts the only shift-commuting continuous maps are blocks-codes. The images of such maps are called topological factors; they are, in general, simpler than the original dynamical systems which may be considered extensions of the factors.

A substitution φ has constant length-q if for all $a \in \mathcal{A}$, $\varphi(a)$ is a word of length q. A famous theorem of Cobham identifies the factors of constant length-q substitutions with the set of q-automatic sequences. In automata theory, q-automatic sequences are a well-studied class of infinite sequences generated by directed graphs having edges labeled $0, 1, \cdots, q-1$ originating from each state in set of states S in the following manner: an "exit map" f

labels the set S with possibly non-unique labels, and the combination of graph and exit map is called a q-automaton. A sequence u_n is q-automatic if there is such an automaton so that whenever any $n \in \mathbb{N}$ is written in base-q expansion, i.e., $n = \sum_{i=0}^{j} n_i q^i$, then, always starting from the same initial state, the edge sequence $n_0,\ n_1,\ \ldots, n_j$ leads to a state $s \in S$ satisfying $f(s) = u_n$.

This dissertation focuses on constant length substitutions and their factors. A well known theorem of Dekking in [8] shows that a pure constant-length substitution has pure-discrete spectrum if and only if a combinatorial condition called "coincidence" is satisfied. A classic example is the Morse substitution: $\kappa(0) = 01$, $\kappa(1) = 10$. The Morse substitution does not have coincidence and therefore does not have pure-discrete spectrum. Dekking also classified the eigenvalues of constant length substitutions. Constant length substitutions always have non-trivial point spectrum, and thus the systems are not weak-mixing. The Toeplitz substitution, $\tau(0) = 01$, $\tau(1) = 00$ is also length-2 but does indeed have coincidence and therefore has pure-discrete spectrum. Now, there is a well-known length-2 block map factoring the Morse system onto the Toeplitz system. Additionally, the spectrum of the Toeplitz substitution is exactly the discrete part of the Morse's spectrum.

The example of the Morse and Toeplitz substitutions poses many questions. First, given a constant-length substitution, it is known that not every subshift factor may be written directly as a substitution; thus the set of sequences defined by substitutions is a strict subset of the set of automatic sequences. Is it the case, however, that every subshift factor is topologically conjugate to a substitution? In this case the block code inducing the factor could be composed with the conjugacy, yielding a block code directly inducing a substitution subshift as a factor. Even if a factor is not a substitution, does every constant-length substitution have a subshift factor with pure-discrete spectrum equal to that of the original substitution subshift?

If non-subshift factors are permitted, every dynamical system has a maximal equicontinuous factor – i.e., a factor where the group of homeomorphisms acting on the space acts equicontinuously. This factor is well-known for all constant-length substitutions, and is not a subshift because subshifts are never equicontinuous. A substitution with pure-discrete spectrum such as the Toeplitz is an almost-everywhere finite-to-one extension of such a system from a measure-theoretic perspective, and the factor itself is an adding machine having the entire point-spectrum of the original system. The Morse substitution, which has the same length, also has the same maximal equicontinuous factor. In this particular case, the Toeplitz system, a topological and substitution factor, is an intermediate factor between the two (see Example 4.11). We do not investigate the measure-theoretic point of view in detail, but it provides reason to believe that a pure-discrete spectrum subshift factor exists for all constant-length substitutions. The problem may be stated as a question of whether all substitutions "factor through" a discrete-spectrum subshift factor as in the diagrams below:

$$\begin{array}{ccc}
X_{\text{Morse}} & \qquad\qquad & X_{\text{arbitrary length-}\ell} \\
\Big\downarrow{\scriptstyle \text{known block code}} & & \Big\downarrow{\scriptstyle \text{unkown block code}} \\
X_{\text{Toeplitz}} & & X_{\text{coincident/discrete}}? \\
\Big\downarrow{\scriptstyle \text{known a.e. extension}} & & \Big\downarrow{\scriptstyle \text{known a.e. extension}} \\
Z_2 & & Z_\ell \times Z(h)
\end{array}$$

This dissertation proves that some classes of substitutions admit substitution representations of all of their topological subshift factors. This complements a result of Mentzen in [20] about measure-theoretic factors of constant-length substitutions. Furthermore, a constructive process is used for building the factors from the original block code, viewing the output of the code as a k-automatic sequence. Next, we create a method of reducing the length of block-codes to length-2 or length-3. The length-2 case strengthens a similar result of Host and Parreau in the measure-theoretic category ([16]). We give a simple and direct topological proof, at the expense of requiring a length-3 code for some substitutions. Finally, the process of looking for a coincidence of an arbitrary constant-length substitution can not be bounded in a simple way. We prove that the structure of bijective substitutions allows for an elegant bound, then we use the bound to demonstrate that there are bijective substitutions which have no pure-discrete spectrum factors.

For a unitary operator U_T on $L^2(X_\varphi)$, the maximal spectral type is the spectral measure of an $L^2(X_\gamma)$ function which contains all the spectral information. Fraczek proved that such a function, called a function of maximal spectral type, can be chosen to be continuous.

Other L^2 functions correspond to a part of the spectrum: they define measures absolutely continuous with respect to the maximal spectral type. Unlike a block code, an arbitrary function $f : L^2(X) \to \mathbb{C}$, even if continuous, is not necessarily finite-valued and may require knowledge of the entire infinite sequence $x \in L^2(X_\gamma)$. Block codes, on the other hand, depend only on a finite-number of coordinates – a "local window" – and take-on only a finite number of values. Eigenfunctions for substitution systems are continuous and are actually block-codes from the right point-of-view. Therefore, eigenfunctions detect their respective eigenvalues only depending on a finite window. Now, the set of eigenvalues of a general substitution is countable: is there a function which, like the maximal spectral type, can detect all eigenvalues, but also, like an eigenfunction, only needs a finite window in order to do so?

In one case we can detect all eigenvalues with a block code: the block code which factors the Morse substitution onto the Toeplitz substitution detects the entire point-spectrum of the Morse substitution (c.f. [3]). Can the Morse case be generalized so that block codes can always be made to detect the point spectrum of substitution subshifts? This is equivalent to asking whether a substitution with eigenvalues always has a topological factor that is isomorphic in the ergodic-theory sense to the maximal discrete spectrum factor. A positive answer to this question was conjectured by Michael Baake in 2010 to E. A. Robinson at a conference on Aperiodic Order at KAIST in Seoul, South Korea. Additionally, Baake and Grimm have shown in [3] that the answer is affirmative in the case of certain two-dimensional analogues of the Morse substitution.

The main result of this dissertation is that there are no topological factors isomorphic to the maximal discrete spectrum factor for some substitutions. In particular we find many examples among the class of bijective substitutions with at least three letters. This provides a negative answer to the conjecture of Baake. Later, we show that if a substitution has irrational eigenvalues, and thus is necessarily non-constant length, then a block coding can retain the entire point spectrum and detect any finite number of eigenvalues.

Chapter 2

Preliminaries

2.1 Topological Dynamical Systems

2.1 Definition (Topological Dynamical System)**.** Let X be a compact metric space, and let $T : X \to X$ be a homeomorphism. The pair (X, T) will be called a *topological dynamical system* or simply, a *dynamical system*.

When there is no ambiguity we may refer to X itself as a dynamical system. We often think of T as producing a $\mathbb{Z}$-action on X via $nx = T^n(x)$ for any $n \in \mathbb{Z}$ and $x \in X$ where $T^0(x) := x$. We may also write $T^k x$ in place of $T^k(x)$ when convenient.

2.2 Definition (Orbit of a Point)**.** Let (X, T) be a topological dynamical system and let $x \in X$. The *orbit* of x (under T) is the set $\text{Orb}_T(x) = \{T^n x : n \in \mathbb{Z}\}$.

When there is no ambiguity we use $\text{Orb}_T(x) = \text{Orb}(x)$. If $B \subset X$, we denote the topological closure of B in X by $\overline{B}$.

2.3 Definition (Invariant Set)**.** Let (X, T) be a topological dynamical system. The set B is called *T-invariant* if $T(B) \subset B$.

2.4 Definition (Minimal Dynamical System)**.** A topological dynamical system (X, T) is called *minimal* if for every $x \in X$, $\overline{\text{Orb}(x)} = X$.

Equivalently, (X, T) is minimal if X admits no closed, T-invariant subsets.

2.5 Definition (Factor Map)**.** Let (X, T) and (Y, S) be dynamical systems. Let $f : X \to Y$ be a surjective continuous map such that $f \circ T = S \circ f$, i.e., the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ \downarrow f & & \downarrow f \\ Y & \xrightarrow{S} & Y \end{array}$$

We call f a *factor map* from (X, T) onto (Y, S). We say that Y is a *(topological) factor* of X and that X is an *extension* of Y. Notice that if f is bijective then it is automatically a homeomorphism, and in this case we say that the systems (X, T) and (Y, S) are *topologically conjugate* or *isomorphic*. If X and Y are conjugate we write $X \equiv Y$ or $(X, T) \equiv (Y, S)$. Topologically conjugate dynamical systems are regarded as indistinguishable, and most dynamical properties, such as minimality, are preserved.

2.2 Subshifts and Sequences

Let $\mathcal{A}$ be a finite set of cardinality d. We call the elements of $\mathcal{A}$ *letters* and $\mathcal{A}$, an *alphabet*. Usually we also identify the set of letters with the first d non-negative integers, i.e., $\mathcal{A} = \{0, 1, 2, \dots d - 1\}$.

We identify the sets $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$ with the sets of all possible one- and two-sided sequences. We often denote elements from these sets by using a radix-point to denote the zero-th position in the following manner:

$$.a_0 a_1 a_2 a_3 \cdots \in \mathcal{A}^{\mathbb{N}},$$

$$\cdots a_{-2} a_{-1} . a_0 a_1 a_2 a_3 \cdots \in \mathcal{A}^{\mathbb{Z}}$$

where each $a_i \in \mathcal{A}$. To select a finite portion of the sequence $x \in \mathcal{A}^{\mathbb{N}}$ (or $\mathcal{A}^{\mathbb{Z}}$) we use the notation $x_{[i,j]} = x_i x_{i+1} \cdots x_j$. We also extend this notation to finite sequences on an alphabet; e.g. $(abcdefg)_{[1,3]} = bcd$.

2.6 Definition (Free Semigroup)**.** If $\mathcal{A}$ is a finite set then the *free semigroup* $\mathcal{A}^*$ on $\mathcal{A}$ is the set of non-empty finite sequences of letters from $\mathcal{A}$ together with the operation of concatenation.

2.7 Definition (Word)**.** An element of $\mathcal{A}^*$, the free semigroup on $\mathcal{A}$, is a *word*. If $x \in \mathcal{A}^{\mathbb{N}}$ (or $\mathcal{A}^{\mathbb{Z}}$) we often speak of $x_{[i,j]}$ as a word appearing in x. To denote this we say $x_{[i,j]} \in x$, and we mean that the finite word appears somewhere in x. The sequence x is also called an *infinite word*.

2.8 Definition (Length of a Word)**.** Let $w = w_0 w_1 \cdots w_{n-1}$ be a finite word in $\mathcal{A}^*$. The *length* of w is n and is denoted $|w|_{\#} = n$. We typically write only $|w| = n$ unless there may be confusion with the standard notation for the modulus of a complex number.

We endow $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$ with the product of the discrete topology on $\mathcal{A}$. This topology may also be generated by the metric defined for $x \neq y$ by $d(x,y) = \frac{1}{2^n}$ where $n = \min\{|k| : x_k \neq y_k\}$. A convergent sequence $\{x_k\} \subset A^{\mathbb{Z}}$ has the property that words near the origin become fixed. A *Cantor space* is a non-empty, compact Hausdorff space with a countable basis of clopen (simultaneously closed and open) sets. With this topology, $\mathcal{A}^{\mathbb{N}}$ and $\mathcal{A}^{\mathbb{Z}}$ are Cantor spaces, and any two Cantor spaces are homeomorphic (c.f. [23]).

2.9 Definition (Full Shift)**.** Let $\mathcal{A}$ be an alphabet. The *full-shift* on $\mathcal{A}$ is the dynamical system $(\mathcal{A}^{\mathbb{Z}}, T)$ where T is the left-shift homeomorphism. That is, for any $a \in \mathcal{A}^{\mathbb{Z}}$:

$$T(\cdots a_{-2}a_{-1}.a_0a_1a_2\cdots) = \cdots a_{-1}a_0.a_1a_2a_3\cdots .$$

2.10 Definition (Subshift)**.** A *subshift* of the full-shift $(\mathcal{A}^{\mathbb{Z}}, T)$ is a dynamical system (Y, S) where $Y \subset \mathcal{A}^{\mathbb{Z}}$ is topologically closed and T-invariant, and $S = T|_Y$.

Instead of S, we usually re-use the symbol T as the homeomorphism of the subshift, unless we wish to identify a particular subshift by its homeomorphism.

2.11 Definition (Language of a Word)**.** The collection of words appearing in an infinite word x is called the *language* of that word. In symbols, $\mathcal{L}(x) = \{w : w \in x\}$.

2.12 Definition (Language of Length-n)**.** The *language of length-n* is the collection of length-n words appearing in x, i.e., $\mathcal{L}_n(x) = \{w : w \in x,\ |w| = n\}$.

2.13 Definition (Language of a Subshift). If $X \subset \mathcal{A}^{\mathbb{Z}}$ is a subshift then the *language of* X is $\mathcal{L}(X) = \{w : w \in x,\ x \in X\}$. The *language of length-n* of X is $\mathcal{L}_n(X) = \{w \in \mathcal{L}(X),\ |w| = n\}$.

As a reminder, the symbol "$\in$" in the definition of $\mathcal{L}_n(X)$ is set-inclusion, but in the definitions 2.11 and 2.12 it refers to a finite word occurring as a subword of infinite word as in definition 2.7.

2.14 Definition (block code). Let $X \subset \mathcal{A}^{\mathbb{Z}}$ be a subshift and $p, q \in \mathbb{N}$. Let $n = p + q + 1$, and $f : \mathcal{L}_n(X) \to \mathcal{B}$. The *block code* of length n with *anticipation* q and *memory* p from X to $\mathcal{B}^{\mathbb{Z}}$ corresponding to f is the map $b : X \to \mathcal{B}^{\mathbb{Z}}$ such that

$$b(x)_i = f(x_{[i-p,i+q]}) \tag{2.15}$$

for $x \in X$.

If w is a finite word with $|w| = n$, it will be convenient to associate b with the *map-on-blocks* f. That is, $b(w) = f(w)$ when $|w| = n$.

The image $b(X)$ is shift-invariant and closed, and b is continuous (c.f. [6], [26]). Therefore, since it also onto and commutes with the shift-operator by construction, $b : X \to b(X)$ is a factor map.

There are several special types block-codes we will consider: If the anticipation $p = 0$ we say that the block code *starts at the origin* or *has no memory*. If $p = q = 0$ we call b a 1-block code. A *1-block code* is equivalent to a re-labeling of the alphabet which may also identify some letters. If $p = 0$ and $q = 1$ we will call the map a *2-block code*. If $p = 1$ and $q = 1$ we will call b a *3-block code*. Importantly, for our purposes 1- 2- and 3-block codes are defined with specific memories and anticipations. In this manner a 3-block code is a particular type of length-3 block code where $p = q = 1$ (not $p = 2$, $q = 0$ or $p = 0$, $q = 2$).

If a block code starts at the origin, then we may also use a block code b as a map $\mathcal{A}^{\mathbb{N}} \to \mathcal{B}^{\mathbb{N}}$ via Equation 2.15. If a block code does not start at the origin, then we occasionally still consider 2.15, but we note that using the equation leaves $b(x)_i$ undefined for $i < p$. If b is such a block code and we say $b(x) = y$, then we mean that $b(x)_i = y_i$ for all $i \geq p$.

2.16 Theorem (Curtis-Lyndon-Hedlund (c.f. [6]))**.** *Every factor map from the subshift $X \subset \mathcal{A}^{\mathbb{Z}}$ onto the subshift $Y \subset \mathcal{B}^{\mathbb{Z}}$ is a block code.*

2.17 Definition (Subshift Generated by a Sequence)**.** Let $x \in \mathcal{A}^{\mathbb{N}}$ or $x \in \mathcal{A}^{\mathbb{Z}}$. The *subshift of* $(\mathcal{A}^{\mathbb{Z}}, T)$ *generated by* x is $[[x]] = \{u \in \mathcal{A}^{\mathbb{Z}} : w \in u \Longrightarrow w \in x\}$. That is, the sequence u is in $[[x]]$ if $\mathcal{L}(u) \subset \mathcal{L}(x)$.

The following lemma justifies the use of the term "subshift":

2.18 Lemma. *If x is an infinite word on a finite alphabet and T is the left-shift,* $([[x]], T)$ *is a subshift.*

Points in the subshift need-not have the entire language of the original point unless the following condition holds:

2.19 Definition (almost-periodic)**.** The infinite sequence x is *almost-periodic* or *uniformly recurrent* if every word in $\mathcal{L}(x)$ appears in x with bounded gaps. That is, if the finite word w appears in x then there is a constant $C \in \mathbb{N}$ (depending on w) such that for all $i \in \mathbb{Z}$, $x_{[i,i+C]}$ contains w.

2.20 Proposition (Minimality and almost-periodicity (c.f. [6]))**.** *If x is an almost-periodic sequence then* $[[x]]$ *is a minimal subshift. Furthermore, if* (X, S) *is a minimal subshift then every $x \in X$ is almost-periodic.*

Thus a one- or two-sided almost-periodic sequence uniquely determines a minimal subshift, and every point in the resulting subshift is a (two-sided) almost-periodic sequence. Thus if $y \in [[x]]$ then $[[x]] = \overline{\mathrm{Orb}(y)}$ by 2.4.

2.3 Substitutions

Minimal subshifts may be created in many ways, but our focus will be on a special class of subshifts defined by a simple rule. As above, let $\mathcal{A}$ denote a finite alphabet, and let $\mathcal{A}^*$ be the free semigroup on the alphabet $\mathcal{A}$.

2.21 Definition (Substitution)**.** A *substitution* on $\mathcal{A}$ is a function $\varphi : \mathcal{A} \to \mathcal{A}^*$.

A substitution φ is called *constant length-n* or simply *length-n* if for all $\alpha \in \mathcal{A}$, $|\varphi(\alpha)| = n$. We also write $|\varphi| = n$ to mean φ is a constant length-n substitution.

2.22 Definition (Abelianization Map)**.** Let φ be a substitution on $\mathcal{A}$. The *abelianization map* $\ell : \mathcal{A}^* \to \mathbb{Z}^d$ is given by the coordinates $\ell(w)_i$, $0 \le i < d$, where $\ell(w)_i$ denotes the number of i's in w.

2.23 Definition (Incidence Matrix)**.** A substitution on $\mathcal{A}$ defines the $d \times d$ matrix

$$M_\varphi = \big(\ell(\varphi(0)), \ell(\varphi(1)), \cdots \ell(\varphi(d-1))\big)$$

where each $\ell(\varphi(i))$ is interpreted as a column vector.

Much can be gleaned about a substitution from its incidence matrix including the frequency of occurrence of letters in typical points of the substitution subshift. A substitution φ is constant length-n if and only if every column of M_φ has sum n.

2.24 Definition (Bijective Substitution)**.** A constant-length substitution φ on $\mathcal{A}$ is called *bijective* if for all $0 \le k < |\varphi| - 1$, $\#\{\varphi(i)_k : i \in \mathcal{A}\} = \#\mathcal{A}$.

It is important to notice the difference between the use of "bijective" and "injective" in reference to substitutions. The latter is sometimes called "injective on letters" or "one-to-one on letters" in the literature:

2.25 Definition (Injective Substitution)**.** A substitution φ on $\mathcal{A}$ is called *injective* if φ is an injective map, i.e., for all $\alpha, \beta \in \mathcal{A}$, $\varphi(\alpha) = \varphi(\beta) \implies \alpha = \beta$.

2.26 Example (Fibonacci Substitution)**.** The Fibonacci Substitution, ϕ, on the alphabet $\mathcal{A} = \{0, 1\}$ is given by: $\phi(0) = 01$ and $\phi(1) = 0$ which we will indicate by

$$\phi \begin{cases} \mathsf{0} \to \mathsf{01} \\ \mathsf{1} \to \mathsf{0} \end{cases}$$

Then we have $\ell(\phi(0)) = [1\ 1]^T$ and $\ell(\phi(1)) = [1\ 0]^T$, so

$$M_\phi = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

We extend a substitution $\varphi : \mathcal{A} \to \mathcal{A}^*$ to a map $\mathcal{A}^* \to \mathcal{A}^*$ or $\mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ by concatenation, i.e., if $a_0 a_1 a_2 \cdots a_n \in \mathcal{A}^*$ then

$$\varphi(a_0 a_1 a_2 \cdots a_n) := \varphi(a_0)\varphi(a_1)\varphi(a_2)\cdots\varphi(a_n) \in \mathcal{A}^*.$$

For example, if ϕ is the Fibonacci Substitution, we have

$$\begin{aligned} \phi(0) &= 01, \\ \phi(\phi(0)) &= 010, \\ \phi(\phi(\phi(0))) &= 01001, \\ \phi(\phi(\phi(\phi(0)))) &= 01001010. \end{aligned}$$

This also shows an interesting phenomenon. If $\varphi(i)$ begins with the letter i then it is easy to show via induction that $\varphi^{n+1}(i)$ begins with $\varphi^n(i)$ for all $n \in \mathbb{N}$ (we define $\varphi^0(w) = w$).

2.27 Definition (Fixed Point of a Substitution)**.** Let φ be a substitution such that $\varphi(i)$ begins with the letter i. Then a *(one-sided) fixed-point* of φ, denoted $\varphi^\infty(i)$, is the unique right-infinite word such that if

$$\varphi^\infty(i) = .u_0 u_1 u_2 u_3 \cdots$$

then for all $n \in \mathbb{N}$, the word $u_0 u_1 \cdots u_n$ is the beginning of $\varphi^k(i)$ for any k such that $|\varphi^k(i)| > n$.

2.28 Lemma. *If φ is a substitution on $\mathcal{A}$, M_φ its incidence matrix, and $w \in \mathcal{A}^*$ then for all $n \in \mathbb{N}$, $\ell(\varphi^n(w)) = (M_\varphi)^n \ell(w)$.*

This means that that the operation of multiplication by M_φ implements the action of the substitution on its abelianization. Also $(M_\varphi)^n = M_{\varphi^n}$.

2.29 Definition (Primitive Substitution)**.** The substitution φ is said to be *primitive* if $(M_\varphi)^n$ is a strictly positive matrix for some $n \in \mathbb{N}$.

Note that given the previous fact, this is equivalent to the requirement that there is a power n such that for all $\alpha \in \mathcal{A}$, $\varphi^n(\alpha)$ contains the entire alphabet. We notice that the Fibonacci substitution is primitive since

$$(M_\phi)^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}.$$

2.30 Definition (Periodic Subshift)**.** A *periodic subshift* (X, T) is a subshift where $\#X < \infty$. If a subshift is not periodic it is called *aperiodic*.

In the language of topological dynamics, all points of a periodic dynamical system, X, have periodic orbits. For a minimal subshift, having a periodic orbit is equivalent to having only one periodic orbit.

2.31 Lemma. *If φ is a substitution, there is an $n \in \mathbb{N}$ and an $a \in \mathcal{A}$ such that $\varphi^n(a)$ begins with a.*

Therefore, φ^n has a one-sided fixed-point generated by a.

2.32 Lemma. *If φ is primitive substitution, then any fixed point of φ is almost-periodic, and all fixed points generate the same subshift.*

The former lemma follows from the fact that given primitivity, any finite word will eventually appear in the expansion of every letter (see [26] for a proof). The fixed point is then viewed as the concatenation of blocks made by expanding letters to some high power, each of which contains the word. The latter is a consequence of the fact that minimality implies the languages of all points in the subshift are identical (c.f. [26], [12], [6]).

It is clear that if a substitution φ is primitive, then any power φ^n is primitive and $(\varphi^n)^\infty(i) = \varphi^\infty(i)$ if the latter is a fixed point. Therefore, we will typically assume that all substitutions φ are primitive and have the property that $\varphi(0)$ begins with 0. Beginning with a primitive substitution we may obtain the desired one by passing to a power with a fixed point generated by the letter i then re-labeling the letters so that the fixed point is generated by 0. We consider non-primitive substitutions as a degenerate case. Now, we can unambiguously define the following:

2.33 Definition (Substitution Subshift)**.** Let φ be a primitive substitution satisfying $\varphi(0)$ begins with 0, then the *substitution subshift generated by* φ is

$$X_\varphi := [[\varphi^\infty(0)]].$$

We sometimes attribute the properties of the substitution subshift to the substitution itself. For example, we may say that a substitution is aperiodic if its subshift is aperiodic.

Recall that block codes provide a way to create factors and conjugacies of subshift dynamical systems. Now that we have defined the substitution subshift we may use block codes to create factors. A block coding of a substitution subshift need-not be a substitution subshift, but Chapter 3 is concerned with proving that such an image is conjugate to a substitution in certain cases.

Some authors define substitution subshifts in an alternative way: If $\varphi(0)$ begins with 0 then there is some integer m and letter j such that $\varphi^m(j)$ ends with j. Without loss of generality, assume $j = 1$. Then the word $\varphi^n(1).\varphi^n(0) \in \mathcal{L}(\varphi^\infty(0))$ for all n, and $\varphi^{n+1}(1).\varphi^{n+1}(0)$ contains $\varphi^n(1).\varphi^n(0)$ if the radix-point is aligned for both words. This generates a two-sided infinite point called a two-sided fixed-point z. If φ is primitive, then $z \in X_\varphi$ as defined above, and $X_\varphi = [[z]] = \overline{\mathrm{Orb}(z)}$.

2.34 Theorem (Frequency of Letters ([26]))**.** *If φ is a primitive substitution, then for every $a \in \mathcal{A}$,*

$$\mathrm{freq}(a) := \lim_{n\to\infty} \frac{\ell(\varphi^n(0))}{|\varphi^n(0)|} = d$$

where d is the right Perron-Frobenius eigenvector of M_φ normalized such that $\sum d_i = 1$. Furthermore, the convergence is geometric.

The Perron-Frobenius theorem is satisfied since M_φ is a primitive matrix. The eigenvector d corresponding to the Perron-Frobenius eigenvalue must be positive. Queffélec has also proved in [26] that the frequency of any word in $\mathcal{L}(X_\varphi)$ must exist and be positive. This gives-rise to a unique invariant measure for a given primitive substitution subshift.

2.4 Recognizability and Desubstitution

Much of the theory of substitutions is built on the notion of desubstitution of words.

2.35 Definition (Desubstitution)**.** If $x \in \mathcal{L}(X_\varphi)$ for some primitive substitution φ, we say $y \in \mathcal{L}(X\varphi)$ is a *desubstitution* of x if $\varphi(y) = x$.

It is clear that not every finite word in $\mathcal{L}(X_\varphi)$ has a desubstitution and that a desubstitution of a word need-not be unique. The notion of recognizability was introduced to address these problems. We use the notion promoted by Brigitte Mossé in [21] and [22]. While weaker than the definition used by Queffélec in [25] and Host in [15], Mossé's notion of recognizability holds with very weak assumptions on the substitution and is now accepted as the "correct" definition by most authors (c.f. [12], [26]):

First, let φ be a primitive substitution with fixed point $u = \varphi^\infty(0)$. We define the set of integers,

$$E_1 = \{0\} \cup \{|\varphi(u_{[0,k]})| : k \geq 0\}.$$

If $w \in \mathcal{L}(X_\varphi)$ and $w = u_{[m,n]}$ where $m, n+1 \in E_1$, then a desubstitution of w can be found as follows: let $m = |\varphi(u_{[0,i]})|$ and $n+1 = |\varphi(u_{[0,j]})|$. Then since $\varphi(u_{[i,j]}) = u_{[m,n]}$, the word $u_{[i,j]}$ is a desubstitution of w.

2.36 Definition (Recognizability)**.** A substitution φ with fixed point u is (bilaterally) *recognizable* if there exists $L > 0$ such that $n \in E_1$ and $u_{[n-L,n+L]} = u_{[m-L,m+L]}$ implies $m \in E_1$. The least such L is called the *index of recognizability*.

Mossé proved the following seminal theorem, improving the earlier results of Host ([15]) and others, about the recognizability and desubstitution of words from substitution subshifts:

2.37 Theorem (Mossé, [22])**.** *Every primitive aperiodic substitution φ is recognizable. Furthermore, any long enough word $w \in \mathcal{L}(X_\varphi)$ can be uniquely desubstituted except for a uniformly bounded length on both ends.*

Given recognizability alone, the process above can be used to desubstitute a word w, except on its ends, by finding any place it occurs in the fixed point and trimming the beginning and end so that the word starts and ends at positions from E_1. Since there are infinitely many places the word occurs (the fixed-point is uniformly recurrent), this process may not yield a unique desubstitution even for constant-length substitutions if φ is not injective. Mossé's result implies that long enough words have, in fact, a unique desubstitution in the center after enough is trimmed, and that the amount which must be removed is bounded for all words (although short words may not be long-enough to remove the sufficient amount on both ends). We will primarily use the recognizability part of the result which was proved first in [21] and assume injectivity for convenience, but Mossé's later result in [22] may make this unnecessary.

2.38 Definition (Cutting of a Word)**.** Assume φ is a primitive, aperiodic substitution and $|\varphi| = n$. A *cutting* of the word $w \in \mathcal{L}(X_\varphi)$ is a number $0 \leq c < n$ such that there is a word w' satisfying $w = w'_{[c,c+|w|]}$ and w' has a desubstitution.

2.39 Lemma. *Assume φ is a primitive, aperiodic substitution and $|\varphi| = n$. There is an L such that all $w \in \mathcal{L}(X_\varphi)$ with $|w| \geq L$ have a unique cutting.*

Proof. Let L be the index of recognizability for φ by Theorem 2.37. Take any word w with $w \geq L + n$. Let u be the fixed-point of φ. Let $0 \leq c < n$ and w' be any extension of w such that $w = w'_{[c,c+|w|]}$ and w' has a desubstitution. Since w' has a desubstitution it must be that $w' = u_{[ns,nt]}$ for some $s < t \in \mathbb{N}$. Then $u_{[n(s+1),n(s+1)+|w|-n+c]}$ begins at $n(s+1) \in E_1$, but this implies any other occurrence of the word $u_{[n(s+1),n(s+1)+|w|-n+c]} \in w$ also occurs at E_1 by recognizability. Thus any other extension of w with $0 \leq c' < n$ has $c' = c$. □

If a cutting of the word 123401123456701 from a substitution of length 3 is 2 we often visualize this with "cutting bars" every 3 letters starting with position $(3 - 2) = 1$: 1|234|011|234|567|01. The length-3 blocks delineated by the cutting bars desubstitute to individual letters for some extension of the word containing 2 additional letters at the beginning. Since the substitution is length-3, the length-3 words between cutting bars must correspond to substitutions of letters from the alphabet.

2.5 Combinatorics of Substitutions

Coincidence is a straightforward combinatorial property of some constant-length substitutions. It provides an easy way to detect a very deep spectral property of the substitution systems:

2.40 Definition (Coincidence)**.** A constant length-l substitution φ on $\mathcal{A}$ has a *coincidence of level* k if for some $0 \leq i \leq l^k - 1$, $\#\{(\varphi^k \alpha)_i : \alpha \in \mathcal{A}\} = 1$. If φ has a coincidence of some level then we say it *has a coincidence.*

2.41 Definition (Height of a Substitution)**.** Let φ be a constant length-l substitution with fixed point u. Define the positive integer $g_0 = \gcd\{k \geq 1 : u_k = u_0\}$. Then the *height* h of φ is the positive integer $h := \max\{k \geq 1 : (k,l) = 1,\ k|g_0\}$.

A substitution with $h = 1$ is called *pure* (not to be confused with *pure*-discrete spectrum). For not-necessarily-constant length substitutions, Host introduced in [15] the notion of a *coboundary* as an analogue of the height for studying eigenvalues of the system. While many authors now replace the theory of coboundaries with the theory of *return words* introduced in [11] by Ferenczi, Mauduit, and Nogueira, Host's original formulation is the most convenient for us.

2.42 Definition (Coboundary)**.** Let φ be a primitive substitution on the alphabet $\mathcal{A}$. The map $h : \mathcal{A} \to \mathbb{T}$ is called a *coboundary* of φ if for every $w \in \mathcal{L}(X_\varphi)$ such that $w_0 = w_{|w|-1}$, we have $h(w_0)h(w_1)\cdots h(w_{|w|-2}) = 1$.

The constant function $h(\cdot) = 1$ is always a coboundary. If this is the only coboundary for φ we say φ has trivial coboundary.

2.6 Spectral Theory

We must now introduce some measure theoretic concepts from ergodic theory.

2.43 Definition (Invariant Measure)**.** Let (X, T) be a dynamical system. The measure μ on X is *T-invariant* if for any Borel set $B \subset X$, $\mu(T^{-1}(B)) = \mu(B)$.

If we have an invariant measure μ for a topological dynamical system (X, T), we call the triple (X, T, μ) a *measure-theoretic dynamical system* or we simply extend the term "dynamical system."

2.44 Theorem (Krylov-Bogolubov)**.** *There is a T-invariant Borel probability measure for (X, T).*

The set of T-invariant Borel probability measures on X is compact in the weak* topology. It is also convex, so the Krein-Milman Theorem implies that it is the closed convex hull of its extreme points which are called *ergodic measures* (c.f. [6]). If μ is ergodic then (X, T, μ) is called an *ergodic dynamical system*. One special case of ergodicity is the following:

2.45 Definition (Unique Ergodicity)**.** If there is only one T-invariant Borel probability measure for the dynamical system (X, T), the system is said to be *uniquely ergodic*.

2.46 Definition (Generic Point)**.** If (X, T, μ) is a dynamical system, a point $x \in X$ is called *T-generic* if for every continuous function $F : X \to \mathbb{C}$,

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{-n}^{n} f(T^n x) = \int_X f \; d\mu. \tag{2.47}$$

2.48 Theorem (The Birkhoff Ergodic Theorem)**.** *The system (X, T, μ) is uniquely ergodic if and only every point is T-generic. Furthermore, if μ is only ergodic then* (2.47) *holds for μ-a.e. x.*

2.49 Theorem (c.f. [26])**.** *If φ is a primitive substitution, then (X_φ, T) is uniquely ergodic.*

As we noted, this is equivalent to having unique positive frequencies for all words.

A *unitary operator* on a Hilbert Space H is a surjective bounded linear operator $U : H \to H$ such that $\langle Ux, Uy\rangle = \langle x, y\rangle$.

2.50 Definition (Unitary Operator Associated with a Dynamical System)**.** Let (X, T, μ) be a dynamical system with ergodic measure μ. The *unitary operator associated with* (X, T) is the map $U_T : L^2(X, \mu) \to L^2(X, \mu)$ given by $U_T f = f \circ T$ for $f \in L^2(X, \mu)$.

A concise exposition of the spectral theory of unitary operators needed for our work may be found in the appendix of [24]. Notice because T is measure-preserving,

$$\langle U_T f, U_T g\rangle = \int (f \circ T)\overline{(g \circ T)}\ d\mu = \int f\bar{g} = \langle f, g\rangle.$$

2.51 Lemma (c.f. [26])**.** *If (X, T, μ) is an ergodic dynamical system, the eigenvalues of U_T form a countable subgroup of the unit circle, $\mathbb{T}$. In particular, 1 is always an eigenvalue and the remaining have modulus 1.*

2.52 Definition. Given a unitary operator on a separable Hilbert space H and $f \in H$ we define the bi-infinite sequence $\hat{\sigma}_f(k) := \langle U^k f, f\rangle$ for $k \in \mathbb{Z}$.

The sequence $\hat{\sigma}_f(k)$ is positive definite (c.f. [24]), so it satisfies the classical Herglotz Theorem:

2.53 Theorem (Herglotz, c.f. [24])**.** *Let $\hat{\mu}(k)$ be a positive definite sequence. There is a unique finite non-negative measure μ on $\mathbb{T}$ such that*

$$\hat{\mu}(k) = \int_{\mathbb{T}} z^k\ d\mu(z). \tag{2.54}$$

Conversely, given a finite, non-negative measure on $\mathbb{T}$, the Fourier coefficients defined by (2.54) *form a positive definite sequence.*

The sequence $\hat{\mu}(k)$ is called the *Fourier transform* of the measure μ. If the positive definite sequence is $\hat{\sigma}_f(k)$, then the corresponding measure σ_f is called the *spectral measure* generated by f.

2.55 Theorem (Maximal Spectral Type, c.f. [24])**.** *Given a unitary operator U on a separable Hilbert space H there is a function $f_{\max} \in L^2(H)$ such that for any $f \in L^2(H)$ if $\sigma_{f_{\max}}$ and σ_f are the spectral measures generated by f and $f_{\max}$ then $\sigma_f << \sigma_{f_{\max}}$.*

The unique measure class of $\sigma_{f_{\max}}$ is called the *maximal spectral type* of U. Any L^2 function which has spectral measure equivalent to the maximal spectral type is called a *function of maximal spectral type* (See [27] for basic measure-theory definitions).

Let (X, T, μ) be a dynamical system with ergodic measure μ. Let f be a normalized eigenfunction of U_T for the eigenvalue λ. It must necessarily span the eigenspace (c.f. [6] and we have

$$\int_{\mathbb{T}} z^k \, d\sigma_f(z) = \hat{\sigma}_f(k) = \langle U_T^k f, f \rangle = \langle \lambda^k f, f \rangle = \lambda^k \langle f, f \rangle = \lambda^k$$

so that the eigenvalues of U_T are point masses of the maximal spectral type. In fact,

2.56 Lemma (Eigenvalues, c.f. [26])**.** *Given a unitary operator U on a separable Hilbert space H with measure of maximal spectral type σ, λ is an eigenvalue of U if and only if $\sigma(\{\lambda\}) > 0$.*

For subshifts of primitive substitutions, we rely on unique ergodicity to unambiguously talk about the eigenvalues of a subshift, since the invariant measure defining the L^2 space is unique. We will make use the following theorem due to Wiener about Fourier transforms of measures on $\mathbb{T}$:

2.57 Theorem (Wiener, c.f. [17])**.** *Let μ be a Borel measure on $\mathbb{T}$ and $t \in \mathbb{T}$, and let $\hat{\mu}(n)$ for $n \in \mathbb{Z}$ be the Fourier coefficients of μ, then*

$$\mu\{t\} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{|n| \leq N} \hat{\mu}(n) t^n.$$

Alexeyev proved the existence of a bounded function of maximal spectral type in [1]. We will use a strengthened version of this result:

2.58 Theorem (Fraczek, [13])**.** *Let X be a compact metric space with a Borel measure α. Let U be a unitary operator on $L^2(X, \alpha)$, then there is a continuous function which realizes the maximal spectral type of U.*

Fraczek also proves that if X is a compact differentiable manifold, then the maximal spectral type can have the same degree of smoothness as the underlying space.

2.59 Definition (Classification by Spectra)**.** A dynamical system (X, T) with ergodic measure μ is said to have *pure discrete spectrum* if the spectral measure σ of U_T is discrete or, equivalently, if the eigenfunctions of U_T span $L^2(X,\mu)$. If the spectral measure is continuous except at 1, then (X,T) is said to have *continuous spectrum*. If the spectral measure has both a continuous and a discrete part then (X,T) is said to have *mixed spectrum*.

In general we refer to the spectral properties of the unitary operator U_T as properties of the corresponding dynamical system.

To some degree, a dynamical system is characterized by its spectral type. This is particularly true if it has pure discrete spectrum: Halmos and von Neumman proved if a dynamical system has pure discrete spectrum then it is metrically isomorphic to a rotation on the dual group of the spectrum. Metric isomorphism is the notion of isomorphism proper to measure-theoretic dynamics; while this fact is central to our motivation, we will not recount the details here. The reader may wish to visit [26] or [6] for basic definitions. If a dynamical system has a continuous part to its spectrum, then a classification by spectral-type alone does not hold.

Every constant-length substitution φ has a *pure base*, denoted φ_p, which is equal to φ if φ is pure and otherwise is a "substitution on blocks" of length h (c.f. chapter 3 and [12]).

2.60 Theorem (Spectrum of Constant Length Substitutions ([7], [26])). *Let φ be a substitution of constant length-l and height h. The dynamical system (X_φ, T) has eigenvalue group*

$$\left\{e^{2\pi i\frac{k}{l^p}} \times e^{2\pi i\frac{j}{h}} : p \in \mathbb{N},\ 0 \leq k < l^p,\ 0 \leq j < h\right\}. \tag{2.61}$$

All eigenvalues have continuous eigenfunctions. Additionally, X_φ has pure discrete spectrum if and only if φ_p has a coincidence at some power.

Thus the divide between pure discrete spectrum and other types of spectra for constant-length substitution systems is just the divide between substitutions with coincidence and those without. As is typical in the literature, we identify the group in 2.61 with the isomorphic group $Z_p \times Z(h)$ and use the two interchangeably.

Eigenfunctions and the notion of recognizability are intertwined in the following way:

2.62 Lemma (Construction of Eigenfunctions)**.** *Given a primitive, aperiodic, constant length-ℓ substitution, we can create an eigenfunction in the following manner: Fix N such that any word of length $2N+1$ can be cut uniquely (2.39). Then the map $f : X_\varphi \to \mathbb{C}$ given by $f(x) = e^{\frac{2\pi i}{\ell}c}$ where c is the cutting of $x_{[-N,N]}$ is an eigenfunction of (X_φ, T) for the eigenvalue $e^{\frac{2\pi i}{\ell}}$.*

Conversely, for such a substitution eigenfunctions are continuous and finitely-valued. Re-labeling the output of an eigenfunction gives the cutting of any $x_{[-N,N]}$ for any x and large-enough N.

The lemma follows from the fact that shifting a word with a unique cutting inside of an infinite word shifts the "cutting bars" along with it. If φ meets the conditions of the lemma, then so does φ^n, although doing-so may increase N. In this manner an eigenfunction for $e^{\frac{2\pi i}{\ell^n}}$ may be created since $X_\varphi = X_{\varphi^n}$.

One may construct an eigenfunction for an extension by composition of the factor's eigenfunction and the factor map itself (c.f. [26], [6], or [19]). We will use this lemma throughout this work:

2.63 Proposition. *If (X, T) is an extension of (Y, S), then the eigenvalues of (Y, S) are contained in (X, T).*

We now move on to the results about factors of certain constant-length substitutions.

Chapter 3

Factors of Pure Substitutions

In this chapter we will prove that factors of a pure length-p substitutions where p is prime must have a very simple form: they must be isomorphic to length-p^n substitutions for some $n \in \mathbb{N}$. The proof will be based on a similar fact about automatic sequences, which asserts that the factors of a constant-length substitution may be represented as a substitution on words or "blocks" of a fixed length from the language.

3.1 Consistency Lemmas

Consider the non-constant length substitution on three letters given by:

$$\delta \begin{cases} \mathsf{0} \to \mathsf{012} \\ \mathsf{1} \to \mathsf{01} \\ \mathsf{2} \to \mathsf{02} \end{cases}$$

The fixed point is $\delta^\infty(0) = 01201020120101202\cdots$. Let the 1-block code $b : \{0,1,2\} \to \{A, B\}$ be defined by $b(0) = A$, $b(1) = B$, $b(2) = B$. In other words, b is a re-labeling which identifies the letters 1 and 2. We attempt to create a substitution δ_b on the new alphabet by $\delta_b(b(\alpha)) = b(\delta(\alpha))$ for all $\alpha \in \{0,1,2\}$. The key ingredient for this new substitution being well-defined is the fact that $b(\delta(1)) = b(\delta(2))$. Because of this, δ_f may be written as

$$\delta_f \begin{cases} \mathtt{A} \to \mathtt{ABB} \\ \mathtt{B} \to \mathtt{AB} \end{cases}$$

The fixed point of δ_f is $\delta_f^\infty(A) = ABBABABABBABABBAB\cdots = f(\delta^\infty(0))$. We wish to generalize this phenomenon.

3.1 Definition (Consistent 1-Block Code)**.** Let φ be a primitive substitution on the alphabet $\mathcal{A}$. We say that the 1-block code f is *consistent* if it is non-trivial (not mapping all letters to a single letter) and for all $\alpha, \beta \in \mathcal{A}$, $f(\alpha) = f(\beta) \Longrightarrow f(\varphi(\alpha)) = f(\varphi(\beta))$.

3.2 Definition (Substitution Defined by a Consistent 1-Block Code)**.** Let φ be a primitive substitution on the alphabet $\mathcal{A}$ and let f be a consistent 1-block code onto the alphabet $\mathcal{B}$. The substitution on $\mathcal{B}$ defined by f and φ, denoted φ_f, is given by $\varphi_f(f(\alpha)) = f(\varphi(\alpha))$ for all $\alpha \in \mathcal{A}$.

The substitution φ_f is well-defined precisely because f is consistent for φ and f maps onto $\mathcal{B}$. Since 1-block codes apply to words via concatenation, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{L}(X_\varphi) & \xrightarrow{\varphi} & \mathcal{L}(X_\varphi) \\ \downarrow f & & \downarrow f \\ \mathcal{L}(X_{\varphi_f}) & \xrightarrow{\varphi_f} & \mathcal{L}(X_{\varphi_f}) \end{array}$$

If $w \in \mathcal{L}(X_\varphi)$ then $\varphi(w) \in \mathcal{L}(X_\varphi)$, so by induction the following diagram commutes:

$$\begin{array}{ccccccc} \mathcal{L}(X_\varphi) & \xrightarrow{\varphi} & \mathcal{L}(X_\varphi) & \xrightarrow{\varphi} & \mathcal{L}(X_\varphi) & \xrightarrow{\varphi} & \cdots \\ \downarrow f & & \downarrow f & & \downarrow f & & \\ \mathcal{L}(X_{\varphi_f}) & \xrightarrow{\varphi_f} & \mathcal{L}(X_{\varphi_f}) & \xrightarrow{\varphi_f} & \mathcal{L}(X_{\varphi_f}) & \xrightarrow{\varphi_f} & \cdots \end{array} \tag{3.3}$$

Equivalently, $\varphi_f^n(f(w)) = f(\varphi^n(w))$ for all $n \in \mathbb{N}$. Diagram 3.3 implies that in the limit, $\varphi^\infty(0) = \varphi_f^\infty(f(0))$, i.e., f maps the fixed-point of φ to the fixed-point of φ_f. This implies $X_{\varphi_f} = f(X_\varphi)$. Additionally, for any word $w \in \mathcal{L}(X_\varphi)$ and $n \in \mathbb{N}$,

$$|\varphi^n(w)| = |\varphi_f^n(f(w))|. \tag{3.4}$$

Consistent 1-block codes preserve primitivity:

3.5 Lemma. *If φ is primitive and f is a consistent 1-block code, then φ_f is primitive.*

Proof. Then since φ is primitive we know there is an N such that $n \geq N$ implies for all $\alpha, \beta \in \mathcal{A}$, $\alpha \in \varphi^n(\beta)$. Since $f(\varphi^n(\beta)) = \varphi_f^n(f(\beta))$, $n \geq N$ implies for all $a, b \in \mathcal{B} = f(\mathcal{A})$, $a \in \varphi_f^n(b)$ □

We need a deep result from Host's theory of coboundaries which is often restated in terms of return-words. As it turns-out, coboundaries are intimately connected with eigenvalues of the substitution.

3.6 Theorem (Host, [15])**.** *Let φ be primitive and recognizable. The number $\lambda \in \mathbb{T}$ is an eigenvalue of (X_φ, T) if and only if there is a $j \geq 1$ and coboundary h such that*

$$h(\alpha) = \lim_{n \to \infty} \lambda^{|\varphi^{jn}(\alpha)|} \tag{3.7}$$

for every $\alpha \in \mathcal{A}$.

Host's theorem allows us to prove a converse to Proposition 2.63 in a special case:

3.8 Lemma (Consistency Lemma)**.** *If the substitution φ is pure and f is a consistent 1-block code then all the eigenvalues of X_φ are eigenvalues of X_{φ_f}.*

Proof. Assume φ has alphabet $\mathcal{A}$ and φ_f has alphabet $\mathcal{B}$. Let λ be an eigenvalue of X_φ. By 2.61, $\lambda = e^{2\pi i \frac{k}{\ell^p}}$ for some $p \in \mathbb{N}$, $0 \leq k < \ell^p$. Since $|\varphi^n(\alpha)|_\# = \ell^n$, we obtain $\lambda^{|\varphi^{jn}(\alpha)|} = 1$ for all j, α when $n \geq p$.

Since we know φ_f has at least the trivial coboundary, $h = 1$, looking at (3.7) we could show $\lim_{n\to\infty} \lambda^{|\varphi_f^{kn}(\beta)|} = 1$ for some k and all β. Let $k = 1$, then for all $f(\alpha) \in \mathcal{B}$,

$$\lim_{n\to\infty} \lambda^{|\varphi_f^n(f(\alpha))|} = \lim_{n\to\infty} \lambda^{|\varphi^n(\alpha)|} = 1 = h(\alpha)$$

by (3.4). Thus λ is an eigenvalue of X_{φ_f}. □

Although we will not make use of it, the consistency lemma could be generalized to state that all eigenvalues coming from the trivial coboundaries are preserved by consistent

1-block factors even if the substitution is not constant-length. The same cannot be said for the so-called "exotic eigenvalues" – those coming from non-trivial coboundaries and which depend on more than the abelianization of the substitution (c.f. [26], [12]). Additionally, consistent 1-block codes can be made to generalize other block codes. For example, see Lemma 4.10. An in-depth look at this phenomenon, and the interplay with the height of the substitution is a possible future direction for this research.

This is a slight rewording of Mentzen's Theorem 8 from [20]:

3.9 Theorem (Mentzen Consistency Theorem)**.** *All the factors of a substitution of constant length-ℓ and height h are, up to conjugacy, either consistent 1-block codes or have pure point spectrum. In the latter case the point spectrum has the form:*

$$\left\{e^{2\pi i \frac{k}{m^j}} \times e^{2\pi i \frac{n}{p}} : j \in \mathbb{N},\ 0 \le k < m^j,\ 0 \le n < p\right\} \tag{3.10}$$

where $m|q^t h$ for some t and $pq = \ell$ (note: this is $Z(p) \times Z_m$).

We will prove later that the pure-point spectrum factors must also have substitutive representation in the pure case. We need some of the following corollaries about pure constant-length substitutions:

3.11 Corollary. *All non-trivial factors of a pure constant length-ℓ substitution either retain the entire point-spectrum or have pure point spectrum of the form* (3.10)*.*

Proof. Let φ be such a substitution, and let b be a block-code inducing the factor Y. By Theorem 3.9 either Y has pure point spectrum of the form (3.10) or Y is conjugate to a consistent 1-block code of X_φ. In the latter case Lemma 3.8 implies no eigenvalues are lost in Y. □

Importantly, the reader is reminded that factors of substitution subshifts need-not be substitution subshifts in general. That is, there is not necessarily a substitution which generates them.

3.12 Corollary. *All non-trivial factors of a pure constant length-ℓ substitution where ℓ is prime either retain the entire point-spectrum or are periodic.*

Proof. Let φ be such a substitution, and let b be a block-code inducing the factor Y. By Theorem 3.9 either Y has pure point spectrum of the form 3.10 or Y is conjugate to a consistent 1-block code of X_φ. In the latter case Lemma 3.8 implies no eigenvalues are lost in Y. In the former case, since ℓ is prime, using the form of eigenvalues of the factor in given in Theorem 3.9, the eigenvalue group of Y is either Z_ℓ or $Z(\ell)$.

The ℓ-adic integers Z_ℓ are the eigenvalue group of X_φ by (2.61). The group Z_ℓ has no infinite subgroups not isomorphic to Z_ℓ, so the only case where the original point spectrum is not retained is if it becomes a finite group $Z(\ell)$. In this case the spectrum is also pure-point, thus the entire system is conjugate to an ℓ point system. □

Finally, we generalize the notion of consistency to longer block-codes of constant-length substitutions.

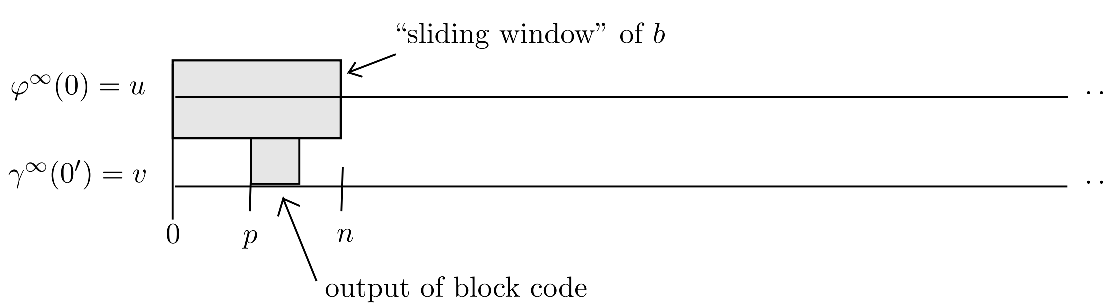
3.13 Definition (Almost-Consistent Block Code)**.** The X_φ be a primitive substitution subshift with $|\varphi| = \ell$. The block code $b : X_\varphi \to Y$ with memory p, and anticipation q is called *almost-consistent* if there is a primitive substitution γ such that if u is the fixed-point of φ and v is the fixed-point of γ then

$$n \geq p \Longrightarrow v_n = b(u_{[n-p,n+q]}), \tag{3.14}$$

and $|\gamma| = \ell^t$ for some t.

Remember that for a block-code b with positive memory we have defined $v = b(u)$ to mean (3.14). Almost-consistent block codes are those that map fixed-points onto fixed-point except at the beginning where they are undefined. See figure 3.1.

Figure 3.1: Almost-consistent block codes as in definition 3.13.



A consistent 1-block code f is almost-consistent with $p = 0$, $\gamma = \varphi_f$, and $t = 1$. Here is the exact meaning of almost-consistency for 1-block codes:

3.15 Lemma. *A 1-block code b of the constant-length substitution φ is almost-consistent if and only if b is consistent for φ^n for some n.*

Proof. Assume b is an almost-consistent 1-block code of φ with fixed-point u. Let γ be the substitution with fixed-point v such that $v = b(u)$, and $|\gamma| = \ell^t$ for some t. Since b is a 1-block code, $v_n = b(u_n)$ for all $n \geq 0$. Thus if $\alpha, \beta \in \mathcal{A}$ and $b(\alpha) = b(\beta)$, find positions i and j such that $\alpha = u_i$ and $\beta = u_j$. Then, $b(\varphi^t(\alpha)) = b(\varphi^t(u_i)) = b(u_{[\ell^t i, \ell^t(i+1)-1]}) = v_{[\ell^t i, \ell^t(i+1)-1]} = \gamma(v_i) = \gamma(b(\alpha)) = \gamma(b(\beta)) = \gamma(v_j) = v_{[\ell^t j, \ell^t(j+1)-1]} = b(u_{[\ell^t j, \ell^t(j+1)-1]}) = b(\varphi^t(u_i)) = b(\varphi^t(\beta))$.

For the converse choose $\gamma = b \circ \varphi^n$ and $t = n$. Then $v = b(u)$ by 3.3 since b is consistent for φ^n. □

The following example demonstrates that there really is a difference between consistency and almost-consistency:

3.16 Example (Almost-consistent but not consistent 1-block code)**.** Consider the 1-block code b defined by $b(0) = A$, $b(1) = B$, $b(2) = B$, and $b(3) = C$ on the substitution subshift X_θ where

$$\theta \begin{cases} \mathtt{0} \to \mathtt{0123} \\ \mathtt{1} \to \mathtt{0001} \\ \mathtt{2} \to \mathtt{0003} \\ \mathtt{3} \to \mathtt{0002} \end{cases}, \quad \text{so} \quad \theta^2 \begin{cases} \mathtt{0} \to \mathtt{0123\ 0001\ 0003\ 0002} \\ \mathtt{1} \to \mathtt{0123\ 0123\ 0123\ 0001} \\ \mathtt{2} \to \mathtt{0123\ 0123\ 0123\ 0002} \\ \mathtt{3} \to \mathtt{0123\ 0123\ 0123\ 0003} \end{cases}.$$

In the above definitions, spaces have been added for clarity. Checking whether b is consistent for θ we see that $b(1) = b(2)$ but $b(\theta(1)) = AAAB \neq AAAC = b(\theta(2))$, so b is not consistent for θ. Moving to θ^2 fixes the problem: $b(\theta^2(1)) = ABBC\ ABBC\ ABBC\ AAAB = b(\theta^2(2))$. Therefore, by Lemma 3.15, b is almost-consistent.

3.17 Lemma. *If b is an almost-consistent block code of X_φ as in Definition 3.13, then $b(X_\varphi) = X_\gamma$.*

This is a consequence of primitivity: the words at the beginning of the fixed-point of γ that do not directly come from the fixed-point of φ must be repeated, hence the language of the subshift does not depend on them. Therefore we may simply say b is an consistent block code of X_φ onto X_γ.

3.18 Definition (Fixed-Point Map)**.** An almost-consistent block code is a *fixed-point map* if it starts at the origin.

Remember that 1- and 2-block codes as defined in chapter 1 have no memory, but 3-block codes have memory 1. Therefore almost-consistent 1- and 2-block codes are fixed-point mappings by definition.

3.2 The Canonical m-Delay

In this section we create some machinery which implements the idea of a time-delay from dynamical systems in the particular situation of subshifts. This was inspired by a construction used by Dekking in [7] to construct an isomorphism between certain constant and non-constant length substitution subshifts.

3.19 Definition (Canonical m-delay)**.** Let X be a primitive aperiodic subshift on $\mathcal{A} = \{0, 1, 2, \cdots, d-1\}$, and let $m \in \mathbb{N}$, $m \geq 2$. We define a new alphabet, $\mathbf{A} = \{a_{i,j} \ : \ 0 \leq i \leq d-1, \ 0 \leq j \leq m-1\}$ and d finite words from $\mathbf{A}$, $A_i = a_{i,0}a_{i,1} \cdots a_{i,m-1}$. Define the mapping $f_{\exp} : \mathcal{A} \to \mathbf{A}^m$ by $f_{\exp}(i) = A_i$. Extend $f_{\exp}$ to infinite words on $\mathcal{A}$ by concatenation. The *canonical m-delay* of X, denoted X^m, is the subshift $X^m := [[f_{\exp}(x)]] \subset \mathbf{A}^{\mathbb{Z}}$ where x is any point in X.

The subshift X^m is well-defined since X is primitive. We point-out that the practical consequence of $X \mapsto X^m$ is a delay of the shift-action by a factor of m, so that $T^m \circ f_{\exp} \equiv T$. If X_γ is a primitive substitution subshift and u is a fixed point, then $X_\gamma^m = [[f_{\exp}(u)]]$.

3.20 Proposition. *The canonical m-delay X_γ^m of the primitive, aperiodic substitution γ of length l on d letters may be generated by a length-l substitution on $d \cdot m$ letters. Furthermore, the map f_{exp} maps the fixed point of X_γ to the fixed point of X_γ^m.*

Proof. We use the notation of Definition 3.19. Define the mapping on blocks $\gamma'(A_i) = f_{\exp}(\gamma(i)) = A_{\gamma(i)_0}A_{\gamma(i)_1}\cdots A_{\gamma(i)_{l-1}}$. Notice that $|\gamma'(A_i)| = l\cdot m$, and define a substitution on $\mathbf{A}$ by $\gamma_*(a_{ij}) = \gamma'(A_i)_{[jl,(j+1)l-1]}$. We then have that $\gamma_*(A_i) = \gamma_*(a_{i0}a_{i1}\cdots a_{i(m-1)}) = \gamma'(A_i)$.

Now, we see that on the level of m-blocks, A_i, the substitution γ' is simply the original substitution γ. Thus $\gamma'^{\infty}(A_0) = f_{\exp}(\gamma^{\infty}(0)) = f_{\exp}(u)$.

We wish to show that the substitution γ_* has $f_{\exp}(u)$ as a fixed point. Since it is assumed that $\gamma(0)$ begins with 0, the construction ensures that eventually $\gamma_*^k(a_{00})$ begins with A_0. Hence

$$\gamma_*^{\infty}(a_{00}) = \gamma_*^{\infty}(A_0) = \gamma'^{\infty}(A_0) = f_{\exp}(u).$$

This implies that the substitution γ_* generates X_γ^m. We note that it inherits primitivity from γ and the fact that each $a_{i,j}$ only appears within the block A_i. □

3.21 Example. Let κ be the Morse substitution ($\kappa(0) = 01$ and $\kappa(1) = 10$) so that $l = 2$. Choose $m = 3$. Our new alphabet will be re-labeled for convenience as follows:

$$\mathbf{A} = \{a_{0,0}, a_{0,1}, a_{0,2}, a_{1,0}, a_{1,1}a_{1,2}\} = \{a, b, c, d, e, f\}.$$

Otherwise using the notation of Proposition 3.20, $A_0 = abc$, $A_1 = def$, so $f_{\text{ext}}(0) = A_0 = abc$ and $f_{\text{ext}}(1) = A_1 = def$. If u is the Morse fixed-point, we know X_κ^3 is generated by the sequence

$$f_{\text{ext}}(u) = f_{\text{ext}}(.01101001\cdots) = .abcdefdefabcdefabc\cdots \in \mathbf{A}^{\mathbb{N}}.$$

We have $\kappa'(A_0) = abcdef$ and $\kappa'(A_1) = defabc$. So the new substitution κ_* is defined by:

$$\kappa_* \begin{cases} \mathtt{a} \to \mathtt{ab} \\ \mathtt{b} \to \mathtt{cd} \\ \mathtt{c} \to \mathtt{ef} \\ \mathtt{d} \to \mathtt{de} \\ \mathtt{e} \to \mathtt{fa} \\ \mathtt{f} \to \mathtt{bc} \end{cases}$$

As expected, $\kappa_*^\infty(a) = .abcdefdefabcdefabc\cdots = f_{\text{ext}}(u)$.

The m-delay has an effect on the height of some substitutions. In the example, it is clear that $g_0 = \gcd\{k \geq 1 : u_k = a\} = 3$, so $h = \max\{k \geq 1 : (k, 2) = 1,\ k|3\} = 3$. In general, if γ has $g_0 = t$ then for the m-delay of γ, $g_0 = mt$, so that if m has no common factor with the original height h, then the new height is mh. The height is always bounded by the number of letters ([26]), but the m-delay also multiplies the size of the alphabet by m.

We will be primarily concerned, in the proof of Theorem 3.33, with p^n-delays of pure substitutions of length-p where p is prime. Although we will not use the fact, it is interesting to point-out that in this case $h = \max\{k \geq 1 : (k, p) = 1,\ k|g_0\} = 1$ for the original substitution which implies $g_0 = p^m$ for some $m \geq 0$. After the delay, $g_0 = p^{mn}$ so $h = \max\{k \geq 1 : (k, p) = 1,\ k|p^{mn}\} = 1$. Therefore, unlike in example 3.21, we do not introduce a height to substitution.

3.3 Automata and Substitution Factors

A *subshift of finite type* X given an alphabet $\mathcal{A}$, is a subshift of $\mathcal{A}^{\mathbb{Z}}$ defined by the property that $\mathcal{L}(X) = \{w \in \mathcal{A}^* : \forall v \in V,\ v \notin w\}$ for some finite set $V \subset A^*$. The set V, called the set of *forbidden words*, is not unique unless it is empty. A topological factor of a subshift of finite type is called a *sofic system*. A subshift is sofic if and only if it admits a presentation as the set of edge-paths through a finite directed graph with possibly non-unique edge labels (c.f. [6], [2]).

Substitution subshifts are another class of subshifts, and in analogy to sofic systems we call a topological factor of a constant length-k substitution subshift an *k-automatic dynamical system*. A more conventional view considers the k-automatic sequence:

3.22 Definition (Automatic Sequence)**.** A sequence $x \in \mathcal{A}^{\mathbb{N}}$ is *k-automatic* if it is the image of the fixed-point of a constant length-k substitution under a block code with no memory.

Completing the analogy, a k-automatic sequence is one which may be generated by a graph automaton of the type described in the introduction, and a k-automatic sequence x generates the automatic dynamical system $[[x]]$. Cobham's Theorem (c.f. [12], [2]) unites the two representations. If the block-code generating the k-automatic sequence is almost-consistent, then the resulting automatic system is also a substitution subshift, but examples are known of automatic systems which are not substitution subshifts. One such case is the system generated by the Rudin-Shapiro sequence (see example 3.35). If we start with an automatic sequence, then the following theorem is a well-known way to find a substitution which generates the automatic sequence in an unconventional way:

3.23 Lemma (c.f. [2])**.** *A sequence $x \in \mathcal{A}^{\mathbb{N}}$ is k-automatic if and only if there exist integers r, s with r, $s \geq 1$ such that for all i, $j \geq 0$ we have*

$$x_{[ik^r,(i+1)k^r]} = x_{[jk^r,(j+1)k^r]} \Longrightarrow x_{[ik^{r+s},(i+1)k^{r+s}]} = x_{[jk^{r+s},(j+1)k^{r+s}]}.$$

3.24 Corollary. *For every k-automatic sequence x with alphabet $\mathcal{A}$, there are positive integers r and s, a substitution γ with alphabet $\mathcal{B}$ and $|\gamma| = k^s$, and an injective map from $f : \mathcal{B} \to \mathcal{A}^{k^r}$ such that $x = f(\gamma^\infty(0))$, where f extends to a map on X_γ via concatenation.*

The substitution γ is commonly referred-to as the "substitution on blocks" for the k-automatic sequence, so that the result states that every k-automatic sequence is the fixed point of some substitution on blocks (c.f. [2]). A "block," of course, is $f(b)$ for some $b \in \mathcal{B}$. We are used-to considering subshifts up-to topological conjugacy, but the subshift point-of-view is not as common with k-automatic shifts as with sofic shifts. We will show that up-to topological conjugacy – necessarily via block-codes – some classes of substitutions admit only substitution factors.

We will need the following basic topological lemma (see [23] for basic topological definitions and results):

3.25 Lemma. *Let X, Y, and Z be topological spaces where X is compact and Y is Hausdorff. Let $b : X \to Y$ and $f : Y \to Z$ be maps. If b and $f \circ b$ are continuous, then f is continuous.*

Proof. Let $U \subset Z$ be closed. Since $f \circ b$ is continuous, $A := (f \circ b)^{-1}(U) \subset X$ is closed. This implies A is compact since X is compact. Now $b(A) = b(b^{-1} \circ f^{-1}(U)) = f^{-1}(U)$ must be compact since b is continuous. Therefore $f^{-1}(U) \subset Y$ is closed because Y is Hausdorff. Hence f is continuous. □

The following encapsulates a nice property of bijective substitutions that we wish to generalize:

3.26 Definition (Left-Resolvable Substitution)**.** A constant-length substitution φ on $\mathcal{A}$ is *left-resolvable* if there exists $N \in \mathbb{N}$ such that any word $w \in \mathcal{L}(X_\varphi)$ with $|w| \geq N$ has a unique desubstitution on the left or extends uniquely on the left to such a word.

The question of left-resolvability is also related to the construction of the so-called "accordion-form" or "prefix/suffix expansion"[12] of an arbitrary word of an injective substitution. Large enough words in recognizable substitutions may be cut into blocks of the form $\varphi(i)$ where $i \in \mathcal{A}$ (see Lemma 2.39). The beginning and end of such long words possibly contain endings and beginnings (suffixes and prefixes) of blocks of the form $\varphi(i)$, where $i \in \mathcal{A}$, instead of whole blocks. The accordion form is made by repeated desubstitution in the center of a word, leaving the prefixes and suffixes which cannot be desubstituted on the ends at each step. A substitution is left-resolvable if for long-enough words the first step in the accordion construction either yields no suffix or if the suffix can be extended uniquely to an entire block.

An example helps to illustrate the point:

3.27 Example (The Morse substitution)**.** Recall that the Morse substitution is given by

$\kappa(0) = 01$, $\kappa(1) = 10$, and that a fixed point is

$$\kappa^{\infty}(0) = 01|10|10|01|10|01|01|10|10|01|01|10|01|10|10|01\cdots$$

where vertical bars are used to delineate the boundaries of blocks of the form $\kappa(i)$, $i \in \{0, 1\}$. We can always determine how to cut long words from the Morse substitution into blocks $\kappa(0)$ and $\kappa(1)$ because the words 00 must appear with bounded gaps. Therefore, long enough words must contain 00 and a cutting should separate the adjacent zeros since the first 0 must be the ending of $\kappa(1)$ and the second, the beginning of $\kappa(0)$. For example, the the word $w = 010011$ cuts uniquely: $0|10|01|1$. We see that the beginning and end of the word contains the suffix of $\kappa(1)$ and the prefix of $\kappa(0)$, respectively. For the Morse substitution it is easy to see that $0|10|01|1$ must appear inside a unique larger word, i.e., $0|10|01|1$ must appear as the center of $10|10|01|10$. Thus we say $0|10|01|1$ uniquely desubstitutes to 1101. The same type of extension of words can be done for all bijective substitutions.

The inference of the first and last letter of the desubstituted word is easy for κ but difficult or impossible for many substitutions. Left-resolvability requires that for long enough words this inference can be made about the first letter of the desubstituted word but not necessarily the last. The Morse substitution exhibits the stronger property that both can be determined. One could simply denote this stronger condition "resolvability."

We show by example that not all primitive, aperiodic subsitutions need-be left-resolvable.

3.28 Example (A Non Left-Resolvable Substitution)**.** Let δ be the following substitution on three letters:

$$\delta \begin{cases} \mathsf{0} \to \mathsf{010} \\ \mathsf{1} \to \mathsf{120} \\ \mathsf{2} \to \mathsf{200} \end{cases}$$

Beginning with $\delta(0) = 010$ we find a sequence of words appearing in $\delta^{\infty}(0)$ by at each step disregarding the first letter of the previous word then applying the substitution. Vertical bars show how the words result from the expansion of letters:

$$\begin{aligned}
\text{Step 1:}\quad 0 &\to^{(\delta)} \mathbf{0}10\\
\text{Step 2:}\quad &\to^{(\delta)} \mathbf{1}20|010\\
\text{Step 3:}\quad &\to^{(\delta)} \mathbf{2}00|010|010|120|010\\
\text{Step 4:}\quad &\to^{(\delta)} \mathbf{0}10|010|010|120|010|010|120|010|120|200|010|010|120|010\\
\text{Step 5:}\quad &\to^{(\delta)} \mathbf{1}20|010|010|120|010|010|120|010|120|200|010|010|120|010\cdots\\
&\quad\vdots
\end{aligned}$$

Therefore, we see that there are arbitrarily long words of the form $0|010|010|\cdots$ in $\mathcal{L}(X_\delta)$. We wish to see if the suffix 0 can be extended to the left so that such a word can be desubstituted uniquely when of sufficient length. But we see that arbitrarily long words of the form $010|010|010|120\cdots$, $120|010|010|120\cdots$, and $200|010|010|120\cdots$ are all in $\mathcal{L}(X_\delta)$. Thus, no matter the length it is impossible to uniquely extend the suffix 0 to an entire block, and thus δ is not left-resolvable.

We see that the last letter of $\delta(i)$ is 0 for all $i \in \mathcal{A}$ for the example. At the other extreme, if the last letter of $\delta(i)$ is unique for every $i \in \mathcal{A}$ then left-resolvability holds:

3.29 Proposition. *Any primitive, aperiodic, length-l substitution φ on $\mathcal{A}$ that has a bijective last column, i.e., $\{\varphi(i)_{l-1} : i \in \mathcal{A}\} = \mathcal{A}$, is left-resolvable.*

Proof. Since φ must be recognizable, there is a length N such that words of at least length N may be uniquely cut into words of the form $\varphi(\alpha)$ where $\alpha \in \mathcal{A}$. Take any word w with $|w| > N$. Uniquely cut w into blocks by the Lemma 2.39. The first block of w contains the last m letters of the block $\varphi(i)$ where $1 \leq m \leq l$ and $i \in \mathcal{A}$. The word w desubstitutes uniquely on the left if there is no other letter $j \neq i$ such that the last m letters of $\varphi(i)$ and $\varphi(j)$ agree. Since φ has a bijective last column, the two endings are identical if and only if $i = j$, thus we are done. □

3.30 Corollary. *Aperiodic bijective substitutions are left-resolvable.*

As previously noted, bijective substitutions are left- and right-resolvable. It is easy to see how, by a symmetric argument to the above lemma, a bijective first column is enough guarantee uniqueness of the desubstitution on the right. We will not need this property.

3.31 Corollary. *If a primitive, aperiodic, length-l substitution φ has a bijective last column, then φ^n is left-resolvable for all n.*

Proof. The substitution φ^n must have a bijective final column since each successive application of φ simply permutes the last column. □

We now invent some terminology for the type of substitution we will consider in the main theorem of this chapter and the next.

3.32 Definition (Basic Substitution)**.** A primitive, pure, aperiodic, injective, constant length-p substitution φ where p is prime, having the property that $\varphi(0)$ begins with 0, is called a *basic substitution*.

The restriction to pure substitutions is perhaps not very strong as substitutions always reduce to a "pure base" times a finite rotation. Additionally, every substitution is conjugate to an injective substitution via a consistent 1-block code (see chapter 4), so this is also an ephemeral restriction. The restriction to prime lengths is perhaps more serious and worth examining in future work.

3.33 Theorem. *Any aperiodic subshift factor of a basic substitution φ of length p is topologically conjugate to a substitution via an almost-consistent block code.*

Additionally, if all powers of φ are left-resolvable then the block-code may be assumed to be a fixed-point mapping.

Proof. Let X_φ factor onto the aperiodic subshift Y via a block code. We may assume the block code has no memory. If it does not, we simply pass to a block code with no memory using the identical map on blocks. It is clear that this new code simply adds a translation to the original mapping, and thus produces the same subshift image Y. Call the block code $b : X_\varphi \to Y$.

Let $x = \varphi^\infty(0)$ be the fixed point of φ, and let $y = b(x)$. By definition the point y is p-automatic, so from Corollary 3.24 we know that there is a substitution γ with length p^s on the alphabet $\mathcal{B}$ and an injective function $f : \mathcal{B} \to \mathcal{A}^{p^r}$ such that $y = f(\gamma^\infty(0))$.

Now, we know that the eigenvalue group of (X_φ, T) is Z_p, and from Corollary 3.12 we know that (Y, T), as an aperiodic factor, must have eigenvalue group Z_p.

Let g be an eigenfunction of Y for the eigenvalue $\lambda = e^{2\pi i/p^r}$. It is clear that for any $z \in Y$, $g(T^{p^r+n}z) = \lambda^{p^r+n}g(z) = \lambda^n g(z)$ so that g takes on the values from the set $\Lambda = \{1, \lambda, \lambda^2, \cdots, \lambda^{p^r-1}\}$.

If $z \in X_\varphi$ then $g(b(Tz)) = g(T(b(z))) = \lambda g(b(z))$. This implies $g \circ b$ is an eigenfunction on X_φ for λ. Therefore, $g \circ b$ must be continuous as an eigenvector of a constant-length substitution subshift (c.f. [26], [15]). Since b is continuous as a factor map, by Lemma 3.25 g must be continuous as well.

Consider Λ to be the alphabet of a shift-space, and, g' as a map $Y \to \Lambda^{\mathbb{Z}}$ where for any $z \in Y$,

$$g'(z) = g(\cdots z_{-1}.z_0 z_1 \cdots) = \cdots g(T^{-1}z).g(z)g(T^1 z)\cdots$$

The shift-space $\Lambda^{\mathbb{Z}}$ is given the usual product topology, so as a product of continuous maps g' is continuous. Furthermore, it is clear from the definition that $g'(Tz) = T(g'(z))$, so that g' is a factor map from (Y, T) onto $(g'(Y), T)$. Therefore, by the Curtis-Hedlund-Lyndon Theorem (2.16), g' can be realized by a block-code. Practically, this means the block code b does not destroy the ability to cut long-enough words from X_φ into substituted blocks. Since the substituted blocks are of constant length we may assume without loss of generality that, as a block code g, starts at the origin (if not, we need only to use a longer map on blocks and look at the rightmost subwords of the blocks). Thus we may let $L \geq 0$ be such that for all $z \in Y$, $g(z)$ depends only on $z_{[0,L]}$.

Since f is injective the blocks $C_b = f(b)$ for $b \in \mathcal{B}$ are distinct. We can construct a block-code h on Y which replaces each C_i with the block $a_{i,0}a_{i,1}\cdots a_{i,(p^r-1)}$. Explicitly, let $n \geq L + p^r - 1$ and $n \geq p^r + p^r - 1$. The block code h has length n, memory p^r, and if

$|w| = n$ then

$$h(w) := a_{i,j} \text{ where } w_{[p^r-1-j,2(p^r-1)-j]} = C_i \text{ and } g(w) = \lambda^j. \tag{3.34}$$

By construction, h is inverted via the 1-block code starting at the origin given by the map $a_{i,j} \mapsto (C_i)_j$. Thus h is a topological conjugacy, but the image of h is also $X_\gamma^{p^r}$ since if $z \in X_\gamma$ then $h(f(z)) = f_{\exp}(z)$. Finally, given Proposition 3.20, $X_\gamma^{p^r}$ may be written as a substitution γ_* of length p^s (the same length as γ).

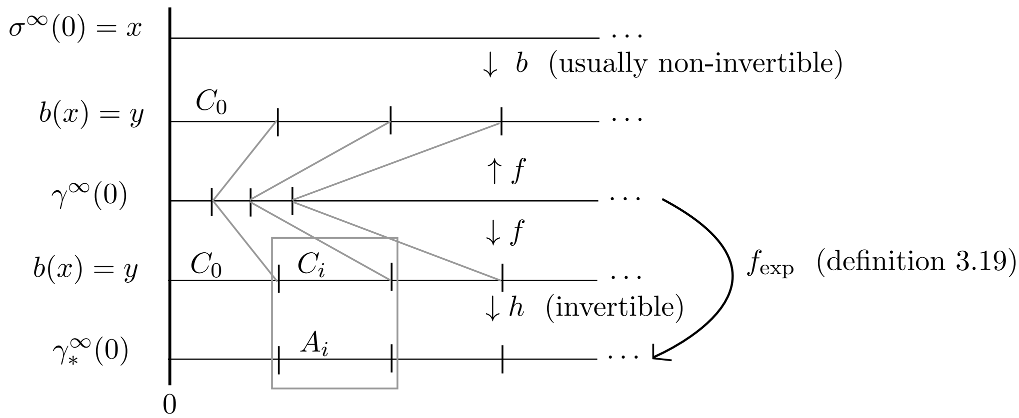
We have shown that the map h is a topological conjugacy between $b(X_\varphi)$ and the substitution system X_{γ_*}. In summary:

$$b(X_\varphi) = Y = f(X_\gamma) \equiv^{(h)} X_\gamma^{p^r} = X_{\gamma_*}$$

where, from left to right we have the image under b of the substitution system X_φ, the dynamical system Y generated as either a factor of X_φ or by the automatic sequence $y = b(x)$, the image of another substitution system X_γ under the expansion-map f (from Corollary 3.24), the canonical p^r delay of γ, and the substitution system X_{γ_*} which is simply $X_\gamma^{p^r}$ generated by a substitution. Figure 3.3 illustrates the construction.

Now, the fixed point of X_{γ_*} can be obtained by the block code $h \circ b$ except at the beginning since even though we assume b starts at the origin, h has memory $p^r - 1$, so $h \circ b$ has memory $p^r - 1$ when constructed directly. If we additionally assume that any power of φ is left-resolvable, we see that the block-code does not need any memory because the length p^r blocks $\varphi^r(i)$ can be inferred from only their final letter. Thus in this case Y is a fixed-point mapping of X_φ. $\square$

Figure 3.2: The construction of Theorem 3.33.



The code h replaces block C_i with block $A_i = a_{i,0}a_{i,1} \cdots a_{i,(p^r-1)}$.

If the length of φ is not prime, then it is unclear if λ is an eigenvalue of the system. This makes it difficult to determine whether or not it is possible to “recognize” the boundaries of the blocks C_i with a block code. In the proof it was critical for the construction of the conjugacy h that the necessary part of the recognizability of the original substitution – the boundaries of expanded blocks – must be retained by the factor.

3.35 Example. We have shown that the procedure used in the following example will always yield a substitution isomorphic to a factor of a height-1, prime length substitution. We demonstrate this procedure on the Rudin-Shapiro sequence as it is a classical example of an automatic sequence not generated by a substitution. A description of the 2-automaton defining the sequence may be found in [2], but we do not give it here and instead emphasize the “substitution on blocks” property of Corollary 3.24.

The sequence

$$r = .0001001000011101\cdots$$

is a well-known 2-automatic sequence called the Rudin-Shapiro sequence. It is given by a block code of the fixed point of some length-2 substitution φ. A lengthy discussion of the properties of the sequence is given in [2]. It is left as an exercise in [2] to prove that the Rudin-Shapiro sequence is not the fixed-point of a substitution.

Let γ be the substitution on 4 letters defined by $\gamma(0) = 01,\ \gamma(1) = 02,\ \gamma(2) = 31$, and

$\gamma(3) = 32$. Let f be the mapping on the alphabet of γ: $f(0) = 00,\ f(1) = 01,\ f(2) = 10,\ f(3) = 11$. Then

$$f(\gamma^{\infty}(0)) = f(.0102013101023202\cdots) = r,$$

and such a substitution and map may be found for any automatic sequence.

We wish to find a substitution which generates a subshift isomorphic to $[[r]]$. To do this we define the blocks $C_i = f(i)$ and the blocks $D_0 = ab,\ D_1 = cd,\ D_2 = ef,\ D_3 = gh$. The result shows that sequences from $[[r]]$, which consist of blocks of C_i's, may be converted back-and-forth to the corresponding sequences of D_i's using a block code. The sequence

$$s = D_0 D_1 D_0 D_2 D_0 D_1 D_3 D_1 \cdots = abcdabefabcdghcd\cdots$$

which corresponds to r is simply the fixed point of X_{γ}^{2} and thus is the fixed point of the substitution

$$\gamma_* \begin{cases} \mathtt{a} \to \mathtt{ab} & \mathtt{e} \to \mathtt{gh} \\ \mathtt{b} \to \mathtt{cd} & \mathtt{f} \to \mathtt{cd} \\ \mathtt{c} \to \mathtt{ab} & \mathtt{g} \to \mathtt{gh} \\ \mathtt{d} \to \mathtt{ef} & \mathtt{h} \to \mathtt{ef} \end{cases}$$

Therefore $[[r]]$ is topologically conjugate to the substitution subshift $[[s]]$.

Chapter 4

Discrete Spectrum Factors

Throughout this chapter we let φ denote a primitive substitution of length-l on the alphabet $\mathcal{A}$. We will show that almost-consistent factors of constant-length substitutions are conjugate to 2-block or 3-block codings. If the substitution is basic, then this result combined with Theorem 3.33 implies that, up-to topological conjugacy, all factors of the substitution may be enumerated by these short codes alone. If the substitution is also bijective then even-more can be deduced about the factors: specifically, there are substitutions having no aperiodic topological factors with pure-discrete spectrum.

4.1 The Map $\varphi \mapsto \varphi_f$

In general, φ is not injective on $\mathcal{A}$. Define $\mathcal{A}' = \{\varphi^{-1}(\varphi(a)) \ : \ a \in \mathcal{A}\}$. The set $\mathcal{A}'$ is the partition of $\mathcal{A}$ corresponding to the relation $a \equiv b$ if $\varphi(a) = \varphi(b)$. We consider A' as a new alphabet which identifies all letters from $\mathcal{A}$ which have the same expansion under φ.

Now define the consistent 1-block code $f : \mathcal{A} \to \mathcal{A}'$ by

$$f(a) = \varphi^{-1}(\varphi(a)),$$

that is, f maps each letter its equivalence class.

As usual, let φ_f denote the substitution on $\mathcal{A}'$ defined by $\varphi_f(f(a)) = f(\varphi(a))$ for all $a \in \mathcal{A}$.

4.1 Lemma. *If X_φ is aperiodic and primitive then X_{φ_f} is aperiodic and primitive.*

Proof. Since X_φ is aperiodic, f cannot reduce the system to one point. Therefore, primitivity is implied by Lemma 3.5. Assume X_{φ_f} is periodic. Let $u = \varphi^\infty(0)$ and $v = f(u)$. Since v is a fixed point of φ_f it must be that v is eventually periodic. Now consider u as one of many preimages of v under f, i.e., $u \in f^{-1}(v)$. The map f simply identifies letters which have the same image under φ, thus $\varphi(u)$ consists of blocks of length $|\varphi|$ arranged in the eventually periodic pattern of v. But $\varphi(u) = u$, thus u is eventually periodic and X_φ is periodic. □

Throughout this chapter, if φ is a substitution, we will use the notation φ_f to denote the substitution induced by this particular consistent 1-block code.

4.2 Lemma. *If φ has a first-order coincidence then so does φ_f, and X_φ is conjugate to X_{φ_f}.*

Proof. First we assume $\varphi(a)_i = \varphi(b)_i$ for all $a, b \in \mathcal{A}$. Then $\varphi_f(f(a))_i = f(\varphi(a))_i = f(\varphi(a)_i) = f(\varphi(b)_i) = f(\varphi(b))_i = \varphi_f(f(a))_i$ for all $a, b \in \mathcal{A}$.

To establish conjugacy, we note that we have shown φ_f is aperiodic and primitive, therefore it is a recognizable substitution. This implies a block code may be created which replaces blocks of the form $\{\varphi_f(i) : i \in \mathcal{A}'\}$ with their preimages from the set $\{\varphi(a) : a \in \mathcal{A}\}$. The block code mapping in the reverse direction is simply f. This conjugacy is discussed in detail in [5]. □

The 1-block map f does not destroy the primitivity, fixed-points, aperiodicity, or coincidence of the substitution φ. In fact, f reduces the number of letters of the substitution φ if and only if it is not injective. Since the new substitution is conjugate to the original, the transformation $\varphi \mapsto \varphi_f$ may be applied repeatedly to produce an injective representation of the substitution φ. Some authors therefore make the assumption that they begin with an injective substitution.

4.2 Reduction to Short Codes

4.3 Theorem. *Let φ be a substitution of length l, and let b be a fixed-point map onto Y. There is a 2-block fixed-point map g such that $g(X_\varphi) \equiv Y$.*

Proof. Let b be a fixed-point map onto X_γ where $|\gamma| = l^k$. Therefore $b(\varphi^\infty(0)) = \gamma^\infty(0)$, but b is not necessarily a 2-block code.

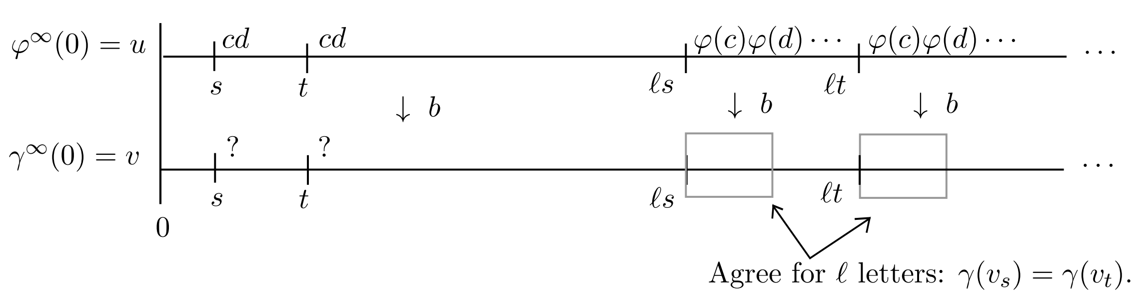
Let $g = f \circ b$. As we have seen, f is also a topological conjugacy so that $g(X_\varphi) \equiv Y$. Assume b (and thus g) has length n, then let $m \in \mathbb{N}$ be such that $l^{mk} \geq n$. If we replace φ with φ^{mk} and γ with γ^m then we may assume $|\varphi| = |\gamma| = l^{mk} =: \ell$.

We now show that g can be reduced to a two-block code, that is, it only depends on the first two letters of its map on blocks. Let $u = \varphi^\infty(0)$ and $v = b(u) = \gamma^\infty(0)$.

Let $x_1,\ x_2 \in \mathcal{L}_n(X_\varphi) = \mathcal{L}_n(u)$ such that x_1 and x_2 begin with the same two-letter word cd. The map g may be reduced to a two-block code if this implies that $f(b(x_1)) = f(b(x_2))$. By the definition of f, $f(b(x_1)) = f(b(x_2))$ when both letters have the same expansion under γ, i.e., when $\gamma(b(x_1)) = \gamma(b(x_2))$.

Now, x_1 and x_2 must appear in u. Let s and t be any positions of the two words respectively. Since u is fixed under φ, we know $\varphi(x_1)$ and $\varphi(x_2)$ appear at positions $s\ell$ and $t\ell$ in u respectively. Thus, since they are both the expansion of the word cd, $u_{[s\ell, s\ell+2\ell-1]} = u_{[t\ell, t\ell+2\ell-1]}$. Since the length of b is less than or equal to ℓ, $v_{[s\ell, s\ell+\ell-1]} = v_{[t\ell, t\ell+\ell-1]}$ (See figure 4.2). Now, since v is the fixed point of γ this implies $\gamma(v_s) = \gamma(v_t)$, but since $b(u) = v$, $b(x_1) = v_s$ and $b(x_2) = v_t$. Therefore $\gamma(b(x_1)) = \gamma(b(x_2))$. Hence g may be reduced to a two-block code and the substitution we are looking for is γ_f. □

Figure 4.1: The construction of Theorem 4.3.



Theorem 4.3 may be applied to any aperiodic factors of bijective basic substitutions by theorem 3.33. Without the left-resolvability property of bijective substitutions we may not necessarily get a fixed-point map from Theorem 3.33, i.e., the block code b does not necessarily begin at the origin. When repeating the argument in this case, one must extend the 2-block code of Theorem 4.3 to a 3-block code – it must look ahead and behind by only one letter:

4.4 Theorem. *Let φ be a substitution of length l, and let b be an almost-consistent block code onto Y. There is an almost-consistent 3-block code g such that $g(X_\varphi) \equiv Y$.*

Proof. We proceed in parallel to 4.3. Let b be map onto X_γ where $|\gamma| = l^k$.

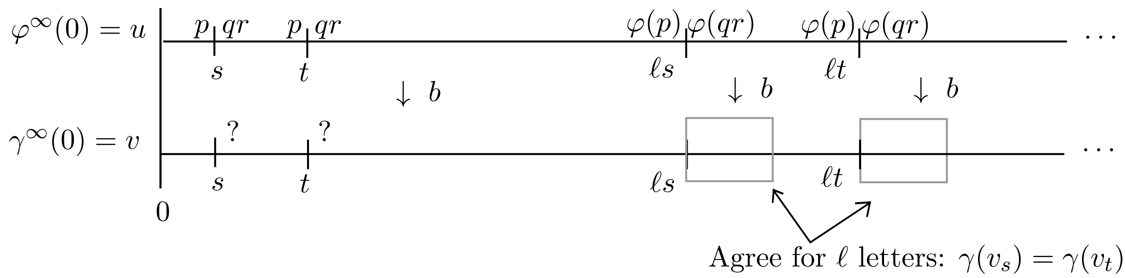
Let $g = f \circ b$. As we have seen, f is also a topological conjugacy so that $g(X_\varphi) \equiv Y$. Assume b (and thus g) has length n, then let $m \in \mathbb{N}$ be such that $l^{mk} \geq n$. If we replace φ with φ^{mk} and γ with γ^m then we may assume $|\varphi| = |\gamma| = l^{mk} =: \ell$.

We now show that g can be reduced to a 3-block code, that is, it only depends on three letters of its map on blocks. Let $u = \varphi^\infty(0)$ and $v = b(u) = \gamma^\infty(0)$.

Let $x_1,\ x_2 \in \mathcal{L}_n(X_\varphi) = \mathcal{L}_n(u)$ such that x_1 and x_2 contain the same word pqr where the middle letter, q, occupies position j where $j \geq 1$ is the memory of b (the $j = 0$ case is implied by Theorem 4.3). The map g may be reduced to a 3-block code if this implies that $f(b(x_1)) = f(b(x_2))$. By the definition of f, $f(b(x_1)) = f(b(x_2))$ when both letters have the same expansion under γ, i.e., when $\gamma(b(x_1)) = \gamma(b(x_2))$.

Now, x_1 and x_2 must appear in u. Find any occurrences of the words x_1 and x_2 in u. Since u is fixed under φ, we know $\varphi(x_1)$ and $\varphi(x_2)$ appear at positions $\ell s - \ell j$ and $\ell t - \ell j$ in u respectively. This implies the expansion of pqr appears at both position $\ell s - \ell$ and $\ell t - \ell$. Thus $u_{[\ell s-\ell, \ell s+2\ell-1]} = u_{[\ell t-\ell, \ell t+2\ell-1]}$. Since the length of b is less than or equal to ℓ, $v_{[\ell s, \ell s+\ell-1]} = v_{[\ell t, \ell t+\ell-1]}$ (See figure 4.2). Now, since v is the fixed point of γ this implies $\gamma(v_s) = \gamma(v_t)$, but then, $b(x_1) = v_s$ and $b(x_2) = v_t$. Therefore $\gamma(b(x_1)) = \gamma(b(x_2))$. Hence g may be reduced to a 3-block code and the substitution we are looking for is γ_f. $\square$

Figure 4.2: The construction of Theorem 4.4.



4.5 Corollary. *Let φ be a basic substitution, and let Y be an aperiodic topological factor. There is an almost-consistent 3-block code g such that $g(X_\varphi) \equiv Y$. If all powers of φ are left-resolvable, there is a 2-block fixed-point map c such that $c(X_\varphi) \equiv Y$.*

Proof. Since φ is basic the factor Y is conjugate to the image of an almost-consistent block code f by Theorem 3.33. Then by Theorem 4.4, $f(X_\varphi) \equiv g(X_\varphi)$ where g is an almost-consistent block code. If φ has left-resolvable powers, then f is a fixed-point map and we apply Theorem 4.3 instead. □

4.3 Bijective Substitutions

In addition to resolvability, bijective substitutions have many other properties which make them an ideal case-study. To help apply Corollary 4.5 we use a construction Queffélec calls the "higher-order substitution." This is a particular case of a "higher-order shift" which can be built starting with any subshift (c.f. [19]). Since we only use the higher-order shift with blocks of length 2, we omit adjectives like "2nd" throughout:

4.6 Definition (Higher-Order Substitution)**.** Let φ be a substitution and $|\varphi| = \ell$. The *(2nd) higher-order substitution* φ_h is the length-ℓ substitution defined on the length-2 "blocks" $\mathcal{L}_2(X_\varphi)$ by the following:

If $[a_0 a_1] \in \mathcal{L}_2(X_\varphi)$. Then $\varphi_h([a_0 a_1]) :=$

$$(4.7)\quad [\varphi(a_0)_0\varphi(a_0)_1][\varphi(a_0)_1\varphi(a_0)_2][\varphi(a_0)_2\varphi(a_0)_3]\cdots[\varphi(a_0)_{\ell-2}\varphi(a_0)_{\ell-1}][\varphi(a_0)_{\ell-1}\varphi(a_1)_0].$$

4.8 Lemma. *The higher-order substitution φ_h is topologically conjugate to φ via a fixed point map in both directions. The block-code $X_\varphi \to X_{\varphi_h}$ is length-2 and the block-code $X_{\varphi_h} \to X_\varphi$ is length-1.*

Queffélec describes higher-order substitutions in detail in [26]. For the length-2 case that we use here, it is important that the second letter defining a block (a_1 in the definition) is only needed in the final block of the expansion while all others adjacent letters of the expansion of a_0. The following well-known fact is a direct consequences of the construction:

4.9 Lemma. *If b is a 2-block code of X_φ, $b(X_\varphi) = c(X_{\varphi_h})$ where c is the 1-block code: $c([a_0a_1]) := b(a_0a_1)$. Furthermore, if c is a 1-block code of X_{φ_h} then $b(X_\varphi) = c(X_{\varphi_h})$ where b is the 2-block code: $b(a_0a_1) := c([a_0a_1])$.*

4.10 Lemma. *If b and c are as in Lemma 4.9, then b and c are either both almost-consistent or both not almost-consistent.*

Proof. By the construction (4.7), $b(\varphi^\infty 0) = c(\varphi_h^\infty 0)$ and $|\varphi| = |\varphi_h| = \ell$. Call the common one-sided sequence u. Either u is the fixed-point of some substitution γ of length ℓ^n or not. □

Note that b and c must be fixed-point mappings if they are almost-consistent. Notice that we have not yet introduced bijective substitutions into the consideration – lemmas 4.8, 4.9, and 4.10 all apply to arbitrary constant-length substitutions. We start the discussion of the particulars of bijective substitutions with a classic example:

4.11 Example (Higher-Order Substitution)**.** Let κ be the Morse bijective substitution: $\kappa(0) = 01$, $\kappa(1) = 10$. Then κ_h has the alphabet $\{[00], [01], [10], [11]\}$. The substitution is

$$\kappa_h \begin{cases} [00] \to [01][10] \\ [01] \to [01][11] \\ [10] \to [10][00] \\ [11] \to [10][01] \end{cases}$$

It is convenient to relabel the blocks when working with the higher-order substitutions, yielding:

$$\kappa_h \begin{cases} \mathtt{a} \to \mathtt{bc} \\ \mathtt{b} \to \mathtt{bd} \\ \mathtt{c} \to \mathtt{ca} \\ \mathtt{d} \to \mathtt{cb} \end{cases}$$

Consider the consistent 1-block code $b \to x$, $c \to x$, $a \to y$, $d \to y$. The corresponding substitution factor would be $x \to xy$, $y \to xx$: the Toeplitz substitution, τ. Creating the 2-block "version" of this code, we get $00 \to y$, $01 \to x$, $10 \to x$, $11 \to y$. This is the well-known block-code which factors the Morse substitution onto the Toeplitz.

If a substitution φ satisfies Theorem 4.3, we desire to inspect all 2-block factors of X_φ. Using Lemma 4.10 we may look at all possible almost-consistent 1-block codes of φ_h instead of the almost-consistent 2-block codes of φ. While this is only a small number of block codes, there are two difficulties: first, it is seemingly difficult to check if a block-code is almost-consistent. Second, if we are trying to find a factor of the substitution which has pure-discrete spectrum, even if we know a block-code is almost consistent we must look for a coincidence in some unknown, possibly high, power of the substitution.

The nature of bijective substitutions solves most of these problems, and we discuss bijective substitutions exclusively for the remainder of the chapter. Assume the bijective substitution φ has length-ℓ and $\#\mathcal{A} = d$. We call a *column* of the substitution any column of letters in the following table:

$$\begin{array}{llll} \varphi(\mathtt{0}) & \to \varphi^2(\mathtt{0}) & \to \varphi^3(\mathtt{0}) & \to \cdots \\ \varphi(\mathtt{1}) & \to \varphi^2(\mathtt{1}) & \to \varphi^3(\mathtt{1}) & \to \cdots \\ \vdots & \vdots & \vdots & \\ \varphi(\ell-\mathtt{1}) & \to \varphi^2(\ell-\mathtt{1}) & \to \varphi^3(\ell-\mathtt{1}) & \to \cdots \end{array} \tag{4.12}$$

The arrows denote expansion by φ and separate the table into "parts." We call the part of the table to the left of all arrows the *first part*, the part between the leftmost arrows and the second leftmost arrows, the *second part*, and so on. The n-th part of the table for φ is the same as the first part of such a table for φ^n.

Since φ is bijective, every column of φ contains all letters from the alphabet, $\mathcal{A}$, of φ. We may thus think of columns as permutations of $\mathcal{A}$. If H and K are two columns, we define the composition of the columns as a composition of permutations in the following way:

$$(HK)_i := K_{H_i}. \tag{4.13}$$

It is instructive to interleave an example: Let Δ denote the bijective substitution given by:

$$\Delta \begin{cases} \mathtt{1} \to \mathtt{121} \\ \mathtt{2} \to \mathtt{312} \\ \mathtt{3} \to \mathtt{233} \end{cases}$$

Then the table 4.12 becomes:

$$\begin{array}{l} \mathtt{121} \to \mathtt{121}\ \mathtt{312}\ \mathtt{121} \to \cdots \\ \mathtt{312} \to \mathtt{233}\ \mathtt{121}\ \mathtt{312} \to \cdots \\ \mathtt{233} \to \mathtt{312}\ \mathtt{233}\ \mathtt{233} \to \cdots \end{array}$$

We will call two adjacent columns considered as a single unit a *bicolumn*. If we consider the first bicolumn in the diagram, $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 3 \end{bmatrix}$, we say that it expands, or substitutes, to

$$\begin{bmatrix} 1 & 2 & \mathbf{1} & \mathbf{3} & 1 & 2 \\ 2 & 3 & \mathbf{3} & \mathbf{1} & 2 & 1 \\ 3 & 1 & \mathbf{2} & \mathbf{2} & 3 & 3 \end{bmatrix}$$

where the bold part represents the central bicolumn of the expansion, $B = \begin{bmatrix} 1 & 3 \\ 3 & 1 \\ 2 & 2 \end{bmatrix}$. Each row of the expansion is created by expanding each row of A by Δ.

Switching back to the general case, consider the first part of 4.12. Let P be the first column of a bijective substitution and let Q denote the last column. If we begin with a bicolumn $C = [C_l\ C_r]$ from some part, we can compute the bicolumn at the center of the expansion of C using composition. Denote the center of the expansion of C by $D = [D_l\ D_r]$. Specifically, $D_l = C_l Q$ and $D_r = C_r P$.

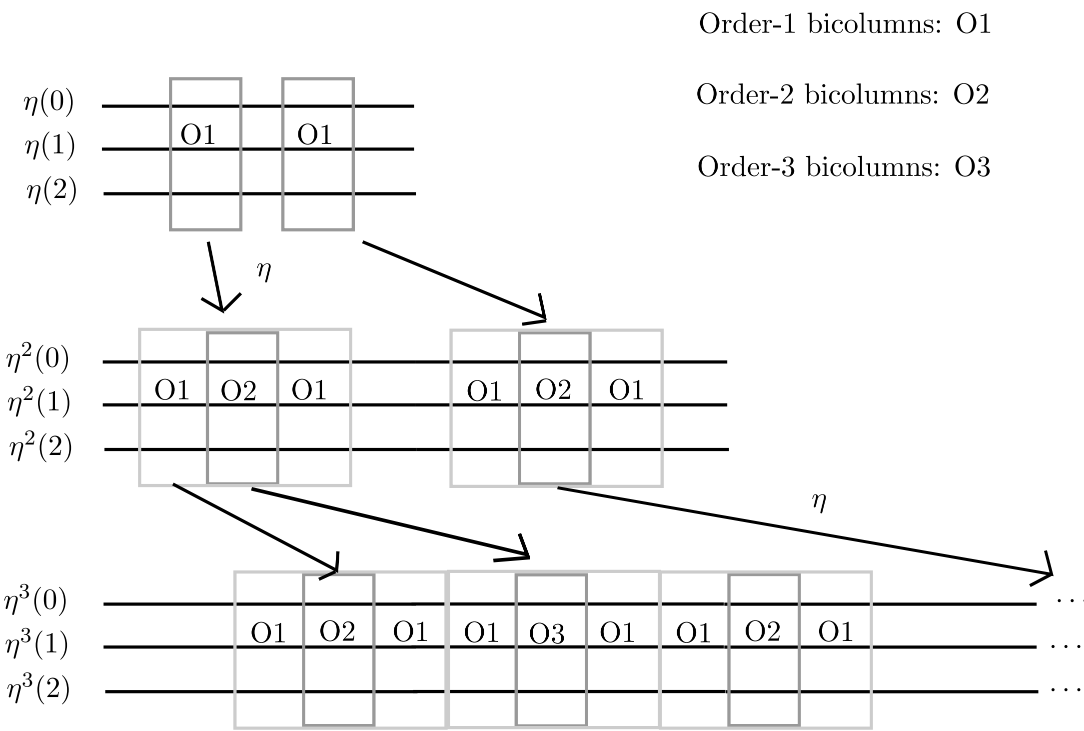
For Δ, $P = [1\ 3\ 2]^T$ and $Q = [1\ 2\ 3]^T$. We see that, in fact, $B_l = A_l Q$ and $B_r = A_r P$. If we were to expand again, we would have center bicolumn $[A_l Q^2\ A_r P^2]$, and so forth.

The expansions of bicolumns of bijective substitutions also contain columns which are not the center bicolumns of expansions. We put the following equivalence relation on bicolumns: two bicolumns are *equivalent* if a permutation of rows changes one into the other. In other words, equivalent bicolumns contain the same set of length-2 words as rows. The bicolumns which are not at the center of an expansion are all equivalent to the bicolumns in the first part of 4.12. This follows from the fact that the expansion of any column of a bijective substitution is just the first part of 4.12 with rows re-ordered by the permutation associated with the column.

We place a hierarchy on the equivalence classes of bicolumns: bicolumns equivalent to the bicolumns of the first part of 4.12 have order-1. The bicolumns equivalent to the central part of the expansion of an order-m bicolumn is said to have order-$(m+1)$. There is nothing preventing a class of bicolumns from having multiple orders, but for a typical bijective substitution, not all bicolumns have order-1.

The first part of 4.12 contains, by definition, all bicolumns of order-1 up to equivalence. For this reason, the second part of the table contains all bicolumns of order-2 up to equivalence at the centers of the expanded bicolumns, but it also contains all order-1 columns in the other locations. Thus the third part of the table will include all order-1, order-2, and order-3 bicolumns, and by induction the n-th part of the table includes, up to equivalence, all bicolumns of all orders n or less. See Figure 4.3 for a visualization.

Figure 4.3: Visualization of bicolumns.



Since there are only a finite number of possible bicolumns, this leads to a natural definition:

4.14 Definition (Exhausted Substitution)**.** Let φ be a bijective substitution. The substitution φ is *exhausted at power* n if the n-th part of the table 4.12 contains, up to equivalence, no more bicolumns than the $(n-1)$-st part.

If φ is exhausted at power n, then the n-th part of the table 4.12 for φ contains all bicolumns up-to a permutation of their rows. Let $|R|$ denote the order of the permutation R, i.e., if E is the identity permutation, then $|R| = \min\{n : R^n = E,\ n \in 1, 2, \cdots\}$.

4.15 Lemma (Exhaustion Lemma)**.** *If φ is a bijective substitution with first column P and final column Q in the first part (corresponding to φ^1), then for all $n \geq \operatorname{lcm}(|P|, |Q|)$, φ is exhausted at power n.*

Proof. If $t = \operatorname{lcm}(|P|, |Q|)$ then for any column R, $RP^t = R$ and $RQ^t = R$. If $[I|J]$ is an order-1 bicolumn built from columns I and J, then the center of the t-th expansion of

$[I|J]$ is $[IQ^t|JP^t] = [I|J]$, but this is an order $t+1$ bicolumn by definition. Therefore, the order-$(t+1)$ bicolumns of φ are simply the order-1 bicolumns of φ, and those bicolumns are already present in the t-th part of the table. Additionally, the order-x bicolumns are the order-y bicolumns if $x \equiv y \mod t$. □

The following proposition improves Lemma 4.10 in the case of bijective substitutions. Since bicolumns in the first part of a substitution's table must be repeated later, the same is true about inconsistencies. Thus consistency for a power is equivalent to consistency for the first power:

4.16 Proposition. *If φ is bijective, a 2-block code b on X_φ is almost-consistent if and only if the corresponding 1-block code c is consistent for φ_h.*

Proof. ($\Leftarrow$) If c is consistent for φ_h then it is almost-consistent and by Lemma 4.10, b is almost-consistent for φ.

($\Rightarrow$) If b is almost-consistent for φ then c is almost-consistent for φ_h by Lemma 4.10. By Lemma 3.15 this means c is consistent for φ_h^n for some n. Let $t = \text{lcm}(|P|, |Q|)$ as in Lemma 4.15, and let $s = nt$. If $c([ij]) = c([kl])$ then we wish to show $c(\varphi_h[ij]) = c(\varphi_h[kl])$. If we assume to the contrary that $\varphi_h([ij])_z \neq \varphi_h([kl])_z$ for some $0 \leq z \leq |\varphi| - 1$, then there are two cases, depending on whether or not z corresponds to a bicolumn of the first part of φ or to the final column of φ_h.

In the first case, $0 \leq z < |\varphi| - 1$. In this case if $z' = (z+1) \cdot |\varphi_h|^s - 1$ (the center of expansion of $[ij]$ and $[kl]$), then $\varphi_h^s([ij])_{z'} \neq \varphi_h^s([kl])_{z'}$ since $t|s$.

In the case $z = |\varphi|$, we are considering the final column which shares the property that it is governed by the permutations P and Q. The permutations act on the set of letters φ_h considered as a bicolumn (with additional rows), and the action must also be the identity map for power t. If $z'' = |\varphi|^s$, we again have $\varphi_h^s([ij])_{z''} \neq \varphi_h^s([kl])_{z''}$ since $t|s$.

Both cases are contradictions since c is consistent for φ_h^s as a power of φ_h^n. This implies c is consistent, so we are done. □

4.17 Corollary. *All aperiodic factors of a basic bijective substitution φ are, up to topological conjugacy, consistent 1-block codes of φ_h.*

Proof. This is immediate from Proposition 4.16 and Corollary 4.5. □

By "*identifying a column*" we mean mapping all letters in the column (or length-2 words in a bicolumn) to the same letter.

4.18 Corollary. *Let φ be a basic bijective substitution which is exhausted at power m. All aperiodic pure-discrete spectrum factors of φ are, up to topological conjugacy, consistent 1-block codes of φ_h that identify all letters in some column of $(\varphi^m)_h$.*

Equivalently, they are almost-consistent 2-block codes that map all rows of some bicolumn of φ^m to the same letter.

Proof. Let the block code c generate such a factor. By 4.17, c can be assumed to be a consistent 1-block code of φ_h. The resulting substitution φ_c has discrete spectrum, so there is an n such that $(\varphi_c)^n$ has a coincidence, i.e., there is a k such that $\#\{((\varphi_c)^n(i))_k : i \in c(\mathcal{A})\} = 1$ where $\mathcal{A}$ is the alphabet of φ_c. But $\#\{((\varphi_c)^n(i))_k : i \in c(\mathcal{A})\} = \#\{(c((\varphi_h)^n(i))_k : i \in c(\mathcal{A})\} = \#\{(c((\varphi^n)_h(i))_k : i \in c(\mathcal{A})\} = 1$. Thus c identifies a column of $(\varphi^n)_h$. Since φ is exhausted at power m, the letters in the column which are identified, up to a permutation of rows, also appear in a column of φ^m. □

After some practice with these constructions, a slightly improved version of this corollary appears as Theorem 4.24.

4.4 Examples and Further Reduction

We now have all the tooling necessary to easily show the existence of bijective substitutions with no aperiodic pure-discrete spectrum factors. The reader may verify that all examples given in this section are primitive, aperiodic, injective, and pure ($h = 1$), and have prime-length 3 or 5. They are also bijective and thus left-resolvable for all powers. Therefore, by Theorems 3.33 and 4.3, all aperiodic subshift factors may be found, up to topological conjugacy, by inspecting almost-consistent 2-block codes. Since the block-codes are almost-consistent, they induce substitution factors. We examine primarily what happens when we assume one of these factors has pure-discrete spectrum, i.e., it has a coincidence at some power by Theorem 2.61.

4.19 Example (Main Example)**.** Let η be the length-5 substitution on 3 letters given by:

$$\eta \begin{cases} \texttt{a} \to \texttt{aabaa} \\ \texttt{b} \to \texttt{bcabb} \\ \texttt{c} \to \texttt{cbccc} \end{cases}$$

The first column is $P = [a\ b\ c]^T$ and the last column is $Q = [a\ b\ c]^T$. Now P and Q are the identity permutations so that $t = \text{lcm}(|P|, |Q|) = \text{lcm}(1, 1) = 1$. Thus by Lemma 4.15, η is exhausted at power 1. The bicolumns of η are

$$\begin{bmatrix} aa \\ bc \\ cb \end{bmatrix}, \begin{bmatrix} ab \\ ca \\ bc \end{bmatrix}, \begin{bmatrix} ba \\ ab \\ cc \end{bmatrix}, \text{and} \begin{bmatrix} aa \\ bb \\ cc \end{bmatrix}.$$

Note that the word ac does not appear in $\mathcal{L}(X_\eta)$. Now to construct η_h we find the expansions of all length-2 blocks in $\mathcal{L}(X_\eta)$ as in 4.7:

$$\begin{cases} \texttt{aa} \to \texttt{aabaa|a} \\ \texttt{ab} \to \texttt{aabaa|b} \\ \texttt{ba} \to \texttt{bcabb|a} \\ \texttt{bb} \to \texttt{bcabb|b} \\ \texttt{bc} \to \texttt{bcabb|c} \\ \texttt{ca} \to \texttt{cbccc|a} \\ \texttt{cb} \to \texttt{cbccc|b} \\ \texttt{cc} \to \texttt{cbccc|c} \end{cases} \tag{4.20}$$

Creating the higher-order substitution and re-labeling blocks:

$$\eta_h \begin{cases} 1 \to 12411 \\ 2 \to 12412 \\ 4 \to 67254 \\ 5 \to 67255 \\ 6 \to 67256 \\ 7 \to 86997 \\ 8 \to 86998 \\ 9 \to 86999 \end{cases} \tag{4.21}$$

Call the new alphabet (for η_h) $\mathcal{B}$. Let c be a block code on X_η which induces an aperiodic pure-discrete spectrum factor. By Corollary 4.18, we can assume c is a consistent 1-block code on η_h which identifies some column of η_h^1. Thus c must identify one of the following sets of letters: $\{1, 2, 4, 5, 6, 7, 8, 9\}$, $\{1, 6, 8\}$, $\{2, 6, 7\}$, $\{4, 2, 9\}$, $\{1, 5, 9\}$. If the final column is identified then the result is the trivial (one-point) subshift. We are looking for aperiodic factors, so consider what happens if a column containing the letters $\{1, 6, 8\}$ is identified by the map c. Since c is consistent, this forces $c(\eta_h(1)) = c(\eta_h(6)) = c(\eta_h(8))$, i.e.,

$$c(12411) = c(67256) = c(86998). \tag{4.22}$$

Therefore, $c(2) = c(7) = c(6)$, $c(4) = c(2) = c(9)$, and $c(1) = c(5) = c(9)$ also, but then $\#c(\mathcal{B}) = 1$. Identifying $\{2, 6, 7\}$, $\{4, 2, 9\}$, or $\{1, 5, 9\}$ also leads to (4.22). Therefore, the identification of any column of η_h is an inconsistent or trivializing 1-block code and thus η has no aperiodic pure-discrete spectrum factors.

4.23 Example. This example of a bijective substitution which does have a 2-block code onto a substitution with coincidence demonstrates how the process may be simplified further.

$$q' \begin{cases} \mathtt{0} \to \mathtt{022} \\ \mathtt{1} \to \mathtt{100} \\ \mathtt{2} \to \mathtt{211} \end{cases}$$

We have $P = [0\ 1\ 2]^T$ and $Q = [2\ 0\ 1]^T$, so $|P| = 1$ and $|Q| = 3$. This implies q'^3 is exhausted, but, in fact, q is exhausted at power 2.

$$q'^2 \begin{cases} \mathtt{0} \to \mathtt{022\ 211\ 211} \\ \mathtt{1} \to \mathtt{100\ 022\ 022} \\ \mathtt{2} \to \mathtt{211\ 100\ 100} \end{cases}$$

The possible bicolumns are

$$\begin{bmatrix} 02 \\ 10 \\ 21 \end{bmatrix}, \begin{bmatrix} 22 \\ 00 \\ 11 \end{bmatrix}, \text{ and } \begin{bmatrix} 12 \\ 20 \\ 01 \end{bmatrix}.$$

For a general bijective substitution, a convenient way to find the alphabet of the higher-order substitution is to look at the length-2 words of the bicolumns in the exhausted substitution. Now compute the higher-order expansions for q'^2:

$$\mathsf{00} \to \mathsf{022\ 211\ 211}\ |\mathsf{0}$$

$$\mathsf{01} \to \mathsf{022\ 211\ 211}\ |\mathsf{1}$$

$$\mathsf{02} \to \mathsf{022\ 211\ 211}\ |\mathsf{2}$$

$$\mathsf{10} \to \mathsf{100\ 022\ 022}\ |\mathsf{0}$$

$$\mathsf{11} \to \mathsf{100\ 022\ 022}\ |\mathsf{1}$$

$$\mathsf{12} \to \mathsf{100\ 022\ 022}\ |\mathsf{2}$$

$$\mathsf{20} \to \mathsf{211\ 100\ 100}\ |\mathsf{0}$$

$$\mathsf{21} \to \mathsf{211\ 100\ 100}\ |\mathsf{1}$$

$$\mathsf{22} \to \mathsf{211\ 100\ 100}\ |\mathsf{2}$$

The higher-order substitution is:

$$(q'^2)_h \begin{cases} [00] = \mathtt{a} \to \mathtt{cii\ hef\ hed} \\ [01] = \mathtt{b} \to \mathtt{cii\ hef\ hee} \\ [02] = \mathtt{c} \to \mathtt{cii\ hef\ hef} \\ [10] = \mathtt{d} \to \mathtt{daa\ cig\ cig} \\ [11] = \mathtt{e} \to \mathtt{daa\ cig\ cih} \\ [12] = \mathtt{f} \to \mathtt{daa\ cig\ cii} \\ [20] = \mathtt{g} \to \mathtt{hee\ dab\ daa} \\ [21] = \mathtt{h} \to \mathtt{hee\ dab\ dab} \\ [22] = \mathtt{i} \to \mathtt{hee\ dab\ dac} \end{cases}$$

This example illustrates several general facts about this process for bijective substitutions: First, the higher-order substitution of any bijective substitution has at least one letter of the form $[ab]$ where a is any letter of the original alphabet – in fact every bicolumn has this property. Therefore differences, insofar as allowing discrete factors, arise from bicolumns sharing or not sharing these higher-order letters. When we identify a column which is not the final column, we are always forced to identify each other non-final column as well through an equation like (4.22). Our verification can therefore be made quicker by starting with this assumption. Thus in this example we assume $c = d = h$, $i = a = e$, $i = a = e$, $h = c = d$, $e = i = a$, $f = g = b$, $h = c = d$, $h = c = d$, and $e = i = a$. We reduce these into disjoint classes: $\{c, d, h\}$, $\{i, a, e\}$, $\{f, g, b\}$. These classes may also be found by examining the list of possible bicolumns. If there is only one class, we are done, the substitution not admitting a pure-discrete factor. Otherwise, we start with a block code which maps each letter to its class. If the resulting substitution is consistent, we have found the pure-discrete spectrum factor. If not, it may still be a composition of this map with a further identification of letters – we are forced to make an identification across columns to achieve consistency.

The final column of the higher-order substitution of any bijective substitution contains all letters: each row's final letter corresponds to a length-2 word which is the center of expansion of the letter, or length-2 word, which defines the row. Every possible length-2 word is contained in the expansion of a length-2 word by the equivalence of bicolumns in the proof of Lemma 4.15. Containing every word, the final column must be bijective on the higher-order alphabet as it has the maximum number of letters. Thus the final column of the higher-order substitution corresponding to the exhausted substitution should never be identified by our procedure lest we trivialize the subshift.

Back to our example, we must check for consistency when applying the 1-block code $b(\{c, d, h\}) = X$, $b(\{i, a, e\}) = Y$, $b(\{f, g, b\}) = Z$ to the higher order substitution. We can check either the exhausted substitution or p'_h itself by Corollary 4.17. Choosing p'_h itself:

$$q'_h \begin{cases} \texttt{a} \to \texttt{cig} \\ \texttt{b} \to \texttt{cih} \\ \texttt{c} \to \texttt{cii} \\ \texttt{d} \to \texttt{daa} \\ \texttt{e} \to \texttt{dab} \\ \texttt{f} \to \texttt{dac} \\ \texttt{g} \to \texttt{hed} \\ \texttt{h} \to \texttt{hee} \\ \texttt{i} \to \texttt{hef} \end{cases}$$

becomes,

$$\begin{cases} \texttt{X} \to \texttt{XYY} \\ \texttt{Y} \to \texttt{XYZ} \\ \texttt{Z} \to \texttt{XYX} \end{cases}$$

which is the pure-discrete spectrum factor of q'. Now, we note that the example may be deceiving: by our argument we have forced the columns of $(q'_h)^2$, except for the final column, to be identified. This, in-turn implies that the columns of q'_h are identified since they must appear in the exhausted power. Furthermore, the identification which we have forced may or may not give the non-final columns different letters – this depends on the relationships present between columns that may only be visible in the exhausted substitution.

We recapitulate our discussion by strengthening 4.18:

4.24 Theorem. *A basic bijective substitution φ has an aperiodic pure-discrete spectrum factor if and only if identifying all non-final columns of φ_h can yield a consistent substitution (possibly needing further identifications between the new letters).*

Thus a basic bijective substitution has an aperiodic discrete spectrum factor if and only if it has a substitution factor of the same length with coincidence in every column except the final one. Another example is the substitution q'' given by:

$$q'' \begin{cases} \mathtt{0} \to \mathtt{020} \\ \mathtt{1} \to \mathtt{101} \\ \mathtt{2} \to \mathtt{212} \end{cases}$$

which factors onto the substitution on two letters:

$$\begin{cases} \mathtt{A} \to \mathtt{ABA} \\ \mathtt{B} \to \mathtt{ABB} \end{cases}$$

The factor has a coincidence in the first two columns as expected.

4.25 Example (Queffélec's Length-3 Bijective Substitution)**.** We now use Theorem 4.24 to analyze the bijective substitution appearing in [26]:

$$q \begin{cases} \mathtt{0} \to \mathtt{021} \\ \mathtt{1} \to \mathtt{100} \\ \mathtt{2} \to \mathtt{212} \end{cases} \tag{4.26}$$

The higher-order substitution is

$$q_h \begin{cases} [\mathtt{00}] \to [\mathtt{02}][\mathtt{21}][\mathtt{10}] \\ [\mathtt{02}] \to [\mathtt{02}][\mathtt{21}][\mathtt{12}] \\ [\mathtt{10}] \to [\mathtt{10}][\mathtt{00}][\mathtt{00}] \\ [\mathtt{12}] \to [\mathtt{10}][\mathtt{00}][\mathtt{02}] \\ [\mathtt{21}] \to [\mathtt{21}][\mathtt{12}][\mathtt{21}] \end{cases}$$

By Theorem 4.24, if there is a pure discrete spectrum factor then the factor requires at least $b([\mathtt{02}]) = b([\mathtt{10}]) = b([\mathtt{21}])$ and $b([\mathtt{21}]) = b([\mathtt{00}]) = b([\mathtt{12}])$. But this identifies all letters. Thus q has no aperiodic pure-discrete spectrum factors.

The example of q suggests a special case of Theorem 4.18 where all letters are immediately conflated by the identification of columns. We recall that if a substitution is exhausted at power 1, then all possible length-2 words appear among the order-1 bicolumns – those in the first part.

4.27 Corollary. *If a basic bijective substitution φ has a length-2 word ab appearing in every bicolumn of its first part, and φ is exhausted at power 1, then φ does not have an aperiodic pure-discrete spectrum factor.*

Using Corollary 4.27 we can tell the following basic bijective substitution ξ has no aperiodic pure-discrete spectrum factors by inspection:

$$\xi \begin{cases} \mathsf{0} \to \underline{\mathsf{03}}\mathsf{110} \\ \mathsf{1} \to \mathsf{12}\underline{\mathsf{03}}\mathsf{1} \\ \mathsf{2} \to \mathsf{2}\underline{\mathsf{03}}\mathsf{22} \\ \mathsf{3} \to \mathsf{312}\underline{\mathsf{03}} \end{cases}$$

The word 03 appears in every non-final bicolumn of the first part and $t = \mathrm{lcm}\{|P|, |Q|\} = 1$, so we are done (assuming we have already checked that ξ is basic and bijective!). It is important to remember that the condition of Corollary 4.27 is not necessary for the result as other examples have shown. For q the word was 21, and the reader may verify that q is exhausted at power 1 even though $t = 2$. Corollary 4.27 is a convenient way to generate examples with no aperiodic pure-discrete spectrum factors.

These examples show that the following previously known fact *cannot* be extended to the three-letter case. We give here an illustrative proof of the fact based on our theory of basic bijective substitutions, but an ad-hoc proof identifying all non-final columns allows the removal of the prime length requirement. As previously mentioned, the removal of this requirement for our results is a possible future direction of this work.

4.28 Proposition. *Every basic bijective substitution on two letters has a discrete spectrum substitution factor given by the block code $b(01) = b(10) = A$, $b(00) = b(11) = B$.*

Proof. Let φ be basic and bijective. If φ is length-2 then it must be the Morse substitution since we assume $\varphi(0)$ begins with 0. This proposition is known for the Morse substitution (see Example 4.11), so assume φ has length greater than two. The fact that φ is aperiodic and bijective implies it has bicolumns

$$\begin{bmatrix} 01 \\ 10 \end{bmatrix} \text{ and } \begin{bmatrix} 00 \\ 11 \end{bmatrix}$$

exactly, up to permutation of rows. Thus by Theorem 4.24 the only possible 2-block code we must check is $b(01) = b(10) = A$, $b(00) = b(11) = B$ since further identification would result in the trivial subshift. Additionally, if b is inconsistent then it must be inconsistent in the last column. Now, remembering 4.7, $b(\varphi([xy])) = \cdots [x'y]$ where $x' = x$ if $Q = [0\ 1]^T$ and $x' \neq x$ if $Q = [1\ 0]^T$. The code b makes the final column consistent in either case.

□

Chapter 5

Diffraction Spectra

5.1 Induced Sequences and Correlation Measures

We start this chapter by reverting to a more general setting. Let T be homeomorphism of a compact metric space X, and let μ be an invariant measure for T. We consider the dynamical system (X, T, μ). The operator U on $L^2(X, \mu)$ defined by $U_T f = f \circ T$ is unitary. For any L^2 function f we define the sequence $\hat{\sigma}_f(k) = \langle U_T^k f, f \rangle$ for $k \in \mathbb{Z}$. This sequence is positive definite, so by the Herglotz Theorem $\hat{\sigma}_f$ is the Fourier transform of a positive measure on the circle, σ_f, which is called the *spectral measure* of U_T corresponding to f. We denote the set of eigenvalues of U_T by Λ.

For any point $u \in X$, function $g : X \to \mathbb{C}$, and $k \in \mathbb{Z}$, formally define the sequence

$$\hat{\zeta}_g^u(k) := \lim_{N \to \infty} \frac{1}{N} \sum_{n<N} g\left(T^{n+k} u\right) \overline{g\left(T^n u\right)}.$$

This sum may not necessarily converge, but recall that if the system (X, T, μ) is ergodic then μ-a.e. x is generic. If it is uniquely ergodic then every point is generic. If v is generic, then $\hat{\zeta}_g^u(k)$ is actually the Fourier transform of the spectral measure corresponding to g:

5.1 Proposition. *If v is a generic point for (X, T, μ) and g is continuous then $\hat{\zeta}_g^v(k)$ is guaranteed to converge and $\hat{\zeta}_g^v(k) = \hat{\varphi}_g(k)$.*

Proof. Since $(g \circ T^k)\overline{(g)}$ is continuous, Definition 2.46 implies

$$\hat{\zeta}_g^v(k) = \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} (g \circ T^k(T^n v))\overline{g(T^n v)} = \int_X g \circ T^k \cdot \overline{g}\, d\mu = \langle U_T^k g, g\rangle = \hat{\sigma}_g(k).$$

□

We also use the notation $\zeta_g^v(k) = \varphi_g(k)$ for the corresponding spectral measures.

In the case that (X, T, μ) arises from a primitive substitution, $u \in X$, and π_0 is the projection $\pi_0(x) := x_0$, we have

$$\hat{\zeta}_{\pi_0}^u(k) = \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} u_{n+k}\overline{u_n}.$$

We assume the alphabet $\mathcal{A} \subset \mathbb{C}$. The corresponding measure $\zeta_{\pi_0}^u$ is called the *correlation measure* of the sequence u. The discrete spectrum of the correlation measure is a subset of Λ since $\sigma_{\pi_0} << \sigma_{\max}$. Inspired by the Morse and Toeplitz substitutions, we would like to create a topological factor of (X, T, μ) onto a subshift which contains in a correlation measure point masses for every $\lambda \in \Lambda$. Such a factor must necessarily be a block coding of the original system given the Curtis-Lyndon-Hedlund theorem.

Again, let (X, T, μ) be a dynamical system. We demonstrate how we will build a subshift factor system, adjusting the usual perspective of block-codes: for any $u \in X$ and finite-valued function $g : X \to \mathbb{C}$ define the *sequence induced by g*

$$u^g := \cdots g(T^{-2}u)g(T^{-1}u).g(u)g(Tu)g(T^2u)\cdots .$$

Consider the d values achieved by g to be letters from a finite alphabet. Denote the mapping $\phi : X \to Y^*$ defined by $x \mapsto x^g$ where Y^* is the full d-shift with the product topology. In the cases we will consider, g is continuous, so, as a product of continuous maps, ϕ is continuous. Since X is compact, $Y = \text{range}(\phi)$ is closed in Y^*, and it is invariant under the left-shift operator S by construction. Thus (Y, S) is a topological dynamical system, and since $\phi \circ T = S \circ \phi$, it is a factor of Y and a subshift.

The following shows how such a map affects the correlation of a sequence:

5.2 Lemma. $\hat{\zeta}_g^u(k) = \hat{\zeta}_{\pi_0}^{u^g}(k)$.

Proof.

$$\begin{aligned}
\hat{\zeta}_g^u(k) &= \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} g\left(T^{n+k}u\right) \overline{g\left(T^n u\right)} \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \pi_0\left(T^{n+k}u^g\right) \overline{\pi_0\left(T^n u^g\right)} \\
&= \hat{\zeta}_{\pi_0}^{u^g}(k).
\end{aligned}$$

□

5.2 Approximation on Cantor Systems

If X is a Cantor space then (X, T, μ) is called a *Cantor system.*

5.3 Lemma. *If X is a Cantor space then a continuous function $f : X \to \mathbb{C}$ can be uniformly approximated by a finitely-valued continuous function f^* such that $||f^*||_\infty \leq ||f||_\infty$.*

Proof. Take any $\epsilon > 0$. Since we assume X is compact, f is uniformly continuous and there is a δ such that for all $x, y \in X$, $d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon$. Assume $\mathcal{C}$ is a basis of clopen sets for X. For each $x \in X$ choose a set $C_x \in \mathcal{C}$ such that $C_x \subset \text{Ball}(x, \epsilon)$. Then $\mathcal{B}'' := \{C_x : x \in X\}$ is an open cover of X. Let $\mathcal{B}'$ be a finite subcover of $\mathcal{B}''$. Finally, let $\mathcal{B}$ be the finite partition of X induced by $\mathcal{B}'$ by taking intersections and complements. Since the elements of B' are clopen the partition elements remain open. Choose a representative x_B for each $B \in \mathcal{B}$. Define $f^*(x) = f(x_B)$ for all $x \in B$. Then for any $x \in X$ if $x \in B \in \mathcal{B}$ then $d(x, x_B) < \delta$ so $|f(x) - f^*(x)| = |f(x) - f(x_B)| < \epsilon$.

Furthermore $|f^*(x)| = |f(x_B)|$ for $x_B \in X$, so f^* uniformly approximates f and $||f^*||_\infty \leq ||f||_\infty$. The function f^* is continuous since the preimage of any point is a clopen set from B. □

If X is a subshift then we identify a finitely-valued continuous function of the type from Lemma 5.3 with a block code. We are now ready for a result.

5.4 Proposition. *Let (X, T, μ) be a Cantor system with generic point $u \in X$. For any $\epsilon > 0$, there is a continuous, finite-valued function $f^* : X \to \mathbb{C}$ which induces a topological factor of the system onto a subshift via $x \mapsto x^{f^*}$ such that the correlation measure of u^{f^*} contains a point mass for each eigenvalue λ satisfying $\sigma_{\max}\{\lambda\} > \epsilon$. In particular, the new system retains any finite number of eigenvalues of U_T.*

Proof. By Fraczek's Theorem (2.58) there is a continuous function $f : X \to \mathbb{C}$ such that $\sigma_f = \sigma_{\max}$. Since f is continuous and u is generic we have that $\sigma_f = \zeta_f^u$.

Let $M = ||f||_\infty$ which is finite since X is compact. By the lemma, choose a finitely-valued continuous function f^* such that $||f - f^*||_\infty < \frac{\epsilon}{2M}$ and $||f^*||_\infty \leq M$. Then for any $k \in \mathbb{Z}$,

$$
\begin{aligned}
&\left|\hat{\zeta}_f^u(k) - \hat{\zeta}_{f^*}^u(k)\right| \\
&= \left|\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} f(T^{n+k}u)\overline{f(T^n u)} - \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} f^*(T^{n+k}u)\overline{f^*(T^n u)}\right| \\
&= \left|\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \left(f(T^{n+k}u)\overline{f(T^n u)} - f^*(T^{n+k}u)\overline{f^*(T^n u)}\right)\right| \\
&\leq \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \left|f(T^{n+k}u)\overline{f(T^n u)} - f^*(T^{n+k}u)\overline{f^*(T^n u)}\right| \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} |f(T^{n+k}u)\overline{f(T^n u)} - f^*(T^{n+k}u)\overline{f(T^n u)} \\
&\qquad\qquad + f^*(T^{n+k}u)\overline{f(T^n u)} - f^*(T^{n+k}u)\overline{f^*(T^n u)}| \\
&\leq \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} (|f(T^{n+k}u)\overline{f(T^n u)} - f^*(T^{n+k}u)\overline{f(T^n u)}| \\
&\qquad\qquad + |f^*(T^{n+k}u)\overline{f(T^n u)} - f^*(T^{n+k}u)\overline{f^*(T^n u)}|) \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \left(\left|(f(T^{n+k}u) - f^*(T^{n+k}u))\overline{f(T^n u)}\right| + \left|f^*(T^{n+k}u)(\overline{f(T^n u)} - \overline{f^*(T^n u)})\right|\right) \\
&\leq \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} \left(\frac{\epsilon}{2M} M + \frac{\epsilon}{2M} M\right) \\
&= \epsilon.
\end{aligned}
$$

Now take any $\lambda \in \mathbb{T}$ such that $\sigma_{\max}\{\lambda\} \geq \epsilon$. By Wiener's Theorem we know

$$\sigma_{\max}\{\lambda\} = \zeta_f^u\{\lambda\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{|k|\leq N} \hat{\zeta}_f^u(k) e^{ik\lambda}.$$

We can now compute

$$\begin{aligned} |\zeta_f^u\{\lambda\} - \zeta_{f^*}^u\{\lambda\}| &= \left| \lim_{N\to\infty} \frac{1}{2N+1} \sum_{|k|\leq N} \hat{\zeta}_f^u(k) e^{ik\lambda} - \lim_{N\to\infty} \frac{1}{2N+1} \sum_{|k|\leq N} \hat{\zeta}_{f^*}^u(k) e^{ik\lambda} \right| \\ &= \left| \lim_{N\to\infty} \frac{1}{2N+1} \sum_{|k|\leq N} (\hat{\zeta}_f^u(k) - \hat{\zeta}_{f^*}^u(k)) e^{ik\lambda} \right| \\ &\leq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{|k|\leq N} \left| \hat{\zeta}_f^u(k) - \hat{\zeta}_{f^*}^u(k) \right| \leq \epsilon. \end{aligned}$$

Hence since $\zeta_f^u\{\lambda\} = \sigma_{\max}\{\lambda\} > \epsilon$, we have $\zeta_{f^*}^u\{\lambda\} > 0$, but also by Lemma 5.2, $\zeta_{\pi_0}^{u^{f^*}} = \zeta_{f^*}^u\{\lambda\}$. That is, the correlation measure of u^{f^*} has a point mass at λ. We identify f^* with a factor onto a new subshift containing the point u^{f^*}. □

5.3 The Substitution Case

We now narrow our focus even further to the case of a subshift generated by a primitive substitution, but we do not assume constant-length as in previous chapters. Such a system is a uniquely ergodic Cantor system. In fact, since constant-length substitutions must have rational eigenvalues by (2.61), the result we obtain will not pertain them.

5.5 Lemma. *A periodic subshift contains rational eigenvalues.*

Proof. Let (X, T, μ) be a periodic with $X = \{x_1, x_2, \ldots, x_p\}$ where $x_n = T^{(n-1 \mod p)} x_1$. Consider a number of the form $\lambda = e^{2\pi i (k/p)}$ where $k \in \{1, 2, \ldots, p\}$ and the function f defined by $f(x_n) = \lambda^n$. For $n \in \{1, 2, \ldots, p-1\}$ we have $f(Tx_n) = f(x_{n+1}) = \lambda^{n+1} = \lambda\lambda^n = \lambda f(x_n)$. Also, $f(Tx_p) = f(x_1) = \lambda = \lambda f(x_p)$ since $f(x_p) = \lambda^p = e^{2\pi i k} = 1$. Therefore λ is a rational eigenvalue of U_T. □

We will use this recent result by of Fabian Durand.

5.6 Theorem (Durand, [9] and [10])**.** *Substitution subshifts have a finite number of non-periodic subshift factors up to topological conjugacy.*

5.7 Theorem. *A primitive substitution subshift* (X, T, μ) *with all eigenvalues irrational has topological factors which contain the entire point spectrum and have arbitrarily many eigenvalues visible to the correlation measure.*

Proof. Pick any $\epsilon > 0$. We will try to capture all eigenvalues with $\sigma_{\max}\{\lambda\} > \epsilon$ on the correlation measure of a factor. Primitivity guarantees that (X, T, μ) is uniquely ergodic. Let Λ denote the group of eigenvalues of U_T. Take any $u \in X$. We usually think of u as being a (two-sided) fixed point of the substitution, although this is not necessary as any point is generic. For each natural number $n > 0$ we denote by Λ_n the set of eigenvalues $\{\lambda : \sigma_{\max}\{\lambda\} > 1/n\}$. By the 5.4 there is a finitely valued function $f_n : X \to \mathbb{C}$ so that u^{f_n} has a correlation measure showing all eigenvalues in Λ_n. Each f_n generates a subshift factor of (X, T, μ), call it (Y_n, S_n).

We know the eigenvalues of (Y_n, S_n), as the eigenvalues of a topological factor, must be a subset of Λ. Therefore, they must also be irrational. In particular, since the correlation measure of u^{f_n} is absolutely continuous with respect to the maximal spectral type of U_{S_n}, U_{S_n} must have all the eigenvalues Λ_n in the correlation measure, although it could have more.

Now, none of the factors on the list $\{(Y_n, S_n)\}$ are periodic since, by the lemma, if one was periodic it would have to have a rational eigenvalue which must be inherited from (X, T, μ). Therefore, by the result of Durand, the list $\{(Y_n, S)\}$ contains a finite number of systems up to topological conjugacy. Therefore, one of these must appear, up to conjugacy, an infinite number of times. Call it (Y, S). Let p be an integer such that $(Y_p, S) \equiv (Y, S)$ and $1/p < \epsilon$. Therefore Λ_p contains all the desired eigenvalues on its correlation measure.

It only remains to show that (Y_p, S) contains all dynamical eigenvalues of (X, T, μ). Take any $\lambda \in \Lambda$. Let m satisfy $\sigma_{\max}\{\lambda\} > 1/m$ so that $\lambda \in \Lambda_m$. Now, let ℓ be the smallest integer such that $\ell \geq m$ and $(Y_\ell, S) \equiv (Y, S)$. Clearly $\lambda \in \Lambda_m \subset \Lambda_\ell$ which implies λ is an eigenvalue of (Y, S) and consequently, of (Y_p, S). Thus the factor (Y_p, S) contains the entire point spectrum of (X, T, μ). □

It is interesting that for a primitive substitution with fixed-point u, it is not necessary to use $f_{\max}$ in the proof of the proposition. There are a countable number of eigenvalues, and we know the eigenfunctions have constant modulus and are continuous (c.f. [26]). Let $\{g_n\}_{n\in\mathbb{N}}$ be the normalized eigenfunctions, then the function $f_{\mathrm{d}} = \sum_{n=0}^{\infty} \frac{g_n}{2^n}$ converges uniformly and is thus continuous. The spectral measure $\sigma_{f_{\mathrm{d}}} = \zeta^u_{f_{\mathrm{d}}}$ contains only the discrete spectrum and is the sum of the delta measures σ_{g_n}. Using f_d in place of $f_{\max}$ yields the same result for this case without use of Fraczek's theorem. The difference between the two resulting approximations would be interesting to analyze, but unfortunately there is no convenient description of $f_{\max}$.

Chapter 6

Spectrum Visualization

6.1 Description of Method

In the course of this work, it has been useful to have some tools available for visualization of the spectral problem. We create some useful images using the following technique:

First, compute a long word, u, for the substitution φ. This is usually just $\varphi^k(0)$ for some large k. Next, approximate the diffraction measure's Fourier coefficients, $\hat{\sigma}_{\pi_0}(k) = \lim_{N\to\infty} \frac{1}{N}\sum_{n<N} u_{n+k}\overline{u_n}$, by choosing N to be a sufficiently small fraction of $|u|$ and limiting k. Our implementation uses $N = |u|/2$ and $k < |u|/2$. Finally, we create a rough picture of the cumulative distribution function of σ_{π_0} via

$$f(x) = \int_0^x \sum_{k=-n}^{n} \hat{\sigma}_{\pi_0}(k)\, e^{(2\pi i k t)}\, dt$$

for $x \in [0, 1]$ and $n = |u|/2$. We subjectively find that without integrating, the distribution function exhibits much Fourier ringing. We vary k depending on the substitution so that it yields $|u| \approx 40,000$. More advanced programming techniques, more patience, or better hardware would allow the approximations to have even more resolution, but we find that very detailed results are possible in real-time. This allows the visualizations to be a useful tool. We remind the reader that the diffraction spectrum of a substitution is pure-discrete if and only if the maximal spectral type of the system is pure-discrete ([26]). If the maximal spectral type is mixed, then it is unclear what the diffraction spectrum will contain.

6.2 Gallery

What follows is a gallery of graphs of cumulative distributions $f(x)$ generated using this technique for well-known substitutions and the main example. The horizontal axis shows the x-range of $[0, 1]$ unless otherwise specified.

In 6.1 and 6.2 we see a clear visualization that the spectrum of the Morse substitution is missing from its diffraction spectrum but is present in its factor's. Additionally, if a substitution has pure-discrete spectrum it generates a distinctive step-function like the Toeplitz substitution in 6.2. This is a one-dimensional version of a "sharp" diffraction. The lack of discontinuities in 6.1 is one of the primary motivations for the investigations in this dissertation since the Toeplitz substitution subshift is an a.e. one-to-one extension of the maximal equicontinuous factor it shares with the Morse system. We have shown the existence of substitutions not admitting such intermediate factors.

Figure 6.1: The Morse Substitution, κ: $0 \to 01$, $1 \to 10$.

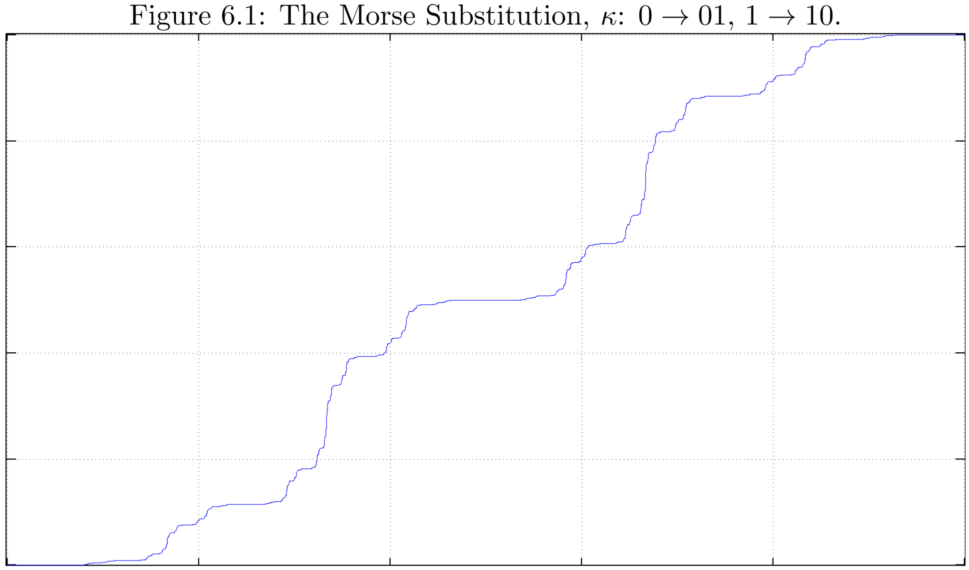
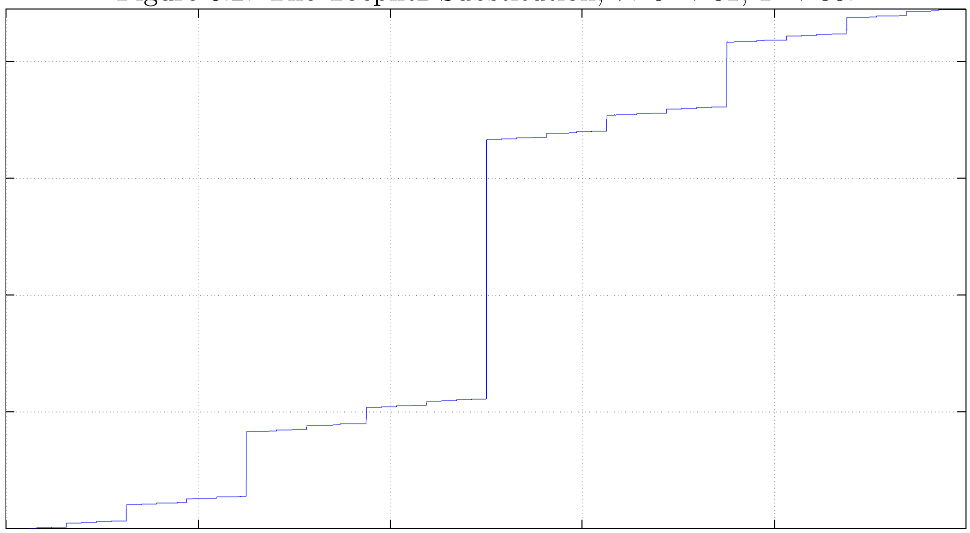


Figure 6.2: The Toeplitz Substitution, τ: $0 \to 01$, $1 \to 00$.



In 6.3 we see the main example, η, used to prove there are bijective substitutions with no pure-discrete spectrum factors. An interesting feature of this correlation spectrum approximation is that it appears continuous but has derivative approaching infinity where its eigenvalues should be. This feature is especially prominent at 0. Figure 6.4 shows a close-up of another eigenvalue. In the Morse substitution's visualization (6.1), points corresponding to eigenvalues appear to have derivative zero. The Morse substitution's diffraction measure is investigated in detail in [3].

Figure 6.3: The Main Example Substitution, η : $0 \to 00100$, $1 \to 12011$, $2 \to 21222$.

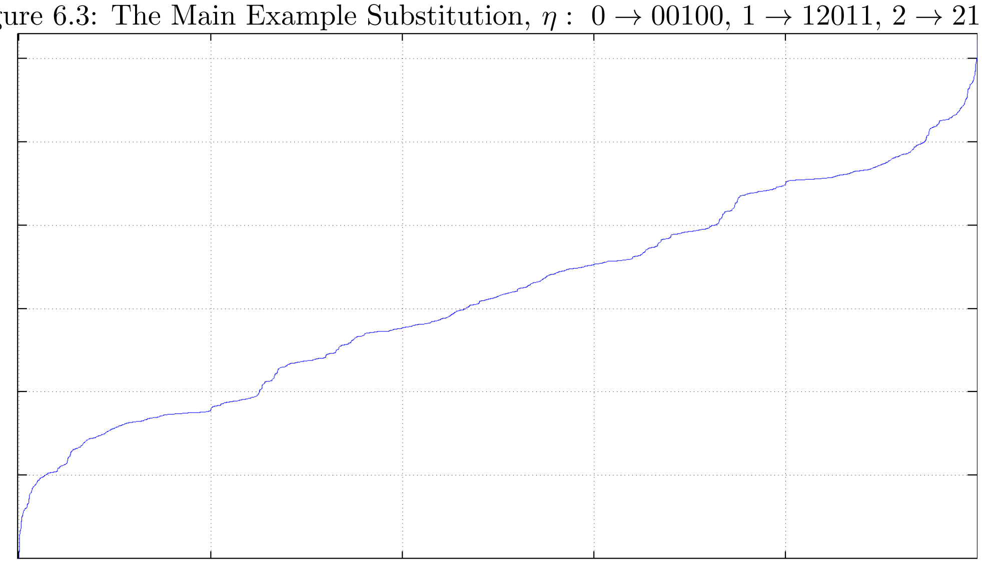


Figure 6.4: A close-up of the interval $[0.1995, 0.2005]$ in 6.3 using $|u| = 5^8 = 390625$.

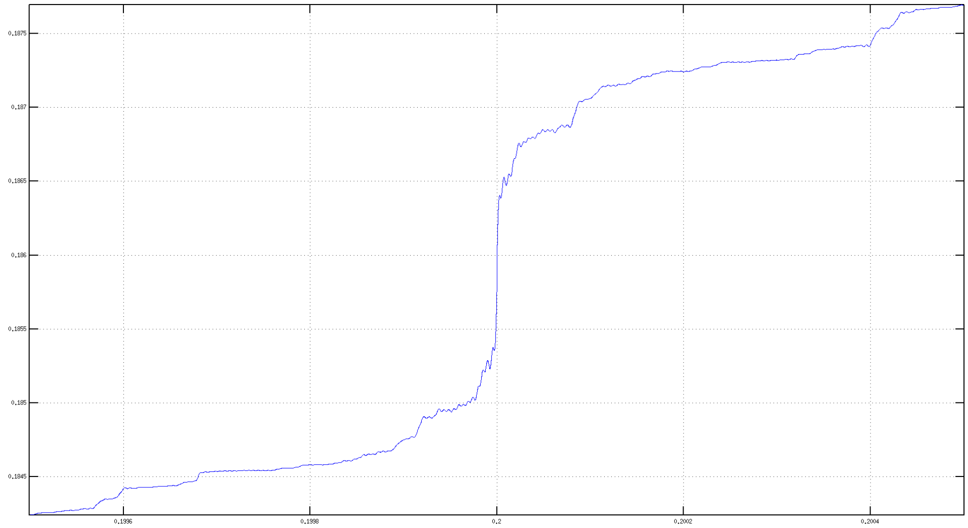


Figure 6.5: Length-3 Bijective, $q'' :\ 0 \to 020,\ 1 \to 101,\ 2 \to 212.$

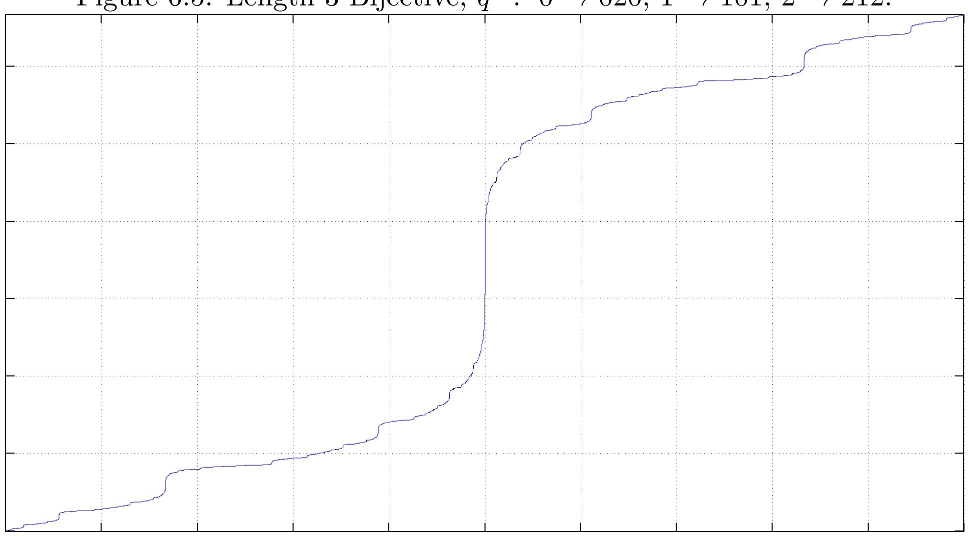


Figure 6.5 shows a length-3 bijective substitution, q'' which does have a pure-discrete spectrum factor. Like in the Morse substitution the diffraction spectrum appears to have derivative zero where the eigenvalues are missing, and there are non-eigenvalue positions which appear to have derivative approaching infinity. The factor has 2-letters and is shown in 6.6.

Figure 6.6: Substitutive 2-block code of 6.5, $0 \to 010$, $1 \to 011$.

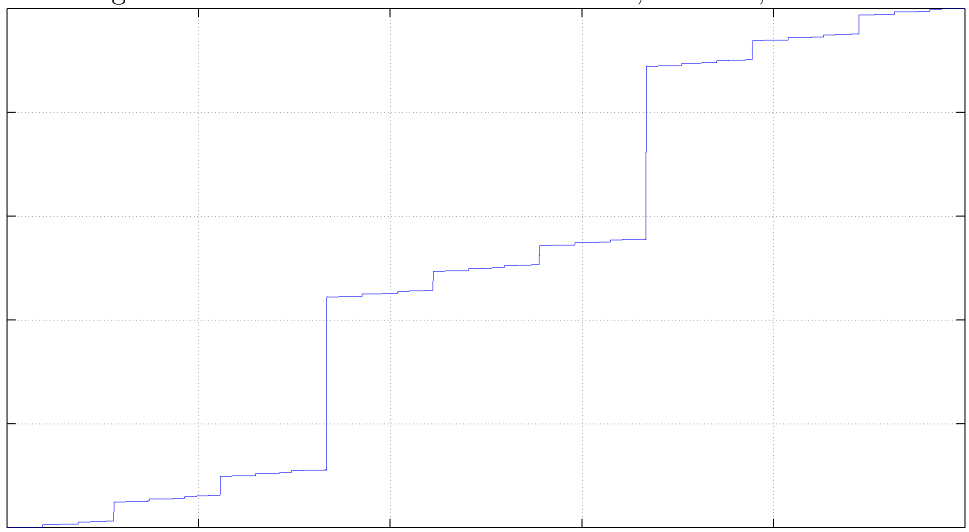
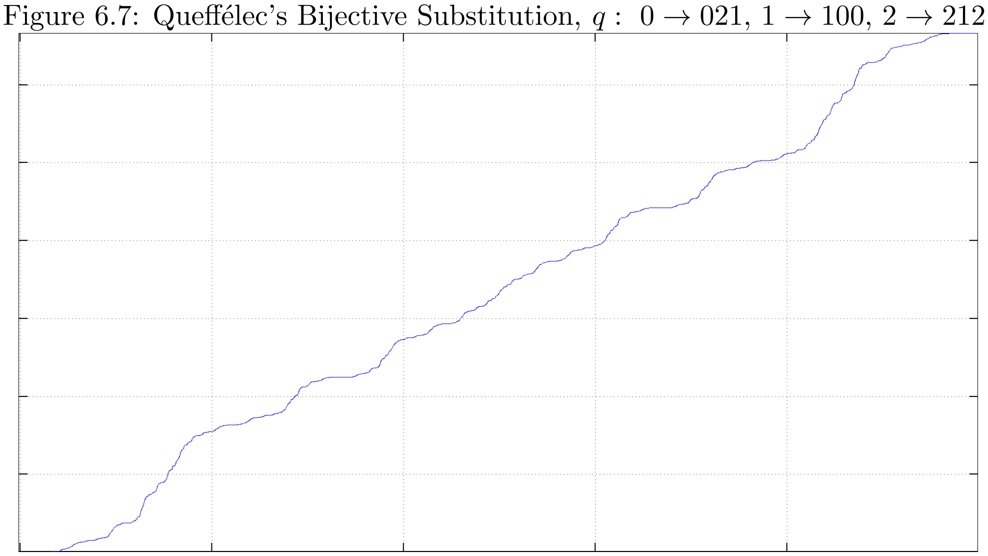


Figure 6.7: Queffélec's Bijective Substitution, $q:\ 0 \to 021$, $1 \to 100$, $2 \to 212$.



The diffraction spectrum shown in 6.7 is from the substitution q which appears in [26]. As it is length-3 and height-1, our theorems apply. The substitution is exhausted at power two and does not have a discrete spectrum factor.

Figure 6.8: Length-3 Bijective, $q' : \ 0 \to 022,\ 1 \to 100,\ 2 \to 211.$

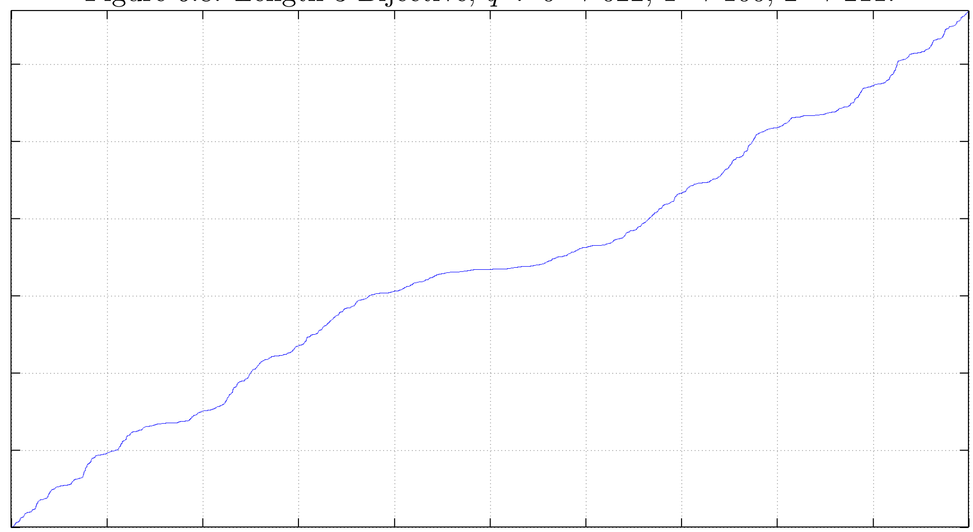
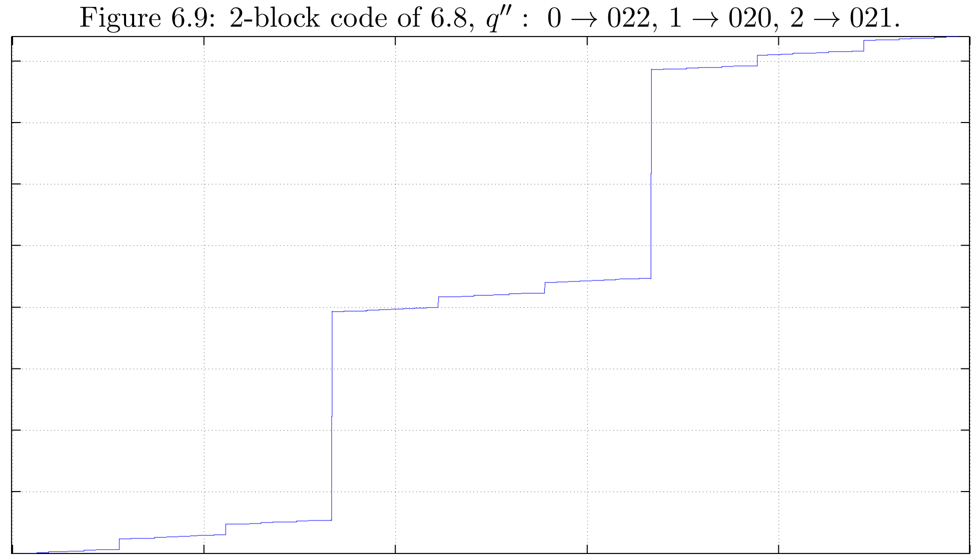


Figure 6.9: 2-block code of 6.8, $q'' : \ 0 \to 022,\ 1 \to 020,\ 2 \to 021.$



Yet another length-3 bijective substitution, q', is shown in 6.8. This one does have a discrete spectrum factor on three letters (6.9), but has diffraction spectrum resembling Queffélec's substitution 6.7 in that it resembles a distribution absolutely continuous with respect to Lebesgue (c.f. [26] for Queffélec's discussion).

Figure 6.10: The Fibonacci Substitution, ϕ: $0 \to 01$, $1 \to 0$.

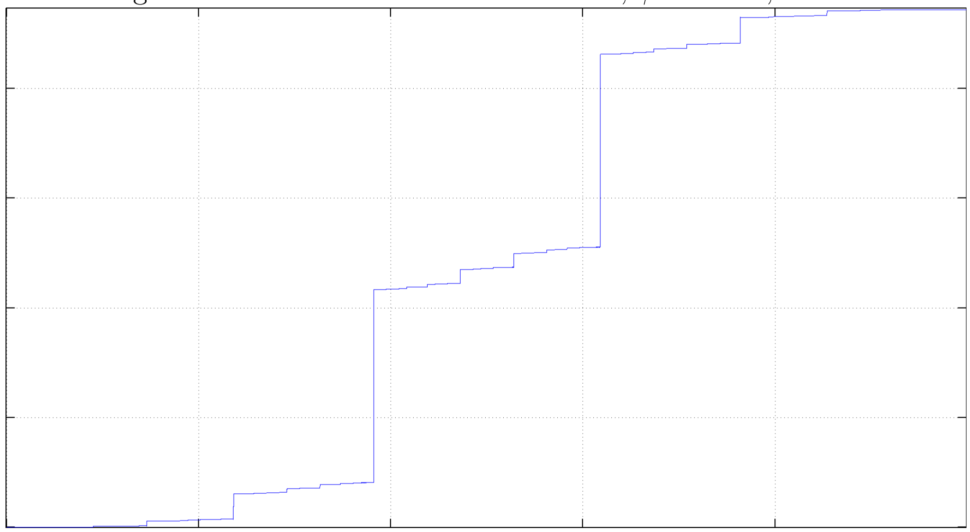
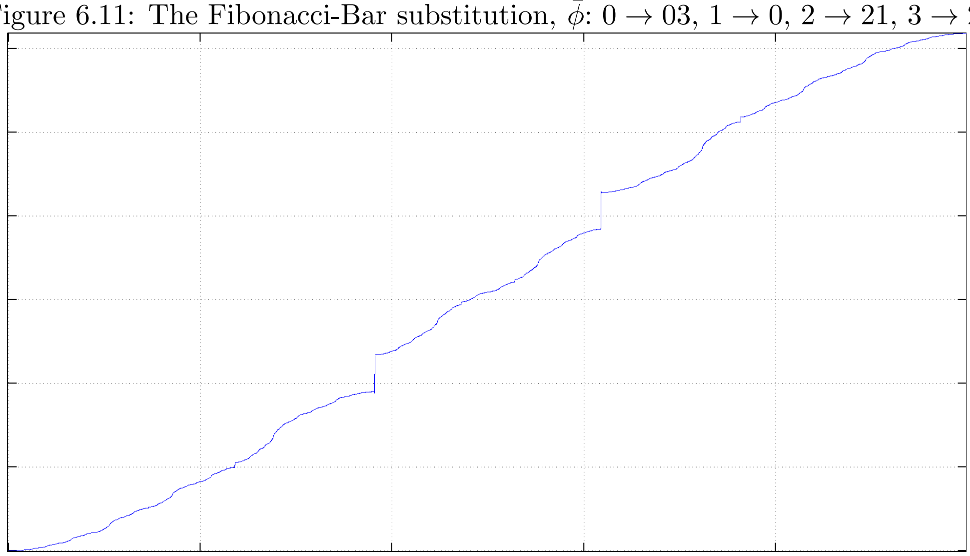


Figure 6.11: The Fibonacci-Bar substitution, $\bar{\phi}$: $0 \to 03$, $1 \to 0$, $2 \to 21$, $3 \to 2$.



The figures 6.10 and 6.11 show the Fibonacci substitution, ϕ, and an extension which is inverted by a 1-block code called the *Fibonacci-Bar*, $\bar{\phi}$, substitution (it is typically written as $0 \to 0\bar{1}$, $1 \to 0$, $\bar{0} \to \bar{0}1$, $\bar{1} \to \bar{0}$). The diffraction spectrum for the Fibonacci-Bar substitution is clearly mixed, and both are the only examples we have shown with irrational eigenvalues.

Figure 6.12: Dekking's Substitution, $0 \to 001$, $1 \to 11100$.

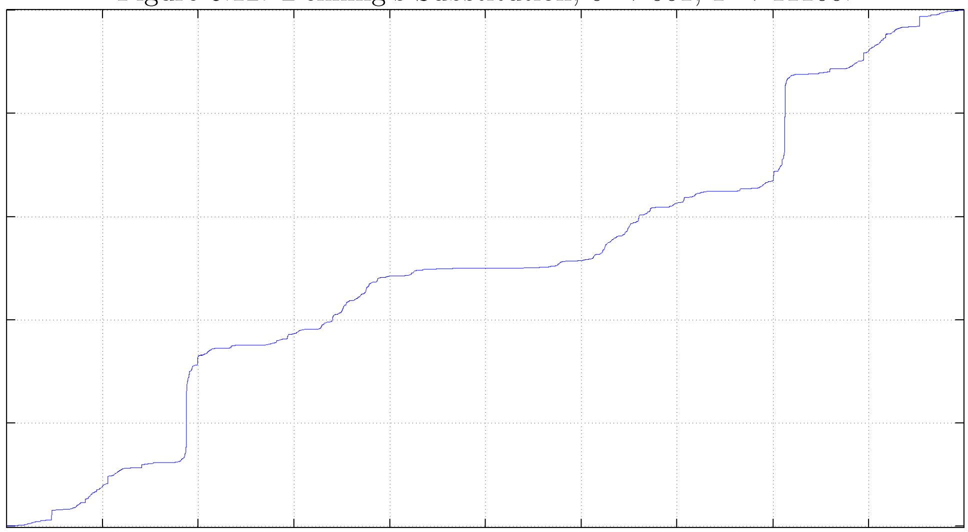


Figure 6.13: A Weak-Mixing Substitution, $0 \to 011$, $1 \to 00$.

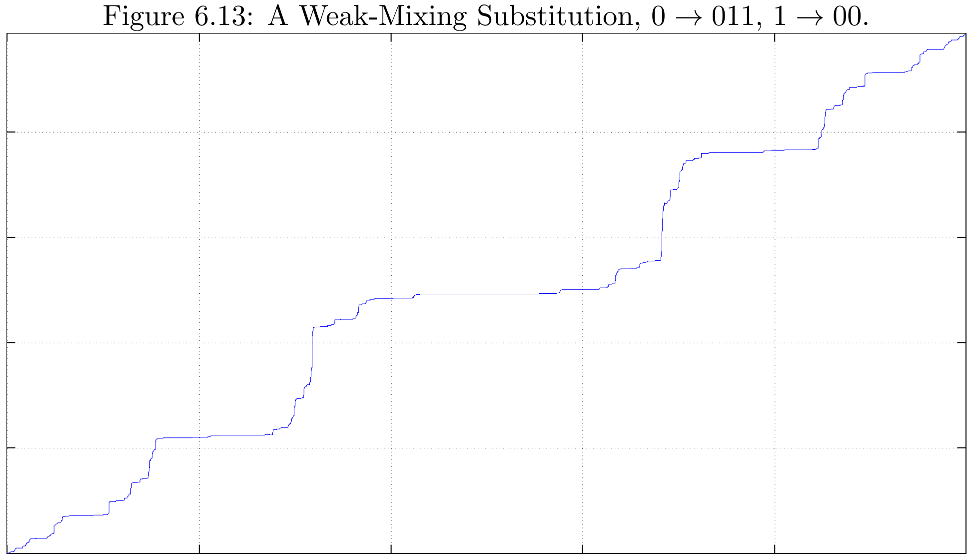
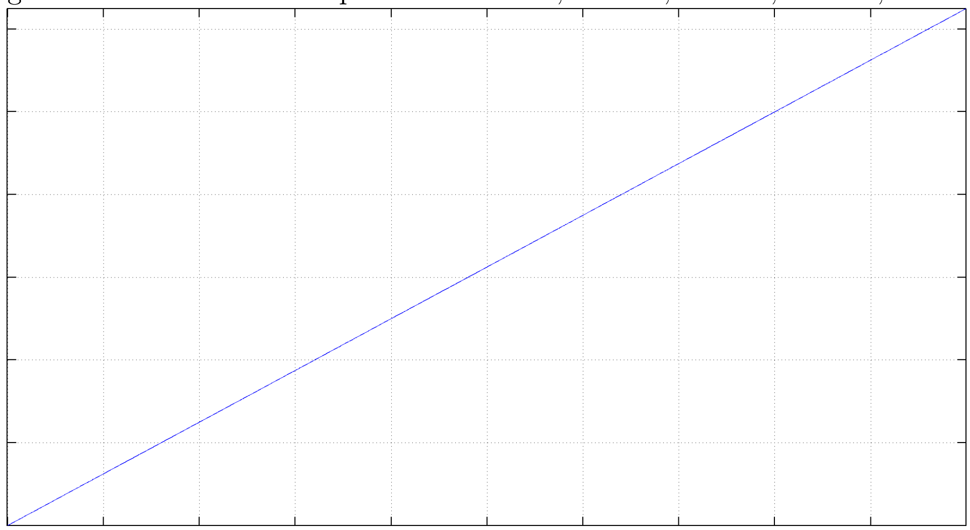


Figure 6.12 was introduced in 1978 by Dekking and Keane in [8] as an example of a weak-mixing substitution. The substitution in 6.13 is another weak-mixing substitution more recently considered by Kenyon, Sadun, and Solomyak in [18]. They discuss weak-mixing of two-letter substitutions in detail.

Figure 6.14: The Rudin-Shapiro Substitution, $0 \to 02$, $1 \to 32$, $2 \to 01$, $3 \to 31$.



Finally, the Rudin-Shapiro substitution is the "substitution on blocks" which generates the Rudin-Shapiro sequence. It is known for having Lebesgue diffraction spectrum.

Index

References

[1] V. M. Alexeyev. Existence of a bounded function of the maximal spectral type. *Ergodic Theory and Dynamical Systems*, 2(3-4):259–261, 1982.

[2] Jean-Paul Allouche and Jeffrey Shallit. *Automatic sequences: theory, applications, generalizations*. Cambridge University Press, 2003.

[3] Michael Baake and Uwe Grimm. Theory of aperiodic order: A mathematical invitation. Unpublished manuscript received by E. A. Robinson; 483 pages, 2013.

[4] Marcy Barge and Beverly Diamond. Coincidence for substitutions of Pisot type. *Bulletin de la Société Mathématique de France*, 130(4):619, 2002.

[5] Francois Blanchard, Fabien Durand, and Alejandro Maass. Constant-length substitutions and countable scrambled sets. *Nonlinearity*, 17(3):817–833, 2004.

[6] Michael Brin and Garrett Stuck. *Introduction to Dynamical Systems*. Cambridge University Press, 2002.

[7] F. M. Dekking. The spectrum of dynamical systems arising from substitutions of constant length. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 41(3):221–239, 1978.

[8] F. M. Dekking and M. Keane. Mixing properties of substitutions. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 42(1):23–33, 1978.

[9] Fabien Durand. Linearly recurrent subshifts have a finite number of non-periodic subshift factors. *ArXiv e-prints*, July 2008.

[10] Fabien Durand, Bernard Host, and Christian Skau. Substitutional dynamical systems, Bratteli diagrams and dimension groups. *ArXiv preprint arXiv:0807.3621*, 2008.

[11] Sébastien Ferenczi, Christian Mauduit, and Arnaldo Nogueira. Substitution dynamical systems: algebraic characterization of eigenvalues. In *Annales scientifiques de l'Ecole normale supérieure*, volume 29, pages 519–534. Paris: Gauthiers-Villars,[1864]-, 1996.

[12] N. Pytheas Fogg. *Substitutions in dynamics, arithmetics and combinatorics*, volume 1794 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002. Edited by V. Berthé, S. Ferenczi, C. Mauduit and A. Siegel.

[13] Krzysztof Fraczek. On a function that realizes the maximal spectral type. *Studia Mathematica*, 124(1):1–7, 1997.

[14] Michael Hollander and Boris Solomyak. Two-symbol pisot substitutions have pure discrete spectrum. *Ergodic Theory and Dynamical Systems*, 23(02):533–540, 2003.

[15] B. Host. Valeurs propres des systemes dynamiques définis par des substitutions de longueur variable. *Ergodic Theory Dynam. Systems*, 6(4):529–540, 1986.

[16] B. Host and F. Parreau. Homomorphisms entre systèmes dynamiques définis par substitutions. *Ergodic Theory Dynam. Systems*, 9(3):469–477, 1989.

[17] Yitzhak Katznelson. *An introduction to harmonic analysis*. Cambridge University Press, 2004.

[18] Richard Kenyon, Lorenzo Sadun, and Boris Solomyak. Topological mixing for substitutions on two letters. *Ergodic Theory and Dynamical Systems*, 25(06):1919–1934, 2005.

[19] Douglas Alan Lind and Brian H Marcus. *An introduction to symbolic dynamics and coding*. Cambridge University Press, 1995.

[20] Mieczysław Mentzen. Invariant sub-σ-algebras for substitutions of constant length. *Studia Mathematica*, 92(3):257–273, 1989.

[21] Brigitte Mossé. Properties of words and recognizability of fixed points of a substitution. *Theoretical Computer Science*, 99(2):327–334, 1992.

[22] Brigitte Mossé. Reconnaissabilité des substitutions et complexité des suites automatiques. *Bulletin de la Société Mathématique de France*, 124(2):329–346, 1996.

[23] James R Munkres. *Topology.* Prentice Hall Upper Saddle River, 2000.

[24] William Parry. *Topics in Ergodic Theory.* Cambridge University Press, 1981.

[25] Martine Queffélec. *Substitution dynamical systems—spectral analysis*, volume 1294 of *Lecture Notes in Mathematics.* Springer-Verlag, New York, first edition, 1987.

[26] Martine Queffélec. *Substitution dynamical systems—spectral analysis*, volume 1294 of *Lecture Notes in Mathematics.* Springer-Verlag, New York, second edition, 2010.

[27] H. L. Royden. *Real Analysis.* Prentice Hall, 1988.