Monte Carlo Methods for Optimizing Finance for Development

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Dedication

To both my grandmothers, Z and L. The former taught me about the beauty I can see, the latter about the one I can feel. It is in the deep mysteries of mathematics, where I hope to find a little bit of both.

My father taught me who I am supposed to be. The measure of one’s character is not in their appearance, nor their happiness can be seen through their smile. He is wise.

My mother taught me about who I could become. Let the complicated become simple when faced with the will to get it right. She is inimitable.

With every day, my brother gives me the security that allows me to follow my dreams safe in the knowledge that someone better watches over our future.

Family keeps me going.

Thank you.
Acknowledgments

This thesis is a continuation of the work by a dear friend and mentor economist Dr. Tito Cordella, adviser to the Acting Chief Economist of the World Bank Group, Dr. Shanta Devarajan, Senior Director of the Development Economics department. Even though my day-job is far from DEC, I was honored to have benefitted from his leadership, encouragement and guidance in making this project come to life. Truth be told, proving he is right, yet again, was not hard only because he is so often correct.

The role of my thesis director, Dr. Tanya Apanasovich cannot be overestimated. Our unending meetings every Thursday are quite literally the heart and soul of this piece. Her enormous patience, candid feedback and open dialogue are what empowered me to deliver.

This work would not have been possible without my thesis committee. I am thankful to Tito, Tanya and Dr. Emre Barut, Assistant Professor of Statistics also at George Washington University for their advice, feedback and support.

This acknowledgement would not be complete without recognizing the unsung heroes that allowed this work to exist in the first place: The World Bank Group team task leaders, who, through their work, enabled mine. There would be no project data without projects, in fact there would be no World Bank Group without you. It is my hope to one day count myself among you.

If there are any valuable insights within these pages they certainly came through my interactions with Tito, Tanya, Emre and the amazing staff of the World Bank Group. All shortcomings are the exclusive fault of the author.
Abstract of Thesis

Monte Carlo Methods for Optimizing Finance for Development

Using Monte Carlo methods, we revisit the normative economic model proposed by economist Tito Cordella to understand the tensions between maximizing private financing and optimizing financing for development. To focus on his comments about the new World Bank Group Cascade Approach’s proposed sequence of interventions, we check the robustness of his main conclusions, particularly his scenarios where subsidies should be considered before reforms to maximize social welfare gains from having the private sector finance international development projects.

More generally, Cordella claims that the cascade approach might not be the optimal sequence of interventions to maximize a project’s social welfare impact because the efficiency of reforms in capturing externalities is a ‘hidden’ societal cost. Multiple scenarios exist where it would be better to first subsidize the private sector before enacting inefficient reforms that waste large portions of a project’s positive externalities to make them profitable enough to receive private sector financing.

His conclusions use a normative economic model that depends on strict independent and uniform distributional assumptions for project characteristics, these results when replicated using Monte Carlo method simulations prove robust when considering (1) non-independent distributions for project characteristics, (2) several different distribution assumptions for project characteristics, and even (3) real-world financial and economic returns data for both public and private sector World Bank Group projects.

**Keywords:** G20 Hamburg Action Plan; multilateral development banks (MDB); World Bank Group (WBG); Maximizing finance for development (MFD); International Development; cost-benefit analysis (CBA); crowding-in private sector finance; mobilizing and catalyzing private capital; decision-making; Hamburg principles
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<td>WBG:</td>
<td>World Bank Group (a collection of 5 different institutions: MIGA, IFC, IDA, IBRD, ICSID)</td>
</tr>
<tr>
<td>IDA:</td>
<td>International Development Association (makes grants to the 77 poorest countries in the world)</td>
</tr>
<tr>
<td>IBRD:</td>
<td>International Bank for Reconstruction and Development</td>
</tr>
<tr>
<td>WB:</td>
<td>World Bank (not to be confused with the larger group. Refers to IBRD / IDA)</td>
</tr>
<tr>
<td>IFC:</td>
<td>International Finance Corporation (the private sector arm of the World Bank Group)</td>
</tr>
<tr>
<td>MIGA:</td>
<td>Multilateral Investment Guarantee Agency</td>
</tr>
<tr>
<td>DEC:</td>
<td>Development Economics (World Bank Group department)</td>
</tr>
<tr>
<td>IEG:</td>
<td>Independent Evaluation Group</td>
</tr>
<tr>
<td>OECD:</td>
<td>Organization for Economic Co-operation and Development</td>
</tr>
<tr>
<td>ICR:</td>
<td>Implementation Completion Report</td>
</tr>
<tr>
<td>ICRR:</td>
<td>Implementation Completion Report (ICR) Review</td>
</tr>
<tr>
<td>XPSRs:</td>
<td>Expanded Project Supervision Report</td>
</tr>
<tr>
<td>FY:</td>
<td>Fiscal Year (Jul 1 to Jun 30 at the World Bank Group)</td>
</tr>
<tr>
<td>ERR:</td>
<td>Economic Rate of Return</td>
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<tr>
<td>IRR:</td>
<td>Internal Rate of Return</td>
</tr>
<tr>
<td>FRR:</td>
<td>Financial Rate of Return</td>
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<tr>
<td>KS:</td>
<td>Kolmogorov-Smirnov statistic</td>
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Thesis Statement

When considering how to crowd-in private sector finance to increase levels of private investment in support of the world’s 2030 Sustainable Development Agenda, the cascade’s ‘reforms before subsidies’ approach might not be the optimal sequence of interventions to maximize a project’s social welfare impact, because the efficiency of reforms in capturing externalities is a ‘hidden’ societal cost. Multiple scenarios where it would be better to first subsidize the private sector before enacting inefficient reforms that waste large portions of a project’s positive externalities to make them profitable enough to receive private sector financing.

Derived from a normative economic model that assumes project characteristics are independent and strictly uniform, these results when replicated using Monte Carlo method simulations prove to also be robust when considering (1) non-independent distributions for project characteristics, (2) several different distributional assumptions for project characteristics, and even (3) real-world financial and economic returns data from World Bank Group project in both the private and public sectors.

Chapter 1. Overview

Using Monte Carlo methods, this thesis’ main purpose is to check the robustness of the conclusions derived from economist Tito Cordella’s normative economic model analyzing the newly created World Bank Group (WBG) cascade framework (Cordella, 2018). The research will begin by contextualizing and laying-out the economic normative model proposed in Cordella’s ‘Optimizing Finance for Development’ paper.

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2 Cordella, “Optimizing finance for development”
Then, the main results discussed in the paper will be replicated using Monte Carlo methods. After this replication using Monte Carlo simulation of his normative model, the focus will shift to (1) changing the independence and distributional assumptions made in the original model, and (2) bringing real data from World Bank Group projects, both public and private, into the model and revising its assumptions accordingly.

Chapter 2 contextualizes the “Optimizing Finance for Development” paper, rehearses brief descriptions of approaches used to tackle objectives 1 and 2, and quickly discusses previous work. Chapter 3 describes the methodology of the different analyses used throughout the thesis. All of those play a role in the main thrust of this thesis: using Monte Carlo methods to simulate normative economic models. Chapter 4 goes over the main ideas in the ‘cascade space’ normative economic model for development projects proposed in the ‘Optimizing Finance for Development’ paper and reviews its main conclusions (Cordella, 2018).

After, chapter 5 replicates the main results from the original normative model using Monte Carlo simulations. Chapter 6 focuses on objective 1; it tests distributional and independence assumptions made in the original model to explore other scenarios and their related conclusions. The expectation is to test the robustness of the main conclusions from Cordella’s original paper related to the tradeoffs between the different sequences, particularly those related to doing reforms or subsidies first.

Chapter 7 begins the work on objective 2. It first describes the Economic Rate of Return dataset for World Bank Group public and private sector projects. Then, it explains a proposed relationship between the dataset and the original model. Chapter 8 extends the model to 3 dimensions to correctly accommodate the observed data. With the
relationships and new normative model described, chapter 9 uses non-parametric Kolmogorov–Smirnov tests to fit the model to the observed data. Descriptions of the resulting parametric distributions will follow. A special chapter discussing the issues of fitting a beta distribution to the data follows the work and ultimately concludes with some of the limitations of the model thanks to the censoring mechanism found in the real data. In chapter 11, the newly derived distributions inform the simulations of the modified normative model. The chapter concludes with a further robustness checks to verify how well Cordella’s conclusions hold-up to assumptions derived from real data, focusing on his conclusions about tradeoff on the efficiency of reforms.

Finally, chapter 11 will summarize the findings of chapters 5-6 for objective 1, and chapters 7-11 for objective 2. Namely, it will review the results from these different robustness checks, and then expand upon this thesis’ contributions and implications for the ‘optimizing finance for development’ paper, and the ‘cascade approach’ more broadly. The expectation is that Cordella’s normative model will maintain most of its deep structure and thus its main conclusions will be deemed robust, even if the numerical trade-off boundaries of the original model shift significantly (i.e. at what level of the efficiency of reforms is the cascade approach of doing reforms first better than using subsidies before reforms).

Chapter 2. Introduction

2.1. Context

The 2030 agenda is an ambitious plan of action for people, planet and prosperity adopted by a United Nations General Assembly resolution on the 25th of September 2015
Ambitious objectives set for the World in this 2030 Sustainable Development Agenda brought renewed interest in increasing cooperation among world-wide international development stakeholders, both public and private. This led to increased interest in bringing the private sector to help finance development projects (Kim, 2018), particularly in the face of (1) a shirking pool of resources for development aid, and (2) trillions of private-sector dollars sitting on the side lines invested in near-zero interest rate bonds (World Bank Group, 2018).

As an institution, the World Bank Group has sought to add to this conversation by helping design a comprehensive project financing framework for multi-lateral development banks (MDBs) that prioritizes using private sector funds. The result is the G20’s “principles of MDBs’ strategy for crowding-in Private Sector Finance for growth and sustainable development” (G20 International Financial Architecture Working Group, 2017). This effort became the Hamburg principles or the ‘cascade approach’ to project financing, and was adopted by the Development Committee (Joint Ministerial Committee of the Boards of Governors of the Bank and the Fund on the Transfer of Real Resources to Developing Countries) during the WBG-IMF joint-Annual meetings of 2017. For the World Bank Group and IMF, the implementation of the ‘cascade framework’ was further documented as the principles for “Maximizing Finance for Development,” or MfD for short (World Bank Group Development Committee, 2017). That said, the cascade approach guidelines can be roughly summarized as (figure 1 diagrams the quote below):

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3 UN General Assembly, 2030 Agenda
4 Jim Yong Kim, Here's how the World Bank is maximizing finance for development
5 World Bank Group, Maximizing Finance for Development infographic
7 G20-IFA WG, “Principles of MDBs’ strategy for crowding-in Private Sector Finance for growth and sustainable development”
8 G20-IFA WG, “Principles of MDBs’ strategy for crowding-in Private Sector Finance for growth and sustainable development”
9 World Bank Group Development Committee, “Maximizing Finance for Development: …”
“When a project is presented ask – ‘Is there a sustainable private sector solution that limits public debt and contingent liabilities?’ […] ‘If the answer is ‘Yes’ - promote such private solutions.’ […] ‘If the answer is ‘No’ - ask whether it is because of:’ […] (1) ‘Policy or regulatory gaps or weakness? If so, provide WBG support for policy and regulatory reforms.’ […] (2) ‘Risks? If so, assess the risks and see whether WBG instruments can address them.’ […] ‘If you conclude that the project requires public funding, pursue that option” (World Bank Group Development Committee, 2017). 10

However, this sequencing of interventions is presented with two important caveats:

1. Countries will make the decision as to whether to follow this cascade, and how

2. The role of the Multilateral Development Banks is to help governments systematically assess options.

Despite point 1 moving the political decisions out of the hands of multilateral development banks, it is still the responsibility of the multilateral banks in question (and thus the World Bank’s) to assess all options thanks to point 2. No work on the trade-offs derived from point 1 vis-à-vis point 2 exists, so it is outside the scope of this paper. This leads us to consider the only research to date on evaluating the cascade namely, ‘Optimizing Finance for Development’ paper by economist Tito Cordella (Cordella, 10)

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10 World Bank Group Development Committee, “Maximizing Finance for Development: …”
The focus was on understanding the tradeoffs in the framework outlined in figure 1, in particular the trade-offs of sequencing (i) reforms, (ii) subsidies (referred above as ‘risk instruments’ & ‘credit enhancements’), and (iii) government/public financing in a given project. Note that the ‘commercial financing’ option is dropped since privately profitable projects are not the focus of this analysis.

2.2. Optimizing Finance for Development

To better understand the cascade approach in practice, a landmark paper by World Bank Economic advisor to the acting-Chief Economist of the World Bank Group, Tito Cordella proposed a normative model explaining:

“(i) how the adoption of the cascade may affect the allocation of finance across projects, and (ii) the condition under which maximizing and optimizing finance for development are likely to coincide, and those under which they are not” (Cordella, 2018).

His normative model proposes a ‘benchmark project space,’ which will be detailed later in chapter 4. For now, the figure above will suffice (and the figure itself will be revisited later). In this new space, the model ponders alternative sequences for project interventions, where one takes as a given that the private sector funded projects will be

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11 Cordella, “Optimizing finance for development”
12 Cordella, “Optimizing finance for development”
13 Cordella, “Optimizing finance for development”
done regardless of intervention by the World Bank Group, but leaves open the possibility that projects could benefit from (1) Reforms or (2) Subsidies and thus be undertaken by the private sector, or (3) be done with government financing.

<table>
<thead>
<tr>
<th>Sequence:</th>
<th>1st Intervention</th>
<th>2nd Intervention</th>
<th>3rd Intervention</th>
</tr>
</thead>
<tbody>
<tr>
<td>GRS</td>
<td>Government / Public Funding</td>
<td>Reforms</td>
<td>Subsidies</td>
</tr>
<tr>
<td>Inverse-Cascade (GSR)</td>
<td>Government / Public Funding</td>
<td>Subsidies</td>
<td>Reforms</td>
</tr>
<tr>
<td>SRG</td>
<td>Subsidies</td>
<td>Reforms</td>
<td>Government / Public Funding</td>
</tr>
<tr>
<td>SGR</td>
<td>Subsidies</td>
<td>Government / Public Funding</td>
<td>Reforms</td>
</tr>
<tr>
<td>Cascade (RSG)</td>
<td>Reforms</td>
<td>Subsidies</td>
<td>Government / Public Funding</td>
</tr>
<tr>
<td>RGS</td>
<td>Reforms</td>
<td>Government / Public Funding</td>
<td>Subsidies</td>
</tr>
</tbody>
</table>

The result is a tally of societal benefits given one of the 6 possible sequences (ordering 3 possible interventions can be done 6 different ways as per table 1). With a total benefit to society given each sequence, the optimal sequence is calculated for a pair of assumptions regarding (a) the effectiveness of reforms, and (b) the overall costs of a project. Ultimately, the model allows economists to understand what assumptions regarding cost and the effectiveness of reforms make the cascade (or RSG) the optimal sequence of interventions for World Bank Group projects.

2.3. Normative Model & Monte Carlos Methods

The normative model presented in Cordella considered a simplified market where projects are defined by their levels of (x) private sector advantage and (y) potential welfare impact, or externality. Since each of the three interventions (reforms, subsidies or gov’t funding), affect all the projects in each ‘market condition’ (i.e. a given fixed value for cost and efficiency of reforms), Cordella performs several integrations and optimizations over his proposed ‘benchmark space’ to obtain net benefits for a given
sequence of intervention. Furthermore, to substitute for the real-world data inputs the model needs, he uses random variables. This means the ‘benchmark space’ proposed by Cordella has a fundamental probabilistic interpretation.

This also means the model’s closed-form mathematical expressions become exponentially more complex as one makes the model more flexible. Since Cordella relies solely on closed form analytical methods to solve the optimization problems, uniformly distributed independent random variables were most convenient and made no assumptions about the real data, while also greatly simplifying the model.

One can avoid most of these complex issues by leveraging numerical approximations derived using Monte Carlo method simulations instead of Cordella’s closed form analytical solutions (Protopapas, 2014). Instead of relying on closed form expressions for each result, Monte Carlo simulations provide arbitrarily correct numeric approximations to each mathematical calculation given a set of probabilistic assumptions. In short, by simulating the same problem Cordella proposes, one can explore a varying set of independence and distributional assumptions, extend the problem to include more complex interventions or even incorporate a larger number of parameters in the calculations. The focus for this piece will be to re-create with computer simulations Cordella’s normative model, then change his model to see how the main conclusions from the original paper hold up, when project characteristics are:

- Uniformly distributed, but not independent
- Independent, but not uniformly distributed
- Modeled against real-world World Bank Group project data

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14 Pavlos, Stochastic Optimization Methods
2.4. Normative Model & Real Data

A further test to the robustness of the conclusions derived by the model can be found by using real-world data. To that effect, a comprehensive dataset on the returns for private and public-sector World Bank Group projects could effectively be “mapped” to the cascade space and thus help change the distributional assumptions to be closer to the real-world, i.e. ‘fit’ the model. This is possible provided one can recreate in the cascade space the data generating process for the returns observed in World Bank Group projects (King, 1998).\(^\text{15}\)

A set of summary statistics will be done in chapter 7, when the data review will take place, here the important note is that the observed data and distributions used by Cordella are not mappable one-to-one. Cordella makes into random variables the overall level of externalities (non-appropriable returns) he calls \( \gamma \) and a private section efficiency advantage he calls \( x \). However, using a function of those random variables and a few other parameters in the model, one can re-create using Cordella’s assumptions the economic returns of all projects.

A further issue with the real data not seen in Cordella’s model is a very complex censoring mechanism. In fact, this censoring mechanism is one of the main reasons for using Monte Carlo methods. The issue is that the observed World Bank Group data is censored because we do not observe any information for projects that were not funded. Again, this issue is easily solved by re-creating the censoring mechanism using Cordella’s assumptions. In this case, the censoring is done on a project being “profitable” to the public or private sectors; in other words, we only observe projects that were

\(^{15}\) King 1998
undertaken by the World Bank Group after review (profitability and further definitions are explained in detail in chapters 4, while the censoring mechanism and its replication is further explained in chapter 7).

Furthermore, Cordella’s normative model makes several assumptions that may not be conducive to using the World Bank Group project’s economic and financial returns data, particularly the assumption in the model that one can take ‘c,’ project cost, to be constant across a given market. The issues surrounding this problem are taken up on chapter 7. The proposed solution is to assume that ‘c’ is also a random variable, as it is detailed in chapter 8. This allows direct comparison between the censored function of the assumed random variables in Cordella to be compared to the data observed, provided we replicate with Monte Carlo method simulations the data generating process outlined (King, 1998).

Comparing ‘apples to apples’ between the Cordella model assumptions and the real data makes for a very complex inference issue. To change the distributional assumptions to better match the data, the histograms of the observed data will be compared to the histogram of the simulated analogs in the Cordella model using Cramér–von Mises statistic (the square of the area between the CDFs) as a goodness of fit statistics. More details on this can be found in the methodology section and the results will be the topic of chapter 9.

In short, the steps are, (1) randomly simulating large numbers of projects within the space according to parametric random variables, (2) re-doing the censoring mechanism, (3) calculating from the now-censored simulated values the corresponding project characteristics we observe from the data, (4) calculating CDFs to compare to the CDFs
overserved in the data using Cramér–von Mises statistics, (5) optimize the parameters of the distributions assumed in (1) by minimizing the Cramér–von Mises statistic.

Since we make several different distributional assumptions that need to be compared to one another, the dataset will also be split into 60/40 between fitting and testing data. To ‘fit’ this model we use Cramér–von Mises test statistic; however, the testing and final display of results will be a comparison of the Kolmogorov-Smirnov statistic (the maximum distance between the two CDFs, the CDF in the test data and the CDF from the best-fit parametric distribution after undergoing the algorithm we just outlined).

Finally, the result of using real-world data will be a ‘fitted’ three-dimensional model, a ‘3d cascade space.’ This means the result for the optimization problems will be a series of one dimensional lines based on the level of efficiency of reforms for each of the parametric models. This one-dimensional solution space will still allow for near direct comparison to some of the most important conclusions made by Cordella regarding the cascade’s effectiveness. In short, checking the robustness of his distributional and model assumptions given real-world data will be the most important point of the work and done in detail in chapter 11.

Chapter 3. Methodology

3.1. Normative Models in Economics

This section will outline in depth the methodology employed to create Cordella’s normative model discussed previously. The methodology of his model, and most models in economics can best be understood as an attempt to illustrate “what is going on” and allow Cordella, or any researcher to be clear about their assumptions, while also

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expressing innovative conclusions extracted from those assumptions (Sen, 2017). More precisely for Cordella’s model, we are looking at a normative model that is concerned with discussing what the “goals of public policy ought to be,” and not necessarily describing reality as it is now. To unpack the concept of normative economics it is useful to also understand its counterpoint, positive economics. This is the subject of the next section (Sen, 2017).

3.1.1. Normative versus Positive Economics

More broadly, a common distinction in economics is made between positive and normative economics, particularly when discussing public policy. Whereas normative economics will put forth and justify policies in terms of its impact in maximizing or simply increasing economic welfare, positive economics concerns itself with describing the world as it is now instead of making efforts to change it. Let us illustrate that point.

Take for example the positive and normative case for the concept of Cost-Benefit Analysis (CBA, also known as BCA or Benefit-Cost Analysis), which we will use quite a bit later in this paper (Hammitt, 2002). Although we will not go into detail on what exactly we mean by Cost-Benefit Analysis, at this point we define the concept in its broadest terms. CBA is a way to systematize a decision evaluation procedure in terms of a decision’s consequences (Dreze & Stern, 1987). Keeping this definition in mind, and focusing on justifying CBA for policy making in terms of positive economics we frame the positive economic argument as a Kaldor-Hicks compensation test (Hammitt, 2002):

\[ \text{this compensation test asks whether, if monetary transfers could be made without cost, those who benefit from a policy change could compensate those who are harmed so that everyone would judge himself better off with the policy change and transfer payments than without.} \]
This is a basic formulation of positive economics because it contains no strong assumptions with regards to how the world ought to be. Instead, it takes at face value the outcomes, and analyzes policy on that basis alone.

On the flip side, normative economics would justify the use of CBA as simply a method by which one identifies actions that are social welfare improving (Hammitt, 2002). A normative economist would point out how rigorous accounting for policy consequences is closest to a “utilitarian calculus” (i.e. welfare gains outweigh the losses), or may even promote more consistent policy-making by relying on rigor to protect against cognitive error (Hammitt, 2002).

What the reader might not realize is that the positive economic case ensures that all are better off, and thus respects individual autonomy by definition, whereas nothing in the argument from the normative economist discusses trade-offs that are bound to occur from the uneven distribution among the affected population of costs and benefits. So, the normative economist would be perfectly satisfied to propose a policy with significant adverse consequences to huge segments of the population if there is a net improvement in the overall social/economic welfare; however, the positive economist would be more concerned about Pareto optimality and thus study how to evenly distribute costs and benefits to pass the compensation test discussed earlier.

With the distinction between positive and normative economics developed, we move back to our discussion about normative economic models, like the one proposed by Cordella, which is the subject of our analysis.

3.1.2. The normative model proposed by Cordella.

Put simply, Cordella is concerned with creating a counter-factual to the common mischaracterization that political reforms have no cost, and thus should always begin any
attempt at maximizing social welfare. However, no policy is completely effective in internalizing all externalities associated with a given transaction (that would mean that there is a hypothetically perfect institutional arrangement, a naïve assumption at best). However, a case could be made that there is a gradient on which to judge different institutional arrangements. This is the foundation of Cordella’s normative model. We will go over the model itself in greater detail on Chapter 4 when we discuss Optimizing Finance for Development’s Normative Model, for now the focus is on his definition of gradient and how his optimizations to arrive at different conclusions in the paper.

In Cordella’s model the gradient is the net-social welfare impact for the 6 different ways of sequencing the 3 interventions that fall in his defined Benchmark Space, namely government funding, reforms, and subsidies. The cascade is one such institutional arrangement that proposes to sequence these three interventions as follows, first do reforms, then use subsidies, and only if they fail use government funding, or RSG for short. As discussed Cordella also defines market conditions. For that he uses two parameters: project cost ‘c’ and the efficiency of reforms ‘r’ (Cordella calls it α).

On the variables side, Cordella populates each market by modeling four project characteristics related to their potentials based on who implements/finances them. So, a project’s returns when implemented by the private sector are either appropriable or non-appropriable; henceforth referred to as (i) appropriable returns for the private sector and (ii) non-appropriable returns for the private sector. If the public sector implements the project its returns are also either appropriable or non-appropriable; we refer to them in turn as (iii) appropriable returns for the public sector and (iv) non-appropriable returns for the private sector. To avoid having to deal with all four at once, he simplifies the
model is two ways. First, he assumes non-appropriable returns will be the same if the project is done by either the private or public sectors and calls it $\gamma$. Second, Cordella normalizes all the projects such that the appropriable returns of a project when done by the public sector equals 1. This creates repercussions to both his parameters and his variables. With the Cordella model sketched out, let us look focus on interpretation.

3.1.3. Interpreting Cordella’s normative model
The Cordella model parametrization makes for some counter-intuitive relationships between his model’s main parameters & variables vis-à-vis the real-world concepts they represent. To illustrate this, let us look at three of them, namely the (1) consequences stemming from the normalization by the appropriable returns to the government, (2) limits imposed on the space by its boundaries, and (3) issues of measurability of variables thanks to how they are defined, particularly ‘x.’ We will take the three in turn.

The first point on interpreting Cordella’s normative model is understanding exactly what each of the different market conditions represent. Because the parameter ‘c,’ project cost, is also a project characteristic it is now normalized by the appropriable returns to the government. This means that when ‘c’ is 2, we are dealing with a Benchmark Space populated only with projects where the project cost is twice as large as the appropriable returns to the government (i.e. government revenues from the project), which means the government took a 50% loss on the project. As ‘c’ decreases, the projects get more and more profitable until the government breaks even from project revenues alone, i.e. government appropriable returns are equal to total project cost. This parametrization creates some interesting issues for this section’s other points, particularly the second (2), since we limit our analysis to only projects that fall between those that can financially
break-even when done by the public sector, up to projects that make a loss amounting to half their revenue. This brings us to the second point.

Second, the boundaries and/or limits imposed on the space might not be justified, particularly when it comes to ‘c.’ As previously mentioned, we are limiting ourselves to projects that in the best-case scenario would break-even when implemented by the government. One can speculate that projects where the government has a positive financial return to be even more likely to be implemented / financed by the private sector. Although we do not have any data on the financial returns from public sector World Bank Group projects, we have several very high outliers when it comes to overall economic returns. As can be seen in the Data appendix, we have quite a few positive outliers in the real-world data that would likely and substantively influence the results of the model if they were considered. We go into detail on this issue in Chapter 8, when consider what data point will be mapped to what random variable in the Cordella’s model.

Finally, the third issue stems from what exactly each variable means. To discuss this, let us look at what ‘x’ means in the real-world and the methodological problem of measuring ‘x’ as a stand-alone variable.

Cordella parametrizes ‘x’ as the private sector’s implementation/financing advantage over a hypothetical public-sector implementation/financing. Then, the variable is further normalized by the appropriable returns to the government. So, when ‘x’ is 1, it means that the private sector advantage is equal to the appropriable returns to the government (i.e. the entire amount of revenues the government would get by undertaking the project is equal to the ad-valorem of having this project be undertaken by the private sector instead of the government).
In short, ‘x’ is the total private sector advantage in terms of a percentage of government revenues had the project been done by the government. This quantity is the counterfactual of a counterfactual and can never be directly measured in a project if we must slice each market condition by cost. This issue of constant ‘c’ will be reviewed in much greater detail in Chapter 7.

For now, the important note to make on interpreting Cordella’s model from its methodology is that a project’s counterfactual, that is a private sector project’s characteristics for a project undertaken by the government, and public sector characteristics for a project undertaken by the private sector are deemed to have the same distribution as their observed correlates (private sector characteristics for projects undertaken by the private sector and public sector characteristics for projects undertaken by the public sector). See the table below that clarifies this point.

*Table 2 Relationships Between Projects in the Normative Model and the Real-World*

<table>
<thead>
<tr>
<th>Observed</th>
<th>Project Undertaken by:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Public</td>
</tr>
<tr>
<td>no</td>
<td>private sector advantage</td>
</tr>
<tr>
<td>yes</td>
<td>appropriable returns to the government</td>
</tr>
</tbody>
</table>

Assuming these definitions, let us now look for methods of measuring them. At first, we attempted to deconvolve ‘x,’ by comparing overall public and private sector returns (see Methodology appendix under Deconvolution Methods). We shelved those efforts in favor of modeling the data generating processes of our observations. In other words, our rendition of Cordella’s normative model only relies on assumptions commonly made of statistical models: this model is a faithful representation of the random variation found in the observations. In short, a link between positive and normative economics.
3.1.4. Normative & positive contributions to Cordella’s Model
As teased in the previous section, our analysis will take both a normative and positive approach. To expand upon the Cordella’s work using normative economics, we will change his model’s distributional assumptions with respects to independence between project characteristics and use different distributions. This will allow for a discussion related to the robustness of his conclusions. The other intervention in this paper is more complicated because it relates to positive economics and empirical claims.

To further expand on our discussion about the robustness of Cordella’s model, real-world data could be brought into the model. Generally, empirical claims require empirical models; however, the proposition here is to take Cordella’s normative model at face value, change its assumptions as little as possible to make sure we can replicate the data generating process observed in the World Bank Group public and private sector projects, and ‘fit’ the parametric distributions from Cordella’s model. We do this by comparing the distributions observed from our data with the distributions recreated using the model that mimic our hypothesis for our observation’s data generating process. Such modeling techniques are commonly the underlying assumptions in most political methodology models, and this paper will re-build Cordella’s normative model by relying on this common set of logical arguments that make a case for how on can map data generating process to real data and build sound model (King, 1998). Put simply, given Cordella’s model is correct, we will check its conclusions against real-world data.

3.2. Monte Carlo Methods
In general, all scientific inquiry depends on some measure of model building, its estimation, and the testing of its underlying assumptions, which needs verification to check how reliably the proposed model is when it (1) departs from its underlying
assumptions or (2) is confronted against real-world data (Hakan Demirtas, 2017). The objective of this paper is to conduct both verifications and rely on Monte Carlo methods to calculate those scenarios and compare results from Cordella’s model to real-world project data.

However, let us first ponder what exactly are so called Monte Carlo methods. This is the topic of this section. The ‘positive-normative’ dichotomy introduced in the previous section is very instructive. In mathematics the dividing line is established differently, there “theoreticians deduce conclusions from postulates whereas experimentalist infer conclusions from observations.” Monte Carlo methods generally falls squarely on the second category, and the bridge we postulated before of bringing some positive economics to Cordella’s normative model could be mimicked here as well under different terminology. So, to conclude our definition using the dichotomy above, “Monte Carlo methods comprise the branch of experimental mathematics which is concerned with experiments on random numbers.” Put differently, the idea behind Monte Carlo methods is to use statistical models to create mathematics that have predictive power. Let us now move to a brief rehearsal of its history for context and more concrete exemplification.

3.2.1. History of Monte Carlo methods
The idea of using Monte Carlo methods in a systematic way only came in 1944, despite various previous commonly used dialectic exercises generally tailored to teach students by providing a direct relationship between the frequency of events and theoretical results. A practical example used to demonstrate this were rigs where several small ball bearings were dropped on top of small wooden pins to show that the arbitrarily random nature of the bouncing balls always seemed to produce a normal curve in the
different boxes holding the balls below (Hammersley & Handcomb, 2013). Real work on
the topic came in the form of the Manhattan project during World War II by scientists
John Von Neumann and Stanislaw Ulam. Despite rigorous formulation coming only later
from Harris and Herman Kahn in 1948, a brief history on the topic might be more
illustrative on the accounts from Ulam himself (Dave, 2017):

The first thoughts and attempts I made to practice [the Monte Carlo Method] were suggested by a
question which occurred to me in 1946 as I was convalescing from an illness and playing solitaires.
The question was what are the chances that a Canfield solitaire laid out with 52 cards will come out
successfully? After spending a lot of time trying to estimate them by pure combinatorial calculations, I
wondered whether a more practical method than “abstract thinking” might not be to lay it out say one
hundred times and simply observe and count the number of successful plays. This was already possible
to envisage with the beginning of the new era of fast computers, and I immediately thought of problems
of neutron diffusion and other questions of mathematical physics, and more generally how to change
processes described by certain differential equations into an equivalent form interpretable as a
succession of random operations. Later [in 1946], I described the idea to John von Neumann, and we
began to plan actual calculations. –Stanislaw Ulam

The name Monte Carlo came from the work in 1944 as well, a kind of ‘war-time’ name
that eventually stuck to the field as a whole (Dave, 2017):

Being secret, the work of von Neumann and Ulam required a code name. A colleague of von Neumann
and Ulam, Nicholas Metropolis, suggested using the name Monte Carlo, which refers to the Monte
Carlo Casino in Monaco where Ulam’s uncle would borrow money from relatives to gamble.

3.2.2. A note on stochastic simulations

With a clearer understanding of the origins and more concrete examples of what is
meant by Monte Carlo methods, it is crucial to note that the process adopted here uses
electronic computers and relies heavily random number generators to accomplish most of
the mathematics involved in the models. We will use so-called “stochastic simulations”
to create as-close-to-real as feasible scenarios but, ultimately, they will be “imperfect
proxies of the perceived underlying truth; iteratively refining and occasionally redefining
the empirical truth to decipher the mechanism by which the process under consideration
is assumed to operate in a repeated manner” (Hakan Demirtas, 2017).
These proxies are generally referred to as simulation studies and the aim of this manuscript is to use simulations to (1) understand how robust is Cordella’s model to changes in distributional assumptions and (2) understand how his model’s conclusions hold up to real-world data.

3.3. World Bank Group Economic Rates of Return (ERR) Analysis

For the World Bank Group projects, one key factor to measure project impact comes in the form of a projects Economic Rate of Return (ERR). ERR is calculated through what is known as a Cost-Benefit Analysis, which we previously alluded to in the start of this chapter. Recall we have discussed in the broadest terms what exactly we mean by CBA (‘it is a way to systematize a decision evaluation procedure in terms of a decision’s consequences (Dreze & Stern, 1987)’). For our purposes, the important issue to understand is where the ERR data comes from by introducing a few basic concepts used in cost-benefit analysis for project evaluation, while also framing CBA in terms of a project’s ‘net effect on social welfare.’

First, a commonly used concept to measure a project’s net effect on social welfare is ‘shadow pricing.’ A shadow price measures the net impact on social welfare of a unit increase in the supply of that good (Dreze & Stern, 1987). Generally, it is used to refer to the pricing process of things that do not have a clear price, such as “greater accessibility / connectivity” or “better health care.” Furthermore, these goods are defined to be first-order measures of a project’s impact on social welfare. This measurement is a guarantee that the World Bank Group projects have a positive profit (in terms of shadow prices) (Dreze & Stern, 1987). Such goods are commonly part of a World Bank Group project and in the Cordella model referred to as an amorphous set of externalities ‘gamma.’ The Independent Evaluation Group is the department responsible for project evaluations, and
they have prepared an informative table that goes over in detail exactly what gets evaluated, and how, as far as World Bank Group projects are concerned (Independent Evaluation Group, 2013).

<table>
<thead>
<tr>
<th>Management Systems</th>
<th>World Bank</th>
<th>IFC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Self-evaluation</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results-based monitoring system and supervision status reports for Bank Group operations</td>
<td>Yes (ISR)</td>
<td>Yes, DOTS</td>
</tr>
<tr>
<td>Self-evaluation of projects</td>
<td>Yes, ICRs</td>
<td>Yes, XPSRs (managed by IEG)</td>
</tr>
</tbody>
</table>

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Results-based monitoring system for advisory services/AAA</td>
<td>Under development</td>
<td>Yes</td>
</tr>
<tr>
<td>Requirement for supervision reports for advisory services/AAA</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Quality assurance of Bank Group’s portfolio</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quality assessment of lending portfolio</td>
<td>Yes, OPCS</td>
<td>Yes, credit review</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Independent Evaluation</th>
<th>World Bank</th>
<th>IFC</th>
</tr>
</thead>
<tbody>
<tr>
<td>System for reviewing self-evaluations of Bank Group operations</td>
<td>Yes, ICR Reviews</td>
<td>Yes, IEG Evaluative Notes</td>
</tr>
<tr>
<td>System for reviewing self-evaluations of country evaluations</td>
<td>Yes, CASCR Reviews</td>
<td>Yes, CASCR Reviews</td>
</tr>
<tr>
<td>System for reviewing self-evaluations of advisory services/AAA</td>
<td>Under development</td>
<td>Yes, PCR EvNotes</td>
</tr>
<tr>
<td>Independent projects evaluations</td>
<td>Yes, PPARs</td>
<td>Yes, PES</td>
</tr>
<tr>
<td>Independent country evaluations</td>
<td>Yes, CPEs</td>
<td>Yes, CPEs</td>
</tr>
</tbody>
</table>

Source: IEG review; Note: AAA = analytic and advisory activities; CASCR = Country Assistance Strategy Completion Report; CPE = Country program evaluation; DEIS = Development Effectiveness Indicator System; DOTS = Development Outcome Tracking System; EAS = environmental and social effects monitoring; EvNote = Evaluative Note; IEG = Independent Evaluation Group; ISR = Implementation Status and Results Report; ICR = Implementation Completion and Results Report; NA = not applicable; PCR = Project Completion Report; PER = Project Evaluation Report; PES = Project Evaluation Summary; PPAR = Project Performance Assessment Review; OPCS = Operations Policy and Country Services; XPSR = Expanded Project Supervision Report.

3.3.1. World Bank (Government Financed Project)

The World Bank Group has several institutions, two of which deal exclusively with public sector projects, the International Development Association (IDA) and the International Bank for Reconstruction and Development (IBRD). There are generally referred to collectively as the ‘World Bank,’ or WB (not to be confused with the World
Bank Group (WBG), the broader institution). Although not mandatory for all projects, a project’s ERR is a crucial piece of information to measure its impact on the broader society (Herrera, 2005). Just like gamma in our model, ERR is an amalgam of the percentage rate of return for a project that accounts for all positive and negative externalities based on the project’s planned interventions.

When a WB project ends, its closing process requires the drafting of a document called an Implementation Completion Report. This document is authored by the project team and submitted to the World Bank Group’s Independent Evaluation Group, so that the later can make their own assessments and issue their review (Independent Evaluation Group, n.d.). During this process a project’s Economic Rate of Return at completion gets calculated to measure the actual impact of a project in terms of a net welfare return. Generally, projects had the same cost-benefit analysis methodology applied before they started. In recent times, fewer and fewer World Bank projects have calculated their ERR (Herrera, 2005). This is due to the added complexity and sophistication of the required calculations as the WB began (1) to undertake more complex projects, (2) to review ERR methodologies to account for green growth and other corporate agendas, and (3) significant changes in project preparation documentation.

For the purposes of this paper, ERR for the public sector is defined as the ERR at completion discussed above.

3.3.2. International Finance Corporation (IFC) (Private Sector)
The World Bank Group has one institution that deals exclusively with private sector projects, the International Financial Corporation (IFC). A random batch of IFC projects
that are in the process of ending require the preparation of an Expanded Project Supervision Reports (XPSRs) all of which IEG reviews; more specifically:

*IFC self-evaluates a random representative sample of projects that reach early operating maturity based on less than 5 percent sampling error associated with estimated development results in the population at the 95 percent confidence level, evaluating 80 projects a year on average* (Independent Evaluation Group, 2013)

Once these projects go through the process, IFC projects have an ex-post ERRs. The project will also have an Internal Rate of Return, which calculates only the profit percent of the project that came to IFC’s coffers from that activity. The ex-post ERR seems to match the normative model’s private sector project’s overall return, while IRR seems to be close to the concept of appropriable returns for private sector projects.

For the purposes of this paper, ERR for the private sector is defined ex-post ERR found in IFC’s XPSRs discussed above, while IRR for private sector projects is defined as project IRR also above.

3.4. Gaussian Copula

Copula is a multi-dimensional probability distribution that has uniform marginals, but correlation between different variables. Copulas can be used to introduce dependence between two random variables without changing the marginals. Different types of copulas exist. Here, we used Gaussian copula to generate a correlated 2D Beta distribution, a distribution that has Beta distributed marginals and correlations between variables (Charpentier, Fermanian, & Scaillet, 2006). See Methodology appendix for more details on correlation.

3.5. Cramér–von Mises statistic

In the manuscript, we need to compare a simulated distribution to the distribution obtained from the data, without assuming any particular form for the distribution. As
such, we need to use non-parametric methods. We used the Cramér–von Mises statistic, which is a measure of how close two probability distributions are to each other. We chose this statistic because it can be easily implemented for two samples and well implemented in Python already; more specifically, as the energy distance between two distributions ‘v’ and ‘u’ with a CDFs V and U equals (The SciPy community, 2018):

\[ D(u, v) = (2\mathbb{E}|X - Y| - \mathbb{E}|X - X'| - \mathbb{E}|Y - Y'|)^{1/2} \]

For real-valued distributions that are one-dimensional it reduces to the distribution dependent Cramér–von Mises statistic, parametrization from web (Mises, 1947):

\[ \omega^2 = \int_{-\infty}^{\infty} [F_n(x) - F^*(x)]^2 dF^*(x) \]

The picture below shows a more intuitive take on the equations above. As we can see it is the ‘area between the two CDFs.’

![Figure 3 Illustration of the Cramér–von Mises Statistic](image_url)
3.6. Kolmogorov Smirnov statistic\textsuperscript{17}

Like the Cramér–von Mises measure, the Kolmogorov-Smirnov statistic (KS) is also a comparison between 2 CDFs (even if the CDFs are an approximation, since it does non-parametric). We chose Kolmogorov-Smirnov statistic because it is a non-parametric way to compare samples from two distributions. It can be applied to any distributions and does not require the sampling distributions to be normally distributed. The test statistic below, assumes $F$ is the expected distribution, $F_n$ is the empirical (in our case the distribution from simulations and the distributions from the data, respectively).

\textit{Equation 3 Kolmogorov-Smirnov Statistic}

\[ D_n = \sup_x |F_n(x) - F(x)| \]

An illustration in the picture below explains the KS statistic visually, KS is the red-line (vertical); it is the maximum vertical distance between the CDFs and depending on the location of each CDF in relation to each other the statistic might even be non-unique with respect to its corresponding value in the ‘x-axis.’ The KS statistic itself, however, is unique by definition (Filion, 2015):

\textit{Figure 4 Kolmogorov-Smirnoff Test Illustration}

\textsuperscript{17} Larry Wasserman, “All of Statistics: ..” page 245
3.7. Censoring Mechanism
As previously mentioned, we only observe a fraction of the benchmark space, because our dataset is limited to completed World Bank Group projects. Reconstructing and estimating distributions from censored data is common in statistics (Wasserman, 2005). The diagram below confronts the data we have vis-à-vis the space proposed by Cordella to define the normative model:

3.8. K-Fold Cross-Validation
We used a hybrid of 60/40 cross-validation and 10-fold cross-validation. Since testing and validation in our approach rely on calculating very similar statistics, we had to use a validation cohort that is like the testing cohort in size. As such, we used 60% of the data for testing and 40% for validation. To repeat the cross-validation, we used the following approach. We separated the data into 10 bins and used 6 bins for testing and 4 for validation. We then repeated it for all 210 ways to select 6 bins out of 10. This allowed us to remain internally consistent, and not to rely on random splitting to perform cross-validation, ensuring the universal replicability of our results. The diagram goes over this in more detail to clarify in Figure 6 K-Fold Validation to Create Variance for KS Statistic. We do this so that the KS statistic can have a confidence interval for each of our
parametric models which, allows us to conduct 2-sample tests to make sure the KS for a given model is indeed different from one another.

Chapter 4. Optimizing Finance for Development’s Normative Model

The paper “Optimizing Finance for Development” seeks to understand what are the assumptions on the efficiency of policy reforms and subsidy levels that make the sequence of interventions suggested by the cascade approach (reforms, subsidy, government) the optimal with respect to net social welfare change. It also considers the other 5 ways to order the 3 interventions (assuming all projects already profitable for private sector will indeed be financed by the private sector without these 3 interventions).

The main question is how to close the “funding gap” - i.e. how to fund projects that are not profitable enough to get private sector funding but could be made more attractive with government support in the form of the 3 proposed interventions (reforms, subsidies, or direct government funding). From that, Cordella creates a simplified model.
The model assumes all projects in the world have 2 types of benefits, those *appropriable* by whomever is undertaking that project, and *non-appropriable* benefits to the wider society (externalities). We also assume that the private sector has an advantage, which translates into higher appropriable returns when compared to the alternative, i.e. had they been implemented by the public sector the appropriable returns would be smaller (this difference is labeled ‘x’). Negative values of ‘x’ are discarded because that would mean there is a detriment to bringing in private sector funding, a scenario outside the scope of this analysis. The paper also assumes that non-appropriable returns of the project are the same whether implemented by the private or public sectors.

As previously mentioned, the paper then normalizes all the project benefits such that the appropriable returns of a project done when done by the public sector equals 1. Below we revise how the non-normalized costs and returns are related to the normalized variables. This will be used later when comparing results of the model to the real data.

Before normalization, each project is characterized by 4 parameters:

- Cost of the project,
- Appropriable returns of the project if implemented by the government ($R_{government}$)
- Appropriable returns of the project if implemented by the private sector ($R_{private}$), (this variable accounts for the private sector advantage as well)
- Non-appropriable returns of the project, externalities ($E$)

We can rescale the project by dividing it by the appropriable returns of the government, in which case we end up with 3 parameters:

*Equation 4 Normalization of Concepts in Cordella Model*

\[
\begin{align*}
    c &= \frac{\text{Cost}}{R_{government}} \\
    x &= \frac{R_{private} - R_{government}}{R_{government}} \\
    \gamma &= \frac{E}{R_{government}}
\end{align*}
\]
4.1. The ‘Benchmark’ Project space
Therefore, in the (x, γ) space, there are four types of projects:

Figure 7 from Cordella, the ‘Benchmark’ Project Space illustration (modified by Author)

<table>
<thead>
<tr>
<th>Category</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Projects that are not worth financing by either the private or the public sector.</td>
</tr>
<tr>
<td>B</td>
<td>Projects, where there is a private sector efficiency advantage, but they are not sufficiently profitable for the private sector to be interested.</td>
</tr>
<tr>
<td>B'</td>
<td>Public investments are welfare improving.</td>
</tr>
<tr>
<td>B''</td>
<td>Public investments are not welfare improving, but private investments would be; however, the projects are not profitable for the private sector to be interested.</td>
</tr>
<tr>
<td>C</td>
<td>Projects, where public sector has an efficiency advantage (i.e. X is negative).</td>
</tr>
<tr>
<td>D</td>
<td>Projects that are quintessentially private.</td>
</tr>
</tbody>
</table>

4.2. Defining the Interventions

Figure 8 Diagram Illustrating the Cascade vis-a-vis sequencing of interventions in the Benchmark Space

Use cascade interventions:
- PRIV: Private sector is in D.
- REFORMS: The Other 3?
- SUBSIDY: All in Benchmark Space!
- GOV’T: But How to Sequence Them??
The two approaches Cordella suggests getting business interested in undertaking projects within the Benchmark space are:

-- **reforms** that increase profits by $\alpha \gamma$ and decrease public good by $\gamma$
-- **subsidies** that increase profits by $s$ and decrease public good by $s(c - 1)$

Finally, when subsidies fail, we apply government financing for projects

-- **government** financing (i.e. public investments)

*Figure 9 from Cordella, illustration of impact of interventions in the Cascade space.*

*Figure 10 Diagram outlining the Welfare Calculation process for each sequence*

Recall the Table of Sequences

<table>
<thead>
<tr>
<th>Sequence</th>
<th>1st Intervention</th>
<th>2nd Intervention</th>
<th>3rd Intervention</th>
</tr>
</thead>
<tbody>
<tr>
<td>reforms</td>
<td>subsidies</td>
<td>government</td>
<td></td>
</tr>
<tr>
<td>subsidies</td>
<td>reforms</td>
<td>subsidies</td>
<td></td>
</tr>
<tr>
<td>subsidies</td>
<td>reforms</td>
<td>subsidies</td>
<td></td>
</tr>
<tr>
<td>subsidies</td>
<td>government</td>
<td>reforms</td>
<td></td>
</tr>
</tbody>
</table>

Given each market (pairs of $\alpha$ and $c$), we can **apply each intervention in the given sequence to the Benchmark space:**

For example: the INVERSE cascade (GSR):

$\alpha = 0.16$; $0.5$; $0.83$

Then we can **calculate the total change in Welfare** for that sequence and compare across.
4.3. Sequencing of Interventions
Cordella shows that if a government can pick a specific instrument for each project, it
does not matter in which order it will offer the instrument to the project. However, the
order of intervention matters when the government can’t match projects with instruments
individually and instead can only offer one reform to all projects; and then offer a subsidy
to all projects that have not been picked; and then provide public funding to the projects
that remained unfunded but are still a net benefit to social welfare.

4.4. Final Results from the Paper:
Subsidies > Bad Reforms
The main result is a tally of what
sequence is best for each market, as seen
in Figure 11 Optimal Sequence given (c, α) from Cordella. More specifically,
Cordella analyses the effect of the
sequential application of
reforms/subsidies/public funding on the
total welfare gains. He assumes that all projects are uniformly distributed (x, γ) ~ Unif (0, 1) range; and that policy makers are ‘myopic,’ which means that policy makers optimize
at the level of the intervention, not the sequence. That means that policy makers set the
level for each of the three interventions in a given sequence of interventions without
considering the others. For example, in the RSG sequence (also known as the cascade),
the first intervention (reforms) will be done to as many projects as possible until ‘break-
even,’ or zero marginal benefit (minus the imperfect targeting of subsidies).
For example, take the cascade sequence (RSG). To calculate the total welfare, the first step is to apply reforms to all projects that can be made profitable through reforms and integrate over the welfare generated by all such projects while subtracting the cost of reforms in the form of total externalities times 1 – the efficiency of reforms.

Note that this is ‘myopic’ in the sense that when reforms are applied first, the legislator does not consider the welfare gain counterfactual of using subsidies or government financing. Even though for some of those projects at the margin it would have be better for the overall social welfare to have those projects be subsidized or even financed by the government directly, they are nevertheless can be made profitable enough to be financed by the private sector by ‘capturing’ a percentage of its positive externalities through reforms (this waste is particularly problematic when we are in a ‘market condition’ that has low levels of efficiency of reform; we will take up this issue again at several points).

Then, of the projects in the benchmark space that are left (i.e. not made profitable enough for the private sector to undertake via making a percentage of its externalities appropriable through reforms), subsidies are applied up to the point where there is again zero marginal benefit (because of imperfect targeting the deadweight loss is a little higher, so we stop giving subsidies a little bit before zero marginal benefit had there been perfect targeting but that is a point we will not take up in this paper, even though we use it in our optimization and Monte Carlo simulations to replicate Cordella’s results). For this, the welfare generated by projects funded because of this intervention are also added up, while the subsidy cost is subtracted. From the projects that are left after the past two interventions, we calculate what projects would be profitable for the government to
undertake and then integrate over the welfare generated by having the government fund those projects, while again subtracting the cost of undertaking the project (in this case the entire project cost).

More importantly, the base assumption of having \((x, \gamma)\) be independent and uniformly distributed leads to simple integrals and interpretable results. With this crucial simplifying assumption, the optimizations can be undertaken for each level of alpha (efficiency of reforms) and cost without too much issue for each sequence. Each sequence can thus be applied to the projects in the benchmark space and thus be compared against all the others. Let us revisit the final result (with the normalization assumptions) in Cordella. See below Figure 12 Final Result from Cordella Model for each Alpha, c pair (modified by author). Note that only when reforms are very efficient, the cascade (RSG) makes sense (i.e. maximizing social welfare). More importantly, some form of subsidies-first approach seems to be more cost effective everywhere else.

![Figure 12 Final Result from Cordella Model for each Alpha, c pair (modified by author)](image_url)
Chapter 5. Monte Carlo Methods & the Normative Model

5.1. General methodology of our simulations

To simulate the model, we used Monte Carlo sampling and dynamic programming approaches. The outline of the simulation is as follows. First, we simulate a set of projects with values of $X$ and $\gamma$ drawn from a given probability distribution (for the base case, this is a Unif $(0,1)$. Second, for each of the 6 sequences of interventions, we find the assignment of projects to one of the 4 possible categories: private-sector funded because of (1) reforms, (2) subsidies, or (3) government-funded and finally (4) non-funded. Finally, we calculate the total welfare of the funded projects and find the optimal sequence of interventions. We repeat these steps for varying values of $c$ and $r$ (or alpha, the efficiency of reforms), then plot the optimal sequence of interventions as a function of $c$ and $r$. Each of the steps is detailed below.

In the first step, we use a random number generator to ‘draw’ values from a random value that has a given probability distribution to simulate project characteristics, namely $X$ and $\gamma$. We use different strategies depending on the distributions considered. Those are described in each corresponding section. We probe both independent distributions of $X$ and $\gamma$, and a case in which $X$ and $\gamma$ may be correlated or anti-correlated. After the draw, we restrict the values of $X$ and $\gamma$ to fall inside the benchmark space.

In the second step, we need to assign projects to funding sources according to the sequences of intervention. Consistent with the paper, we assume that each project may be funded only by one source of funding: government, reforms or subsidies and we do not take other interventions into account when applying each intervention sequentially. Note as well that the assignment of projects to government funding, or private sector funding thanks to reforms/subsidies is done according to the equations described in the paper.
For subsidies, we need to determine the optimal value of the subsidy that maximizes total welfare (despite imperfect targeting), which requires maximizing a value of an integral with respect to the parameter ‘s.’ A brute-force solution to this problem would be inefficient (and would have a compute-time that is quadratically proportional to the number of projects), to solve this optimization problem. Hence, we designed a dynamic programming approach that finds the exact solution in linear time.

To calculate the optimal subsidy parameter, we use a dynamic programming algorithm that finds the solution reasonably fast (linearly proportional to the number of the projects). We first order all not-yet-funded projects in the order of decreasing x. We then calculate total welfare if only the project with the largest private sector advantage was able to take the subsidy; then we add the project with the second-largest private sector advantage and re-calculate the total welfare. We then keep increasing the subsidy, which adds more projects one-by-one. At each step, we record the total welfare, and in the end, we find a subsidy that maximizes total welfare. We note that as we increase the subsidy, the welfare cost is increasing for all projects; however, this cost has a simple form of \( s \ast (c - 1) \ast N \). Furthermore, this has the virtue of also considering the imperfect targeting of subsides, since the level of subsidy is set to be the same for a given level of ‘x’ regardless of how much more or less positive externalities the project generates.

Finally, we find the most efficient sequence of interventions. We note that when we apply 6 different sequences of interventions, we use the same draw of x and \( \gamma \) for consistency. We then repeat the process of finding the optimal sequence of interventions for a different set of values of c and r. We use a 20x20 grid for r from 0 to 1, and for c from 1 to 2. Then, we exclude c = 1 because it has no projects in the benchmark space.
5.2. Formulas for interventions in our model

We followed the Cordella paper to define interventions in our model. The cutoff for government funding was $\gamma > c - 1$, and the welfare returns of the project were $\gamma - c + 1$. For modelling reforms, we used the same cutoff formula: $x > c - 1 - \alpha \gamma$, with welfare returns being $1 + x + \alpha \gamma - c$. Finally, for subsidies, we defined a subsidy level $s$. We then defined welfare returns of a project according to Cordella, as $WR_i = 1 + x_i + \gamma_i - s * (c - 1) - c$. We then maximized total welfare:

\[
WR_{tot} = \text{SUP}_s \left( \sum_i WR_i \cdot \theta(1 + x + s - c) \right)
\]

Where $SUP_s$ denotes maximum over $s$, $WR_i$ is as defined above, and $\theta(y)$ denotes the step function, that equals 1 for positive values of $y$, and 0 for negative values of $y$. We note that the only change from the formulas in Cordella is a natural change from integration to find optimal level of subsidies to the summation over simulated projects.

5.3. Replicating Closed-Form Results Using Simulations

To replicate the closed form solution of the paper, we performed our Monte Carlo simulation starting with the uniform distribution of $x$ and $\gamma$. We generated $N = 200,000$ draws from the uniform distribution for $x$ and $\gamma$, and repeated this for each combination of $(c, r)$ in the 20x20 grid as described above. For each value of $(c, r)$, we found the optimal sequence of intervention.

A challenge was to distinguish between regions where one sequence of intervention is clearly better, and where two or three sequences of intervention provide the same or similar results. We designed a way to display this using a combination of color shading and dot plots. We call this dots-and-square plot. The plot is made as follows:
• If one sequence of interventions is clearly better, shading of the corresponding color is displayed in a form of a square.

• If another sequence or two sequences come within 0.3% of the best sequence in terms of total welfare, they are displayed as a dot inside of a square, and as a smaller dot inside the dot.

• If two or three sequences are the same, then the primary shading is chosen according to the following sequence: SRG, SGR, RSG, RGS, GRS, GSR; favoring the former elements to the latter. This order was designed to put sequences that are typically present in the plot, such as SRG, SGR, RSG, first, so that the coloring is more continuous.

• If the distribution is such that, for a particular \((c, r)\) combination, less than 0.1% of the projects are being funded, then nothing is displayed, and the space is left white. This happens, for example, when distribution of \(x\) is heavily skewed towards \(x=1\), and small \(c\) only selects for projects near \(x=0\), thus selecting only a few projects.

Resulting dot plot is shown below. Note that ‘r’ in simulations is the ‘same’ as \(\alpha\), i.e. It is the efficiency of reforms:

*Figure 13 Optimal Sequence given \((c, \alpha)\) from Simulations*

Another way to visualize the relative efficiency of each of the 6 sequences of interventions is to show how well is a sequence doing compared to the best sequence for the \((c,r)\) pair. To visualize this, we plot the 6 heatmaps side by side below.
These plots offer us a unique perspective and allows us to see an overall picture for the different sequences of intervention and their comparative performance in each market even when they are sub-optimal. We can see that SGR/SRG performs well even when reforms are efficient. However, RGS/RSG perform very poorly if reforms are inefficient. More importantly, the figure above suggests that if we cannot calculate the efficiency of reforms well enough, SRG/SGR could be the more conservative approach in the scenarios assumed here.

Chapter 6. Changing Assumptions of Normative Model

6.1. Independence assumption: Robust, Subsidies > Bad Reforms

With the simulation replicated, we first tested if the results of the model would hold if $x$ and $\gamma$ were no longer independent. Instead of independent draws from the uniform distribution of $x$ and $\gamma$, we designed a way to keep $x$ and $\gamma$ uniform but introduce correlation between them. To impose correlation, we generated a sample of bivariate normal distribution with $\text{var}(x) = \text{var}(\gamma) = 1$, and $\text{cov}(x, \gamma)$ equal to values ranging from -0.8 to 0.8 with a step of 0.2. We then applied a rank transformation to the $x$ and $\gamma$, which made each of them uniformly distributed, while keeping them correlated or
anticorrelated. Note that taking the rank transformation changes the Pearson correlation between $x$ and $\gamma$ by a small amount: a correlation of 0.2 in multivariate normal becomes about 0.18 after the rank transformation, while correlation of 0.8 becomes about 0.78, as expected (Hauke & Kossowski, 2011). See methodology appendix for more details.

We find that the diagram is largely unperturbed by introducing dependence between $x$ and $\gamma$. While boundaries of different regions shift, overall regions remain stable.

Note that the diagrams for the case of anti-correlation show a narrow region of $c$ for which GSR becomes favorable over SRG/SGR strategy. This effect is true; indeed, for values of ‘c’ around 1.4, GSR becomes almost as good as SRG/SGR strategies, and for a moment may become minimally advantageous to SRG/SGR. The effect is more
pronounced with stronger anti-correlation between x and \( \gamma \), although it is visible even for the basic model with uniform distribution of X and \( \gamma \).

6.2. Distributional assumptions: Robust except for GSR < SGR

Beta distribution is a natural choice for a variable distributed within a [0,1] interval. It allows us to investigate a broad range of behaviors, including bimodality, and probability density shifted towards 0 or 1. We focused on 6 beta distributions as per the figure below. First is a classic U-shaped Beta (0.5, 0.5). This distribution assumes that values are either close to 0, or close to 1. For x, it would mean that private sector advantage is more likely to be close to 0, or 1, than to be around 0.5. Second distribution is a parabolic-shaped Beta (2,2). It is the opposite of the U-shaped distribution and has a peak at around 0.5.

Third and fourth distributions are Beta (0.5, 2) and Beta (2, 0.5). They assume that the probability mass is heavily centered around 0, or around 1, and decays gradually towards 1 and 0 respectively. Finally, fifth and sixth distributions are Beta (2,8) and Beta (8,2). They are not divergent at zero or one, and have a mean of 0.2 and 0.8 respectively. They

![Figure 16 Illustration of Beta Distributions given (a, b) parameters](image-url)
decay quicker towards the other end (1 and 0 respectively) than the Beta (0.5, 2) distribution.

We then assumed that \( X \) and \( \gamma \) are distributed independently, and we tried all pairwise combinations of the beta distributions for \( x \) and \( \gamma \). This yielded 36 pairwise combinations. The complete set of plots for all possible combinations organized by the Beta distribution of the ‘\( x \)’ can be found in the Figures & miscellaneous appendix under the Distribution Assumption: Plots for Different Beta Distributions. Below we show only a few examples. All the examples show the same basic features: RSG/RGS strategy is optimal if reforms are efficient. SRG is optimal if project cost is high and reforms are medium; SGR is optimal if project cost is low. And SRG/SGR are optimal if reforms are inefficient. We also notice that for some distributions, GSR may become a best strategy.

![Figure 17 Various Plots of Benchmark Space given Different Beta distributions](image)
We also note that for Beta (2,8) and for Beta (8,2), some parts of the (c, r) space are inaccessible because there may be no projects in the benchmark space, or all/most of the projects may be non-profitable for a particular value of ‘c.’

6.3. Why sometimes GSR strategy better than SGR?

We noticed that for (x, γ) that are either (1) anti-correlated uniform distributions, (2) non-uniform Beta distributions where ‘x’ is right-skewed, and even (3) in some cases where ‘x’ is distributed Beta (0.5, 0.5), which is ‘U-shaped,’ GSR (orange) has higher total net-social welfare gain than SGR (green). See below. Note that for Figure 18 Uniform and Anti-Correlated (x, γ) we see a small range in which GSR strategy is better than SGR (corr = −0.4 has the ‘GSR line,’ but it is between the lines 1.35 and 1.4)

![Figure 18 Uniform and Anti-Correlated (x, γ)](image-url)
We focus here on the case of Beta (0.5, 0.5), \( c=1.3, r=0.4 \), that also favors SGR sequence. We compare it to GSR sequence. We find that for the case of SGR, all projects are funded by subsidies alone, including many projects that are not worth financing. This is a consequence of imperfect targeting of subsidies. This distribution presents an example of when imperfect targeting of subsidies becomes a burden. However, if government funding was applied first, many projects with high \( \gamma \) are funded by the government, and therefore the dead weight loss of applying subsidies to the entire square is no longer efficient, since the high \( \gamma \) projects that offset subsidy costs were funded by government.

To see if a better strategy for allocation of subsidies exists, we focused on this example, and manually swept through a range of subsidy levels ranging from 0 to 1 with a step of 0.01. After applying the subsidy, we applied government funding and reforms as usual. We then selected the subsidy level that maximizes the total welfare. Interestingly, we found that the total welfare produced by this modified subsidy selection method was much higher than for a naïve SRG method, and even surpassed all other sequences including the GSR.

This result suggests a very important conclusion: when setting the subsidy threshold, one should consider future interventions. In fact, if subsidies are first in a sequence, subsidies should always stop slightly before they become welfare damaging. This would
ensure that other interventions have the potential to generate additional welfare above what we can get for simply providing a subsidy to an ever-increasing section of the benchmark space.

More broadly, what we seem to have stumbled upon here is the drawback of myopic policy-making, i.e. when policy makers don’t consider the counter-factual social welfare impact of other possible interventions when applying a given intervention to a project. In fact, when one considers just one other intervention before applying subsidies, as we did here (Figure 20 Benchmark space (x,γ): SGR versus GSR: welfare and project outcome graphs), the results clearly favor subsidies. One might even speculate that because subsidy costs are a second order term while reform costs are a first order term that a non-myopic legislator will always opt for subsidies first in the scenario built by Cordella.

This accounting for ‘non-myopic legislating’ and the results from setting a subsidy threshold that considers other possible interventions will be addressed in future studies. For now, the important economic point here is that the ‘orange space’ appears and then disappears because the deadweight loss from subsidizing projects with small positive externalities are not enough to offset the very high positive externality projects at the high end. In other words, although low externality projects would be better served by a different intervention, the subsidies are imperfectly targeted, so one must set the same level of subsidies for each level of private sector advantage ‘x’. Ally that with the high density of large positive externality projects at all levels of ‘x,’ and the entire benchmark space receives subsidies because any added cost from inefficiently distributing subsidies is more than offset by the social welfare added from the higher density of large
externality projects. However, eventually the costs become so low that we are back to subsidies-first approach being more effective again.

Figure 20 Benchmark space \((x, y)\): SGR versus GSR: welfare and project outcome graphs
Chapter 7. Real-Data & Optimizing Finance for Development Model

7.1. ERR Data from World Bank and IFC

- For the World Bank projects (IBRD/IDA) we have economic rate of returns (ERR) data at appraisal (before the projects is undertaken) for 4,001 projects, and 3,523 projects ERR at completion (measured after the project closes to estimate the impact of the project), for 3,310 we have both. The dates range from fiscal year 1956 to fiscal year 2011 for project board approval dates (the go-ahead by the institution to allow the project to be undertaken). Ultimately, the focus was on ERR measured at completion. A histogram of that data can be seen on Figure 21 Government (IDA/IBRD - World Bank) projects ERR.

- Of the 3523, we have project cost and net commitments (amounting to 235bi but only 1,092 have cost data which makes it a 299bi vs 114bi split) However, we will not use cost or commitment data for the analysis here, opting instead to focus exclusively on project returns data for the projects undertaken by the public sector. The idea is to avoid throwing away 75% of our project data, and focusing on the best-measured data possible, which is the economic rate of return analysis done after project completion.

- For the private-sector projects (undertaken by the World Bank Group’s private sector arm, the International Financial Corporation, or IFC), we have around 20 years’ worth of data from IFC’s XPSRs (which have ERR and IRR for projects taken after the project was completed) which translates to 1150 IFC projects from FY91-FY09, of which 467 have a valid ERR (even if sometimes negative) and 477 have FRRs (or financial rates of return, our discussed IRR).

<table>
<thead>
<tr>
<th>Summary Statistics of WBG Data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>FRR (private)</td>
</tr>
<tr>
<td>ERR (private)</td>
</tr>
<tr>
<td>ERR (public)</td>
</tr>
<tr>
<td>ERR-FRR (private)</td>
</tr>
</tbody>
</table>
7.2. Censoring Mechanism in the data & project impact assumptions
Before even attempting to reconstruct Cordella model correlated to the returns data
using Monte Carlo simulations, it is crucial to note that we observe only a fraction of the
space, since the returns data for the IFC and World Bank are only for projects that were
undertaken, which means they were profitable. See the diagram below.

The data has several projects with negative returns since it is measured after project
completion. *Those projects with negative returns were excluded.* This allows the strong
assumption that we are looking at the actual, as opposed to projected, returns for
profitable projects and do not have to rely on the accuracy of the World Bank Group
projections for their project’s returns. In fact, the only important assumption that one
must make is that the World Bank Group is correctly measuring the impact of their
projects after they are closed. This is a trivial assumption, since for a lot of these project
evaluation / measurement methods and evaluation frameworks, the World Bank Group
staff around the world has written the manual on how to apply economic theory into
practice when it comes to measuring project impact (Duvigneau & Prasad, 1984) and
their evaluations and project documents are completely public, allowing for verification and scrutiny.

7.3. Problem of varying c between the projects

In the normalization chosen in the paper, c is a total cost of the project. However, the meaning of c is more complex, because all the values are normalized to the appropriable return of the government. Therefore, mathematically \( c = \frac{\text{total cost of the project}}{\text{appropriable returns if the project is done by the government}} \). This value is hard to measure in the real-world. The manuscript assumes that for a given market, c is constant. It then applies different interventions in a particular order and finds the optimal sequence of interventions. Finally, it spans through the different values of ‘c,’ and selects the optimal level of intervention at each value of c.

In the real-world, projects have different cost, and different appropriable return of the public sector. Therefore, in each market, there will be projects with different values of c. As a result, different interventions will be applied to the projects with different levels of c. Reforms and government are allocated based on defined mathematical cutoffs (e.g. \( \gamma > c-1 \) for the government), and therefore would be the same between the 2D model with constant c, and the 3D model where c is a random variable. However, subsidies rely on the subsidy level set by the government to maximize total welfare. In 2D model, for each value of c there will be a different subsidy level chosen. However, in 3D model there will be only one subsidy level chosen, which will be shared across different values of c. We note that since c is intrinsically not observable, it would be impossible to have a subsidy targeting that allows the subsidy level to depend on c. As a conclusion, when projects with different values of c are present in the market, subsidies should be shared between them, and ‘c’ should be considered as a random variable.
7.4. Estimating distribution of \( x \) and \( \gamma \) assuming constant \( c \)
We first thought to use the ERR and FRR data for the private and public sector to infer the distributions of \( X \) and \( \gamma \). To this end, we made the following assumptions:

- As assumed by the paper, \( c \) is a constant for all projects
- The distribution of \( X \) and \( \gamma \) is the same for private and public projects

Under these assumptions the values of ERR and FRR should be able to give us the distribution of \( X \) and \( \gamma \). Indeed, we find that from here, we can compare the distributions in the following table:

<table>
<thead>
<tr>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( FRR_{private} )</td>
<td>( \left( \frac{1 + x}{c} \right) - 1; \text{private} )</td>
</tr>
<tr>
<td>( ERR_{private} - FRR_{private} )</td>
<td>( \frac{\gamma}{c}; \text{private} )</td>
</tr>
<tr>
<td>( ERR_{public} )</td>
<td>( \left( \frac{1 + \gamma}{c} \right) - 1; \text{public} )</td>
</tr>
</tbody>
</table>

Theoretically, if we assume that \( c \) is a constant, it leaves us with two ways to obtain \( X \) and \( \gamma \).

7.4.1. Obtaining \( x \) from FRR for private projects
The formula for FRR for private projects depends only on \( x \) and \( c \). If one assumes that \( c \) is constant, then \( x \) could be expressed in terms of FRR for private projects as follows:

\[
\begin{align*}
1 + x &= c + c * FRR_{private} \\
x &= (c - 1) + c * FRR_{private}
\end{align*}
\]

This states that \( x \) must be less than \((c-1)\) for private projects. This is obviously true for quintessentially private projects. However, in this study we are interested in the distribution of \( x \) in the benchmark space, i.e. \( x < c-1 \). Therefore, there is no way to obtain the distribution of \( x \) from FRR for private projects because we need the distribution of \( x \)
in the benchmark space, and the above formula can give us the distribution of \( x \) only outside of the benchmark space (\( x > c-1 \)).

We could use projects with negative FRR for this estimate. However, we quickly learn that there are only a few projects with negative FRR in the data, their FRR is very close to zero, and their ERR is also generally very low. Therefore, it does not look feasible to use them for our purpose.

7.4.2. Obtaining \( x \) by comparing private and public projects

If we compare equations for private and public ERR, we can see that they are only different by a factor of \( x/c \).

\[
\begin{align*}
1 + x + \gamma &= c + c \cdot \text{ERR}_{\text{private}} \\
1 + \gamma &= (c - 1) + c \cdot \text{ERR}_{\text{public}}
\end{align*}
\]

Therefore, we may be able to extract the distribution of \( x \) by analyzing the distribution of \( 1+x+\gamma \) and \( 1+\gamma \). It is not possible to subtract the distributions directly, and deconvolution-based approaches are needed to make this inference.

We obtained histograms of ERR for public and private projects and noted that ERR for public projects has a wider distribution than for the private projects. This contradicts our assumption, because the distribution of the sum of two independent random variables is always wider than each variable, and therefore the distribution of \( 1+x+\gamma \) should be wider than \( 1+\gamma \).
Therefore, we conclude from this section that deconvolving ERR (private) and ERR (public) is impossible because the distribution of ERR for the private sector is narrower than the distribution of the ERR for public sector.

7.4.3. Estimating \( \gamma \) from public projects

Estimation of \( \gamma \) from the data for public projects is possible. From the ERR for only the public-sector World Bank projects, we can express \( \gamma \) as seen below:

\[
\gamma = \text{ERR}_{\text{public}} \times c + (c - 1)
\]

Although this formula expresses \( \gamma \) in terms of ERR for public projects, it also implies that \( \gamma \) should be more than \((c-1)\). Indeed, this is true for projects that are worth funding by the government. However, this does not tell us anything about the distribution of \( \gamma \) for projects that are not worth funding by the government but may be funded by private sector leveraging the private sector advantage. As such, it may only give us information about the part of the distribution of \( \gamma \), and not the whole distribution. Indeed, a large part of the benchmark space contains projects that are not worth being funded by the government but could leverage private sector advantage with reforms and subsidies. It is not possible to obtain the distribution of \( \gamma \) for these projects using only government-funded projects.

7.4.4. Estimating \( \gamma \) from private projects

We can use the welfare returns of the private projects to estimate \( \gamma \) for the private projects. Indeed, we can see that:

\[
\frac{\gamma}{c} = \text{ERR}_{\text{private}} - \text{FRR}_{\text{private}}
\]

\[
\gamma = c \times (\text{ERR}_{\text{private}} - \text{FRR}_{\text{private}})
\]
This tells us that the distribution of $\gamma$ is very narrow, since $(ERR_{private} - FRR_{private})$ is very narrow (see below). This is probably the best estimate for the distribution of $\gamma$ that we have, and we will make use of it in the next chapters.

7.5. Takeaway: model cannot accommodate data if $c$ is fixed. Under the assumption of fixed $c$, we cannot leverage existing data to obtain information about the distribution of $x$. Censoring of real projects does not let us access projects in the benchmark space, which are inaccessible without reforms and subsidies. Moreover, we find that assuming a fixed distribution of $x$ and $\gamma$, and a constant $c$ leads to an inconsistency within the real data. The only useful conclusion we can draw from this data is about the distribution of $\gamma$: it is heavily biased towards 0 and is a narrow distribution.

In the next two chapters we show that extending the model to three dimensions, with $c$ being a variable, will resolve these contradictions, and allow for a much better fit between the model and the real data.
Chapter 8. Extending the Model to Map to Three Dimensions

8.1. Need for three-dimensional model
In the paper, and in our previous simulations, we fixed one important parameter: c. The meaning of c is the ratio of the cost of the project to the appropriable returns for the public sector. In the “real-world”, this variable is not fixed, and would naturally vary from project to project.

This calls for an improved model in which c is a variable, and not a fixed parameter. We implemented the model using the same principles of Monte Carlo, now treating c as a random variable as well. This extends the two-dimensional model with variables x and γ into a three-dimensional model in the space of \((x, c, γ)\).

Adapting our model for the case of c being a variable was relatively easy. The only notable difference is in the dynamic programming formulas for finding optimal subsidy, which now requires ordering the data in \((x - c)\) and not in x.

8.2. Finding the distribution of x, c, and γ from the data available
Before running the model to determine the optimal sequence of interventions, we first aimed at finding the probability density of projects in the space of \((x, c, γ)\). We show below that the assumption of a uniform distribution in \((x, c, γ)\) space is insufficient to match the histograms of ERR/FRR from the real-world data to their simulated counterparts. We also provide evidence that independent distributions of x, c, and γ are not consistent with the data.

We first searched for a way to use the data to extract the three-dimensional probability density function \(p(x, c, γ)\) directly. However, as the previous section shows, we cannot extract x, c, or γ from the data directly even if their distributions are independent. We can only see relatively complex functions of x, c, and γ, and only for the
subset of the projects (private sector or public sector). Moreover, the space of possible $p(x, c, \gamma)$ functions is very large. Therefore, as we cannot extract $p(x, c, \gamma)$ directly from the data, and to do so we need to make additional assumptions.

To this end, we chose a following approach. We first analyze distributions of FRR and ERR and see what they can tell us about the probability density of the projects in the $(x, c, \gamma)$ space. We then attempt to construct a probability distribution $p(x, c, \gamma)$ using our knowledge and compare if the distribution fits observations based in real data. We then revise our distribution until we reach convergence.

Variables $x$, $c$, and $\gamma$ are not measured directly in the data; however, what is measured are financial return rate (ERR) and economic return rate (FRR). We next try to express FRR and ERR for private/public projects in terms of $x$, $c$, $\gamma$ in our model.

8.2.1. Comparing ERR/FRR data to the model

In the real-world, each project is characterized by 4 parameters as mentioned before.
- Cost of the project
- Appropriable returns of the project if implemented by the government ($R_{government}$)
- Appropriable returns of the project if implemented by the private sector ($R_{private}$), (this variable accounts for the private sector advantage)
- Non-appropriable returns of the project, externalities ($E$)

Note that $R_g$, $R_p$, $E$, and cost are actual costs, not renormalized. So, we can express:

<table>
<thead>
<tr>
<th>Data Concepts</th>
<th>Model Concepts</th>
</tr>
</thead>
<tbody>
<tr>
<td>$FRR_{public}$</td>
<td>$(R_{government}/\text{cost}) - 1$</td>
</tr>
<tr>
<td>$FRR_{private}$</td>
<td>$(R_{private}/\text{cost}) - 1$</td>
</tr>
<tr>
<td>$ERR_{public}$</td>
<td>$(R_{government}/\text{cost} + E)/\text{cost} - 1$</td>
</tr>
<tr>
<td>$ERR_{private}$</td>
<td>$(R_{private}/\text{cost} + E)/\text{cost} - 1$</td>
</tr>
</tbody>
</table>
When we do a change of variables, we end up with three renormalized variables:

\[
\begin{align*}
\text{Equation 10 The } (x, c, \gamma) \text{ space mapped to model concepts} \\
\quad c &= \frac{\text{cost}}{R_{\text{government}}} \\
\quad x &= \frac{R_{\text{private}} - R_{\text{government}}}{R_{\text{government}}} \\
\quad \gamma &= \frac{E}{R_{\text{government}}} \\
\end{align*}
\]

Using these, we can now write ERR and FRR in terms of \(x, c,\) and \(\gamma\). The last column indicates if we have data for a given variable:

<table>
<thead>
<tr>
<th>Data</th>
<th>Model</th>
<th>Have data?</th>
</tr>
</thead>
<tbody>
<tr>
<td>(FRR_{\text{private}})</td>
<td>(\left(\frac{1+x}{c}\right) - 1; \text{private})</td>
<td>Yes</td>
</tr>
<tr>
<td>(ERR_{\text{private}})</td>
<td>(\left(\frac{1+x+\gamma}{c}\right) - 1; \text{private})</td>
<td>Yes</td>
</tr>
<tr>
<td>(FRR_{\text{public}})</td>
<td>(\left(\frac{1}{c}\right) - 1; \text{public})</td>
<td>No</td>
</tr>
<tr>
<td>(ERR_{\text{public}})</td>
<td>(\left(\frac{1+\gamma}{c}\right) - 1; \text{public})</td>
<td>Yes</td>
</tr>
</tbody>
</table>

From this table, we have three potential variables to compare between the model and the data. ERR is the only measurement we have for the public sector. For the private sector, we have two measurements: ERR and FRR. However, we note that ERR is a complex function of all 3 variables: \(x, c,\) and \(\gamma\). If we however subtract ERR from FRR, the difference has a very simple form. Indeed:

\[
\text{Equation 11 Deriving } \gamma/c \text{ result from returns data} \\
\quad \left(\frac{\text{ERR}_{\text{private}}}{\text{FRR}_{\text{private}}}\right) = \\
\quad = \left(\frac{1+x+\gamma}{c}\right) - \left(\frac{1+x}{c}\right) = \\
\quad = \frac{\gamma}{c}
\]

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So, the difference between ERR and FRR is only a function of two variables. Moreover, FRR depends on x and not on γ, while (ERR – FRR) depends on γ and not on x, separating the two important variables. As such, rather than comparing ERR and FRR for the private sector, we compare FRR, and (ERR – FRR).

Table 8 Distributions to Compare: Data <-> Model

<table>
<thead>
<tr>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$FRR_{private}$</td>
<td>$\left( \frac{1 + x}{c} \right) - 1$; private</td>
</tr>
<tr>
<td>$ERR_{private} - FRR_{private}$</td>
<td>$\frac{\gamma}{c}$; private</td>
</tr>
<tr>
<td>$ERR_{public}$</td>
<td>$\left( \frac{1 + \gamma}{c} \right) - 1$; public</td>
</tr>
</tbody>
</table>

8.2.2. Replicating the censoring mechanism: private and public sectors in the model

Since the distributions for government projects and for the private projects come from different projects, and have different distributions of ERR, we need to be able to separate the private sector and the public-sector projects in the space of $(x, c, \gamma)$.

It is unknown what kind of rules or interventions were used in real markets to assign projects to the private or public sector. Therefore, we decided to use very conservative ways of assigning them.
• ERR and FRR for private projects were taken from projects that are quintessentially private \((1+x > c)\)

• ERR for government projects was taken from the projects that should be funded by the government \((1 + \gamma > c)\) and are not quintessentially private.

8.3. Describing distributions of ERR/FRR in the data

We obtained distributions of ERR and FRR in the WBG data. A salient feature of the distributions of return rates is that the histograms are relatively narrow. Return rates, both ERR and FRR, are rarely over 40%. When we plotted two relevant distributions for the private sector: FRR and (ERR – FRR), we found that they have a mode below 0.2. For the public sector, we only have the distribution of the economic return rate. That distribution was somewhat wider than for the private sector and has much heavier tail.

The unique form of the distributions above allows us to infer what the probability distribution of projects looks like in a space of \(x, c, \text{and } \gamma\). Since distributions are very narrow, and centered around small positive values, we can treat ERR and FRR from these distributions as small positive numbers. We use epsilon to denote such a small number below.
8.3.1. Private sector ERR

As shown above, we do not directly compare the ERR data for the private sector due to its complicated mathematical form. Instead, we study the distribution of (ERR – FRR). We observe that the distribution of ERR - FRR for private projects is very narrow and has most of the weight around zero. Since \( \text{ERR-FRR} = \gamma/c \), we know that the distribution of \( \gamma/c \) is narrow. Since \( c \) is from 1 to 2, and \( \gamma/c \) is around 0 to 0.1, the distribution of \( \gamma \) itself should be also very narrow and close to 0. Mathematically, this will be written as:

\[
\frac{\gamma}{c} = \frac{\varepsilon_{\text{ERR-FRR}}}{c} = \varepsilon_{\gamma/c}
\]

Since \( c \) is between 1 and 2, \( c \times \text{epsilon} \) is still a small positive number, let’s call it “epsilon’”. Therefore:

\[
\gamma = \varepsilon_{\gamma/c}'
\]

where \( \varepsilon_{\text{ERR-FRR}} \) and \( \varepsilon_{\gamma/c}' \) are small positive numbers. We will now use similar logic for the other distributions.
8.3.2. Private sector FRR

Figure 28 Histogram of FRR for Observed Projects undertaken by the Private Sector

FRR in our model corresponds to \( \left( \frac{1+x}{c} \right) - 1 \). We will now use the fact that FRR distribution for the private sector is very narrow and is on the positive side of zero. We use the same mathematical approach: denote that \( FRR = \varepsilon_{FRR} \), and treat epsilon as a small positive number.

\[ FRR = \varepsilon_{FRR} \]
\[ \left( \frac{1+x}{c} \right) - 1 = \varepsilon_{FRR} \]
\[ \left( \frac{1+x-c}{c} \right) = \varepsilon_{FRR} \]
\[ (1+x-c) = c \times \varepsilon_{FRR} \]

Since \( c \) is between 1 and 2, \( c \times \text{epsilon} \) is still a small positive number, let’s call it “epsilon’”. Therefore:

\[ (1+x-c) = \varepsilon'_{FRR} \]
\[ (1+x) = c + \varepsilon'_{FRR} \]

This implies that the private projects are clustered around the plane \( (1+x) = c \), on the side of \( (1+x) > c \) from this plane.

We note that the plane \( (1+x) = c \) is the plane of profitability. Therefore, the conclusion of this part is that most private projects are “barely profitable”, which is consistent with the original observation that the FRR is very small. This also implies that the distribution of \( x \) and \( c \) are not independent. Instead, since the projects are clustered
around \((1 + x) = c\) plane, then \(x\), and \(c\) must be correlated. We estimate this correlation later. Indeed, it is natural to assume that appropriable returns of the private sector must be correlated with the project cost.

8.3.3. Public sector ERR

Distribution of ERR for the public sector is less narrow than private sector ERR, but also mostly centered towards 0. This implies that the distribution of \(\gamma\) for public sector projects is wider than the distribution of \(\gamma\) for private sector projects. We cannot say at this point how this can be used when constructing the probability distribution. However, if we observe this effect naturally, it will reinforce an idea that we inferred distribution correctly.

8.3.4. Correlation between \(x\) and \(\gamma\)

The last question to address is if there is any correlation between \(x\) and \(\gamma\). Since we have a matched data for the ERR and FRR for the public sector, then comparing FRR and \((\text{ERR} - \text{FRR})\) should give us some information about the correlation between \(x\) and \(\gamma\).

\[\text{ERR}_{\text{private}} - \text{FRR}_{\text{private}} = \frac{\gamma}{c}\]

\[\text{FRR}_{\text{private}} = \left(\frac{1 + x}{c}\right) - 1 = \left(\frac{1 + x - c}{c}\right)\]
Since $c$ is the same for each project, it suggests that correlation between FRR and (ERR-FRR) should be like the correlation between $(1 + x - c)$ and $\gamma$.

When we plot FRR and (ERR-FRR) for private projects, we can see that there is no visual correlation between the two. Spearman correlation is very weak, although significant ($\text{Spearman } r = 0.15$, $p = 0.0015$). While we can’t draw any conclusions from it directly, it gives us another quantity to compare between the model and the real data.

![Figure 30 ERR vs. FRR Spearman Correlation plot](image)

**8.3.5. Projects below profitability line**

Most of the private sector data contains successful projects that were intrinsically profitable, and therefore lay in the space $(1 + x) > c$. However, in this study we are interested in the projects that are below the profitability plane; $(1 + x) < c$. We do not have any data about private projects that are not intrinsically profitable but can be profitable due to reforms or subsidies. Therefore, we initially assume that the density of projects is symmetric around the profitability plane. In the sections that follow we also assume that the number of unprofitable projects is exponentially increasing.
8.4. Takeaway: 4 parameters to compare

We can learn general features of the distribution of \((x, c, \gamma)\) from the distributions of ERR and FRR for the private and public projects. From the private projects, we learn that \(\gamma\) is a narrow distribution centered towards zero. From public projects, we learn that projects are tightly distributed around the \((1 + x) = c\) line. We also note the weak positive correlation between \(\gamma\) and \((x - c)\).

This gives us four parameters to compare between the model and the real data. Three are the distributions of (a) FRR, (b) (ERR - FRR) for private projects, and (c) ERR for public projects. The fourth is the fact that a weak correlation is present between FRR and (ERR-FRR) for private projects. We will be comparing these 4 observables when constructing the distributions in the next chapter.

Chapter 9. Selecting the 3D Model’s Probability Distribution

9.1. Uniform distributions

First, we tried the naive assumption of \(x, \gamma\) and \(c\) being uniformly distributed. We used our model to calculate simulated distributions of FRR for the private sector, ERR for the public sector, and ERR-FRR for the private sector. We compared them with distributions measured from the real data.
We found that distributions from the model were very different from the real data, as seen in Figure 32 Model fit assuming Uniform Distributions ($KS = 1.78$; ▼ better); $KS$ stands for the sum of the three two-sample Kolmogorov-Smirnov statistics.

9.2. Difficulty of inferring a distribution

The goal of this section is constructing a probability distribution of projects in a space of the spanned by $x$, $c$, and $\gamma$. We will construct a distribution, and then verify that it fits the data well by comparing the FRR and ERR for private and public projects to the ones inferred from the model. This comparison is only done over the censored part of the $(x, c, \gamma)$ space: either quintessentially private part (D), or projects which are worth funding by the government (above the $'1 + \gamma - x > 0'\ line$).
This leaves us with a third part of the space: projects that are not quintessentially private and are not worth funding by the government. Many of these projects, however, will be welfare improving if implemented by the private sector with help of reforms or subsidies. This space is essential for the model, and projects funded by reforms and subsidies reside in that part of the \((x, c, \gamma)\) space.

We therefore can separate the space of \((x, c, \gamma)\) into four parts. We have data for the two parts of the space: quintessentially private projects and projects worth funding by the government, green and orange points on the 3D plot. We are not interested in projects that are not welfare improving (gray). We are however interested in the blue projects, which are not profitable on their own but could be welfare improving.
In the next parts, we will construct a distribution of projects that covers the entire space. We will then see how the distribution fits the observed data in the green and orange parts of the space, quintessentially private or quintessentially public projects. Since we cannot fit projects in the blue space, we will make assumptions about how the density of projects in the green/orange parts of the space is related to the blue part of the space.

9.3. Constructing the data-driven distribution

We then turned to design a better probability distribution for projects $p(x, c, \gamma)$. We know that the distribution of $\gamma$ is narrow and centered towards zero. Therefore, for $\gamma$ we choose a truncated Gaussian $N(0, \sigma_\gamma)$, with varying values of the standard deviation $\sigma_\gamma$, and truncated to the interval $[0,1]$. This is consistent with the fact that distribution of $\gamma/c$ is narrow and close to zero, but positive. We assumed that $\gamma$ is independent from $x$ and $c$. This yields the following distribution for $\gamma$ (non-normalized)
Equation 15 PDF for γ after censoring assuming normal distribution

\[ P_\gamma \sim \exp \left( -\frac{\gamma^2}{2 \sigma_\gamma^2} \right), \gamma \in [0,1] \]

Since the distributions of ERR and FRR do not provide any information about c or about x, we have no knowledge about the probability distribution of c and x. We therefore make no assumptions about these distributions and focus on the joint distribution \((x, c)\).

Equation 16 PDF for x and c after censoring assuming bivariate normal

\[ P_{xc} \sim \exp \left( -\frac{(x - c + 1)^2}{2 \sigma_{xc}^2} \right), x \in [0,1], c \in [1,2] \]

Then, we assumed that \((x - c)\) is distributed as a Gaussian \(N(0, \sigma_{xc}^2)\) and truncated such that both x and c are in the [0,1] range. This models the fact that projects are tightly clustered around the profitability line, and our assumption that density of projects is symmetric around the profitability line. We assume no correlation between x and γ, as well as between c and γ, consistent with the fact that γ and \((x - c)\) were found to be very weakly correlated. This resulted in a two-parameter distribution characterized by \(\sigma_{xc}\), and \(\sigma_\gamma\). Resulting probability density (unnormalized) may be written as:

Equation 17 PDF for x, c and γ after censoring assuming normal distributions

\[ P_{xc, \gamma} \sim \exp \left( -\frac{(x - c + 1)^2}{2 \sigma_{xc}^2} \right) \ast \exp \left( -\frac{\gamma^2}{2 \sigma_\gamma^2} \right), x \in [0,1], \gamma \in [0,1], c \in [1,2] \]

9.4. Fitting the distribution to the data & cross-validation

We then used the Cramér–von Mises test statistic to find the goodness-of-fit between the distribution from the data, and from the simulations. Namely, we defined the goodness-of-fit as the sum of the three Cramér–von Mises statistics for the three histograms. We chose Cramér–von Mises the two-sample test statistic, and it is equivalent to a so-called energy distance, implemented in the `scipy.stats.energy_distance` function in Python as discussed in Chapter 3.
To separate training and validation datasets, we used 10-fold cross-validation, and 60/40 split into training and validation datasets. Specifically, we separated each data sample into 10 bins. We then used all possible combinations of 6 bins for training, and remaining 4 bins for validation, yielding 210 combinations.

We first generate simulated training data. We take all possible combinations of $\sigma_{xc}$ and $\sigma_{\gamma}$ from a 31x31 grid from 0.05 to 0.35 for $\sigma_{\gamma}$, and from 0.15 to 0.45 for $\sigma_{xc}$ with a step of 0.01. During the first run, a 100x100 grid was used from 0.01 to 1 for each variable. We noted that values of $\sigma_{xc}$ and $\sigma_{\gamma}$ reside well inside the grid defined above. As such, for a production run a smaller grid was used, allowing us to simulate larger cohorts. For the $(\sigma_{xc}, \sigma_{\gamma})$ in the 31 x 31 grid, we perform a simulation with $N = 200,000$ projects. We record simulated ERR and (FRR-ERR) data corresponding to 3 histograms from each simulation.

The validation part is performed 210 times for different splits into training and validation datasets as described above. For a training dataset, we find the goodness-of-fit with each of the simulated datasets. We then find the best-fitting values of $\sigma_{\gamma}$ and $\sigma_{xc}$ that maximize goodness-of-fit. For this value, we generate a large simulated validation dataset consisting of $N = 2,000,000$ projects. We evaluated the goodness-of-fit for the “validation World Bank data,” and the “validation simulation data.” The resulting data contains 210 optimal values of $\sigma_{xc}$ and $\sigma_{\gamma}$, as well as matching KS divergence and the correlations within the simulated data. Shown below are mean values and standard deviations over 20 replications. We found that estimates for $\sigma_{xc}$ and $\sigma_{\gamma}$ were very tight, with the standard deviation of 0.013 and 0.014 respectively. Mean values of $\sigma_{xc}$ and $\sigma_{\gamma}$ were 0.27 and 0.17 respectively:
Table 9 Overview of First Monte Carlo Cross Validation Results.

<table>
<thead>
<tr>
<th>sigma_xc</th>
<th>sigma_\gamma</th>
<th>KS (ERR-FRR), private</th>
<th>KS ERR, public</th>
<th>KS FRR, private</th>
<th>KS sum</th>
<th>Spearman corr. FRR and (ERR-FRR), private</th>
<th>0.077</th>
<th>0.634</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean</td>
<td>0.266</td>
<td>0.171</td>
<td>0.203</td>
<td>0.208</td>
<td>0.185</td>
<td>0.677</td>
<td></td>
<td></td>
</tr>
<tr>
<td>std</td>
<td>0.013</td>
<td>0.014</td>
<td>0.042</td>
<td>0.075</td>
<td>0.052</td>
<td>0.095</td>
<td>0.002</td>
<td>0.026</td>
</tr>
</tbody>
</table>

We found the resulting KS divergence acceptable, although not ideal. As expected, we found that x and c are correlated (Pearson r = 0.64, p < 1e-3). Interestingly, we also find that FRR and (ERR – FRR) are weakly but significantly correlated (Spearman r = 0.08, p < 1e-3), consistent with the weak correlation observed in the data (Spearman = 0.15, p = 0.0015). This result appeared naturally, as information about this weak correlation was not used in constructing the distribution. It supports our assumption that distribution of \( \gamma \) and x are largely uncorrelated.

We used the mean values of \( \sigma_{xc} \) and \( \sigma_\gamma \) for visualization purposes below. The resulting 3D distribution is hard to visualize. However, we can visualize distributions for x, \( \gamma \), c, as well as 2D histograms, and a distribution of (x – c).

Figure 35 Model fit assuming Data-Driven Symmetric Normal Distributions (KS = 0.6; ▼ better)
When we inspected the distributions carefully, we found that the mean of the second two distributions (ERR-FRR for the private sector and ERR for the public sector) were correct, but the tail of the distribution in the data is wider. Since both plots are related to γ, we thought that making the distribution of γ wider would increase the goodness-of-fit.

9.5. Improving the data-driven distribution

Exponential distribution has a heavier tail than Gaussian, so we tried a truncated exponential as the distribution for γ. We then repeated the same steps as in the previous part, including selection of the optimal parameters. Resulting probability density function (unnormalized) can be written as:

\[
P_{xc, \gamma} \sim \exp \left( -\frac{(x - c + 1)^2}{2 \sigma_{xc}^2} \right) \ast \exp \left( -\frac{\gamma}{\sigma_{\gamma}} \right), x \in [0,1], \gamma \in [0,1], c \in [1,2]
\]

Results are presented in the table below. All p-values in the table and below are for a 2-sided t-test with assumption of different variance. We find that KS divergence decreased for ERR of public projects (0.18 to 0.11; p=5e-7), and for (ERR – FRR) (0.26 to 0.09, p<2e-16). For financial return rate of private projects, KS divergence did not change significantly (p=0.45), consistent with the fact that it is related to x and not to γ. Total KS divergence decreased significantly (0.68 to 0.47, p=3e-81). The weak correlation between FRR and (ERR – FRR) for private projects remained.

### Table 10 Comparing Results from Monte Carlo Cross-Validation: Symmetry Around Profitability Line

<table>
<thead>
<tr>
<th></th>
<th>(\sigma_{xc})</th>
<th>(\sigma_{\gamma})</th>
<th>KS (ERR-FRR) private</th>
<th>KS ERR public</th>
<th>KS FRR private</th>
<th>KS sum</th>
<th>Spearman corr. FRR and (ERR-FRR); private</th>
<th>x and c correlation; all projects</th>
</tr>
</thead>
<tbody>
<tr>
<td>mean (Exponential γ)</td>
<td>0.263</td>
<td>0.106</td>
<td>0.122</td>
<td>0.161</td>
<td>0.187</td>
<td>0.471</td>
<td>0.061</td>
<td>0.639</td>
</tr>
<tr>
<td>std</td>
<td>0.014</td>
<td>0.010</td>
<td>0.037</td>
<td>0.064</td>
<td>0.032</td>
<td>0.077</td>
<td>0.002</td>
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</tr>
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<td>mean (Gaussian γ)</td>
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<td>0.171</td>
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<td>0.208</td>
<td>0.185</td>
<td>0.677</td>
<td>0.077</td>
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<td>std</td>
<td>0.013</td>
<td>0.014</td>
<td>0.042</td>
<td>0.073</td>
<td>0.032</td>
<td>0.095</td>
<td>0.002</td>
<td>0.026</td>
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<td>pvalue_exp_vs_gaus</td>
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<td>8.4E-12</td>
<td>4.5E-01</td>
<td>3.3E-81</td>
<td>2.1E-282</td>
<td>3.6E-02</td>
</tr>
</tbody>
</table>
Properties of the resulting distribution are presented below. We find that distribution for ERR of public projects, and (ERR – FRR) for private projects show extremely good match, both visually and in terms of KS divergence. We however note that FRR for private projects in the real-world has a mode at a slight positive value, and smaller probability of having FRR near zero. In our simulations, by construction, it has higher density at profitability line, FRR=0.

This effect in the real data may manifest the fact that selection in the real-world does not happen exactly at profitability line 1+x=\(c\). Instead, projects with very small profits would be present in the set more rarely, because investors would reject them more often. While such fuzziness of the project selection criteria is undoubtedly interesting, it is not in the original paper and is out of the scope of this study. Hence, we conclude that our model captures FRR for private sufficiently well.
9.5.1. Note on the variances in the distributions of ERR for private and public sector

We were originally puzzled by the observation that the ERR for the public sector has a wider distribution than ERR for the private sector. However, we were surprised to see that our model reconstructed both distributions naturally. It would be interesting to explore this effect in the future studies.

Figure 37 Comparing Distributions between Model Vs. Data for Width

9.6. Model with exponentially more non-profitable projects

In the previous section we assumed symmetry around the profitability line. Here, we replace this assumption with a more plausible one, from an economic perspective.

We notice that economic return rate is sharply peaked towards zero: there are many projects with slightly positive economic return. However, the WBG project data only includes the projects that were profitable on their own, or more precisely, that were calculated profitable after being undertaken by the public (World Bank) or private sector (IFC). In this paper, we are interested in the density of all projects, and not only the projects that were profitable on their own. Therefore, we can argue that if the density of projects increases as the project becomes less profitable, it should keep increasing as
projects become non-profitable. To put this assumption into math, we assumed a very simple form: density of projects is exponential in a distance from profitability line.

*Equation 19* PDF for x, c and γ after censoring assuming exponential distributions

\[ P_{x, c, \gamma} \sim \exp \left( -\frac{c - x}{\sigma_x} \right) \ast \exp \left( -\frac{\gamma}{\sigma_\gamma} \right), x \in [0,1], \gamma \in [0,1], c \in [1,2] \]

We repeated similar fitting procedure for this distribution. The most-optimal distribution and its marginals are visualized below.

*Figure 38 Model fit assuming Data-Driven Exponential Distributions (KS = 0.34; ▼ better)*

Interestingly, the model with more non-profitable projects performed better than two other models in a cross-validation test: sum of 3 KS statistic was 0.440, compared to 0.471 for a model with exponential γ (two-sided t-test p = 2e-4).
9.7. Takeaway: Model fits the data, but revisit symmetry assumption
Using our knowledge of ERR and FRR from the real data, we constructed the probability distributions of projects in the \((\chi, c, \gamma)\) space that match the real data well. We note, however, that we were only able to match the distribution for the quintessentially private projects \((1 + x > c)\), and for the projects that are worth doing by the public sector \((1 + \gamma > c)\). We do not have any data about projects that are strictly inside the “gray area” and thus are never profitable on their own but can be made profitable using financial instruments (reforms or subsidies). We needed to use additional assumptions to create the density of projects in this gray area. We assumed that density of projects is symmetric around the profitability line. We also assumed that distribution of \(\gamma\) for funded projects is the same as the distribution of \(\gamma\) for the projects in the gray area.

One assumption that it is worth revisiting in the future is the symmetry of the density of projects around the profitability line. One may expect that there should be many more non-profitable than profitable projects. In the future, we will revisit the model in which the density of projects increases as they become less and less profitable.
Chapter 10. Beta Distributions & the 3D Model

10.1. Constructing beta distributions for the 3D model

We then aimed to design a 3D probability distribution based in beta distributions. We first assumed that x, c, and γ are all distributed as beta distribution:

\[ P_{xc}, \gamma \sim \text{Beta}_x(a, b) \times \text{Beta}_y(c, d) \times \text{Beta}_C(e, f) \]

where a, b, c, d, e, f are parameters of the beta distributions.

We however note that in previous chapters we heavily relied on the fact that the projects are clustered around the profitability line, and hence x and c may be correlated. To account for this, we added correlation between x and c using a Gaussian copula (Charpentier, Fermanian, & Scaillet, 2006).

To generate a correlated 2D Beta distribution, we first sampled x and c from their beta distribution. We also created a bivariate normal distribution of the same length to implement the copula with the correlation between the two variables. We then rank-transformed the two Gaussian variables, which created a copula, which is a 2D distribution with uniform marginals and with correlation between the variables. To apply this copula to the beta distributions, we first sorted x and c, and then sampled x and c using ranks from a 2D copula. This introduced the desired correlation between x and c, while keeping marginal distributions of x and c as betas.

10.2. Fitting beta distributions & parameter search spaces

Fitting a distribution with 7 parameters is a very challenging problem. In addition, the goodness-of-fit measure that we are trying to optimize relies on our Monte Carlo simulation, and hence is itself intrinsically stochastic. Therefore, typical optimization routines such as gradient descent are not applicable to our problem because they will be
easily misled by stochasticity. We therefore turned to approaches that do not rely on
gradients and used random search as an optimization method.

Our function depends on 7 parameters: 6 parameters governing beta distributions that
range from 0 to infinity, and correlation that ranges from -1 to 1. We first aimed to
transform the correlation such that it belongs to the same range as parameters of beta
distribution and behaves similarly to them. We chose correlation to be governed by a
parameter g and defined it as: $\text{corr} = 1 - \frac{2}{(1 + g^2)}$. This would yield $\text{corr} = -1$ for
g=0, $\text{corr} = 0$ for $g = 1$, and $\text{corr} = 1$ for $g = +\infty$.

We then noted that parameters of beta distribution are best randomized by
multiplication, and not by addition. Indeed, a and b are strictly positive, and Beta (a, b)
has the same mean as $\text{Beta} (x * a, x * b)$, suggesting that multiplying parameters by a
factor is preferable to adding a factor to them. We therefore turned to log-normal
distributions to describe both the starting point for our search, and for the randomizing
procedure. Correlation between a and b resides on a different interval, namely from -1 to
1. Therefore, we transformed correlation to the parameter g that resides between 0 and
infinity as described above. This made it possible to apply the same mechanism to
transforming parameters of the beta distribution and the correlation.

To generate the random set of parameters, we utilized the following strategy. First,
we select a parameter that controls a scale of our starting guess, that we call “spread”. It
was selected from a uniform distribution Unif(0.5, 2). We then selected initial values of
parameters a, b, c, d, e, f, g as a log-normal distribution $\text{exp} ( N(0,1) * \text{spread})$. 

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We employed similar strategy for the random search. We modified the value of parameters in a similar way, but with using a smaller spread. We also chose a relatively long-tail distribution for spread, so that the random search would include both large and small steps. We chose ‘spread’ from the exponential distribution with $\lambda = 0.3$. We then multiplied each of parameters $a, b, c, d, e, f, g$ by $\exp(\mathcal{N}(0,1) \times \text{spread})$. If the spread was selected to be small, it multiplied the parameters by a random number very close to 1, effectively exploring the vicinity of the previous guess. If the spread was randomly selected to be large (e.g. 2) the beta parameters were multiplied by a large random number, attempting to push the solution to a new local minimum with a large jump.

Some parameters of the beta distribution may generate a distribution so extreme that none of the projects may end up in a quintessentially private space. To combat that, we rejected all the parameters sets that had less than 400 quintessentially private projects, or less than 400 government projects.

We perform the search by making a jump from a current set of parameters to a randomly shifted parameter set as described above. Then we simulate $N = 100,000$ samples for that parameter set and evaluated goodness-of-fit. If the goodness-of-fit became worse compared to the current set, we rejected that jump and tried again. If the goodness-of-fit became better, we re-simulated the parameter set with $N = 2,000,000$ samples. If the goodness-of-fit remained better than the previous point, we jumped to the new parameter set. If not, we rejected the jump. Since our search algorithm rejects most jumps, this strategy leads to about 20-fold increase in speed without sacrificing precision.

We validate the early stoppage time of our optimization by plotting the average Cramér–von Mises test statistic as a function of the iteration. We find that we reach
relatively good agreement after 100 attempts, and the statistic nearly stops changing after 1000 steps. The relative improvement at 3000 steps compared to 1500 is about 1%, which is well within the difference between two testing datasets.

10.3. Results using naive random search

Naïve search with random starting points generated a vast number of very different nearly-optimal solutions (high goodness-of-fit). Moreover, many solutions tended to converge to extreme cases of beta (skewed all the way to 0 or 1, or both; i.e. a u-shape).

Four distinct solutions chosen from the top 30 below.
Such high variability of results combined with high goodness-of-fit suggests that we do not have enough data to reliably fit distributions with 7 parameters. We therefore need to choose a different distribution, or to restrain our search space.

In addition, most of the results showed mean KS divergence in validation stage of 0.53, with a standard deviation of 0.21. We note that the average KS divergence between validation and training datasets is 0.32, which is comparable to the best KS divergence obtained using the Beta model (0.33). Taken together, it suggests that the Beta model has too many parameters and is overfitting the training data.

10.4. Takeaway: Overparametrized, Model Depends on Starting Seed

We then attempted to narrow the search space for the beta distribution by seeding it with a staring seed resembling the distributions we used before. We generated a Beta distribution like the one that is symmetric around the profitability line distribution, and the distribution with exponentially many non-profitable projects. The parameters of the Beta distributions to start them were as follows. For the exponentially more non-profitable projects distribution, we chose parameters (0.8, 1.5, 1.5, 0.8, 0.8, 6, 1): $x = \text{Beta}(0.8, 1.5), c = \text{Beta}(1.5, 0.8), \gamma = \text{Beta}(0.8, 6), \text{corr}(x, c) = 0$. For the symmetric around profitability line distribution, we chose $x = \text{Beta}(1.2, 1.2), c = \text{Beta}(1.2, 1.2), \gamma = \text{Beta}(0.8, 6), \text{corr}(x, c) = 0.5$.

We were surprised to see that the random search process converged to the parameters of beta distribution that were not too far from the starting position, and generally belonged to the same class.
Both distributions resemble the distribution they started with, and have slightly worse fit, potentially due to over-fitting the Beta distribution, as it has too many parameters.

Chapter 11. Simulating the Data-Driven 3D Model

From the previous chapter, we fit a probability distribution of projects that matches existing data well. In this chapter, we take those distributional assumptions from the past chapter to re-construct the results found by Cordella, with one small change. With c, project cost as a random variable, we are left with only one free parameter, efficiency of the reforms, r, so the final results plot is the optimal sequence of interventions given a level of efficiency of reforms.

11.1. From 2D to the 3D model

As in the 2D model, we start our Monte Carlo simulation by making random draws for a set of projects. However, we now are making draws in a 3D space. First, we remove all quintessentially private projects: 1+x > c. We then proceed as before, just treating c as a per-project variable rather than a constant. Finally, we then apply different interventions in the different proposed orders and calculate total welfare.
Both reforms and government funding are modelled based on strict cutoffs and did not change between the 2D and 3D model. However, subsidies require finding the optimal subsidy level. To simulate that, we had to modify the dynamic programming part of the code. Luckily, we were able to find the mathematical transformation that allowed us to calculate the optimal subsidy still in linear time.

11.2. Displaying results of the data-driven 3D model
In the main model, diagrams displaying project spaces used to be two-dimensional plots in (x, γ) space. For a 3D model these are now three-dimensional scatter plots. Visualizing 3D scatter plots is challenging so we used two ways to visualize each plot. For each sequence of interventions, we constructed two plots. First, we made a 3D scatter plot, with color representing the results of the intervention for each project. Second, we generated slices of the scatter plot in c direction, which generated more familiar scatter plots of slices in the (x, γ) space.

Below is an example of RSG and SRG cascade implemented on simulated data for r=0.7. As can be seen from the images, most interventions cluster around profitability line, x=c-1. We can also see that for this value of reforms, there are almost no projects left for subsidies. This is consistent with the behavior observed in the 2d model. Indeed, RSG and RGS strategies may be equal only if one of the two instruments, government or subsidies, leads to zero projects. Government funding, however, always picks up a small number of projects with nearly-zero x and large values of γ. However, subsidies usually have no impact because there are no projects left where applying subsidies would be welfare improving, because reforms are so efficient.
Finally, for a model with exponentially more non-profitable projects, the results look very similar. Abundance of non-profitable projects in the plots below can be seen as a large mass of non-funded projects around x=0, c=2. Interestingly, it did not change the allocation of project outcomes. The issue is best presented with plots of 2D cuts over ‘c’ to “flatten” the 3D model plot for Exponential γ Model. As above, we have r = 0.7, and present SRG and RGS sequence.
There are similar such plots generated from all the different sequences for each of the different models (i.e. distributional assumptions). A final note here is that we generated and inspected these graphs as fully interactive visualizations using the python package plotly and custom code helper package pyscatter3D (Plotly Technologies Inc., 2015).18

11.3. Optimal sequence of interventions

Since we now included $c$ into the model as a random variable, we can now plot the optimal sequence of interventions as a function of efficiency of the reforms, $r$, only. To this end, we run the model for each sequence of interventions, and for 50 different values of $r$ over the space $[0,1]$. We then calculate the total welfare for each sequence, rescaled such that the worst sequence is at 0, and the best sequence is at 1. The sequence at the top is therefore the most optimal relative to all others. For this model, we simulated 3 million draws of $(x, c, \gamma)$ for each value of $r$. 

18 Plot.ly & pyscatter3d
We find that if reforms are efficient, RGS/RSG are most optimal sequences, producing the same results. If reforms are inefficient, SRG/SGR are equivalent for small r, then SGR is better, and later in a small region SRG becomes the best solution. The general form of this solution closely resembles the main model for larger values of c.

Figure 48 Best Sequence given Efficiency of Reforms for Various Distributional Assumptions

Table 12 Overview of Results from Each Distributional Assumption

<table>
<thead>
<tr>
<th></th>
<th>SGR, SRG</th>
<th>SGR</th>
<th>SRG</th>
<th>RGS, RSG</th>
</tr>
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<tbody>
<tr>
<td><strong>Uniform distribution</strong></td>
<td>0.005 to 0.22</td>
<td>0.225 to 0.73</td>
<td>N/A</td>
<td>0.735 to 1</td>
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<tr>
<td><strong>Data-driven with gaussian gamma</strong></td>
<td>0.005 to 0.1</td>
<td>0.105 to 0.7</td>
<td>0.705 to 0.72</td>
<td>0.725 to 1</td>
</tr>
<tr>
<td><strong>Data-driven with exponential gamma</strong></td>
<td>0.005 to 0.055</td>
<td>0.06 to 0.67</td>
<td>0.675 to 0.715</td>
<td>0.72 to 1</td>
</tr>
<tr>
<td><strong>Data-driven with more non-profitable projects</strong></td>
<td>0.005 to 0.085</td>
<td>0.09 to 0.68</td>
<td>N/A</td>
<td>0.685 to 1</td>
</tr>
</tbody>
</table>

After we summarize transition points in the table above, we see that the results are surprisingly robust. The transition between subsidies-first and reforms-first occurs
between $r=0.685$ and $r=0.735$, which is a narrow range. The only potential difference is that SRG alone may or may not become advantageous right before the transition point.

11.4. Takeaway: Again, Subsidies > Bad Reforms
We found that the main result of the 2D model robust even when the model was extended to 3D. Indeed, we find that SGR provides much better results than the cascade if reforms are inefficient. If reforms are efficient, they perform better than SRG. Our analyses additionally suggest that SRG/SGR are universally acceptable sequences of interventions regardless of the efficiency of reforms, while RSG becomes the worst sequence of interventions if reforms are not efficient. Given the data, we also note that the trade off point where RSG (the cascade) overtakes SRG (subsidy first cascade) moves down from 0.735 in the original model to roughly 0.72 or 0.685 depending on the model.

Chapter 12. Concluding Remarks
In this study, we used Monte Carlo methods to replicate closed form results found in the normative model proposed by Cordella to evaluate the efficiency of World Bank Group project allocations using the sequence of interventions proposed by the Cascade. Cordella shows that the Cascade is only efficient if one believes there is a high effectiveness of reforms.

Once the results were replicated, this paper moved to test the robustness of the model with a variety of perturbations. We studied the effects of introducing positive and negative correlation between the two main input variables used in Cordella’s “Cascade Space,” namely $x$ and $\gamma$ variables, as well as the effect of changing distributions of $x$ and $\gamma$. We found that the results of the model were universally robust to all perturbations. By and large the results would shift numerically, but the overarching conclusions regarding the presumption of efficient reforms precluded the optimality of using the cascade. In
fact, to the untrained eye, it is not trivial to find the difference between the plots, all of which are put forth in the Figures & miscellaneous appendix under “Plots for correlation or anticorrelation between x and γ.”

With the help of our modifications to the model, we re-affirmed a set of observations proposed by Cordella when discussing the fact of using a “myopic financing model,” where interventions cannot be forward looking, particularly for subsidy allocations. For example, we found that the naïve selection of the subsidy level could lead to the subsidies being too generous, and as a result GSR strategy being more advantageous than SRG. Proving and simulating Cordella’s conclusions, we discuss the possibility that the selection of subsidies should consider future interventions. With this modification, SRG strategy could become optimal under most conditions and overtake the cascade in almost all cases, even with efficient reforms. Of course, targeting subsidies is a well-known problem in economics, and discussing it further is outside the scope of this paper. More importantly, it is often not trivial to assume legislators are indeed forward or backward looking, since calculating ex-ante the impact of interventions is not obvious.

To further investigate Cordella’s conclusions, we additionally evaluated the performance of each sequence of interventions relative to the best sequence. Here again Cordella’s conclusions were robust. Following his conclusions, we also found that SRG and SGR perform reasonably well under most circumstances (optimal, or >70% of optimal sequence). Sequences RSG/RGS were optimal if the reforms are efficient, but quickly became sub-optimal if the reforms are inefficient. GRS/GSR were rarely the best and performed sub-optimally compared to SRG/SRG in almost all cases.
In attempting to compare the model to the real data, we faced a set of obstacles not found by Cordella. We found that the variable ‘c’, defined as a ratio of the cost of the project to the appropriable returns of the public sector, cannot be held constant given our data analysis of the different characteristics from our sample’s distribution. More to the point, this is expected, since in the real-world ‘c’ would naturally vary between projects. We therefore extended the model to include ‘c’ as a random variable, thus extending the model to three dimensions. At the crux of the model are projects that are not normally accessible by the private sector without interventions. However, all the data we have is about successful projects (i.e. profitable public and private sector projects). We therefore had to make assumptions about the density of projects in the inaccessible area of the cascade space.

This allowed us to carefully construct the probability distribution from our model to match the Economic and Financial return data observed in successful World Bank Group projects in both public (World Bank) and private sector (IFC). We re-create from the normative economic model the data generating process for the 3 variables we observe in the real-world, despite them being censored.

Reconstructing the probability density of projects in the space of x, c, and γ was particularly challenging for a set of reasons. First, the ERR/FRR data we have translated to relatively complex functions of x, c, and γ. Second, we only have data for the projects that were ultimately carried-out by the World Bank Group, while the space of projects that are not undertaken is much larger, and potentially infinite (there are a myriad of ways to use resources ineffectively, but only a few ways to use them ‘wisely/effectively’). The latter required us to make strong assumption about how the density of successful
(profitable) projects relates to the density of projects that are not profitable. We used two assumptions: (1) the density of projects is symmetric around the profitability line, or (2) the density of projects increases exponentially as projects become more and more bad/unprofitable. The first assumption (1) seemed the most mathematically convenient but did not consider the economic underpinnings of this problem. The second assumption (2) stems from the economics idea that there are many possible ways to make wasteful interventions in the world and call them a ‘project.’

It remains unclear how to verify the distributional assumption, specially the density of projects in the inaccessible space. One potential way would be to use a market where subsidies or reforms were applied. However, it would require a lot of manual data curation, particularly creating by hand a project’s counter-factual or obtaining information about how successful a project was based on changes in our interventions.

Data generating processes aside, to fit model to our data, we used a Cramér–von Mises statistic and compared the resulting models against each other using a Kolmogorov-Smirnov statistic using a 10-fold 60/40 cross-validation for the KS statistics. This fitting and validation processes allowed us to leverage and compare several models with different distributional assumptions.

Assuming a Beta probability distribution for projects characteristics, we found that the optimization procedure produced many distinct solutions with high goodness-of-fit. This indicates that many possible probability distributions can fit the real-world data equally well. This effect mainly stems from data censoring. Since we can only match real-world data to our observed, but highly censored data, the distribution can take several different equally well-fitting shapes that are drastically different over the space
we do not observe. This is particularly problematic because of important parts of the Benchmark Space are unobserved. As such, additional assumptions were needed to match the density of profitable projects to the density of non-profitable projects. We began with a mathematically convenient symmetry assumption over the profitability line. However, a more reasonable assumption from an economics standpoint was to make distribution with an exponentially increasing number of non-profitable projects.

Another factor that leads to having too many well-fitting solutions when trying to fit a Beta distribution is that we have very limited information about the distribution of c. Indeed, most variables that we consider include either \( \gamma/c \), or \( x/c \), but not \( c \) itself. However, if we had FRR information for public sector projects, we would directly obtain information about the distribution of \( c \), since \( FRR = 1/c-1 \) for public projects.

We simulated the resulting 3D model and found that the major result of Cordella’s normative held. Indeed, RSG/RGS sequences were optimal if the reforms are efficient, while SGR was optimal otherwise. Our observation of SRG/SGR being optimal or close to optimal for all parameters also held for the data-driven 3D model.

Put simply, we started with Cordella’s conclusions and we can now safely re-affirm them thanks to a thorough statistical analysis of all his model’s main conclusions.
13. Bibliography


14. Appendices

14.1. Methodology appendix

14.1.1. Deconvolution Methods

Previous research sought to use statistical deconvolution to recover from the sum of appropriable returns to the private sector, only the private sector advantage. However, we shelved those efforts in favor or focusing on data generating processes for observed variables.

This did not work because the variance of the overall public-sector returns was larger than the private sector one. In fact, we were originally puzzled to measure that the ERR for the public sector has a wider distribution than ERR for the private sector. However, we were surprised to see that our model reconstructed both distributions naturally.

This is evidence that most likely the deconvolution was just not the correct method

14.1.2. Pearson Versus Spearman Correlation

One must define what we mean by correlation. Generally, when researchers discuss correlation what they are actually measuring is the Pearson correlation, defined as a linear
relationship between two normally distributed random variables. Once we are no longer using purely normally distributed random variables, the Pearson relationship between two variables will fluctuate.

Case in point, take the work in Chapter 6 discussing ‘Changing Assumptions of Normative Model,’ specifically the section on the ‘Independence assumption.’ There we have a simulation that first generates two correlated normally distributed random variables (measured using Pearson correlation). Then, we rank-order and transform them to get the desired pair of correlated Beta distributions. Once we measure the Pearson correlation of the Beta distributions, the initial correlation seems to have changed slightly. This is expected.

The Pearson correlation measures the linear relationship between two normally distributed random variables. The two simulated normal variables, once rank-ordered and transformed into a Beta distribution, are no longer even expected to be normally distributed. As such, measuring the linear relationship between the resulting Beta distributions is a different matter best served by measuring the Spearman correlation that focuses on the monotonic relationship between two random variables (Hauke & Kossowski, 2011).

More importantly, as Hauke and Kossowski discuss at length in their 2011 paper using real-world data, researchers should not see “the Spearman’s rank correlation coefficient as a significant measure of the strength of the associations between two variables” [when dealing with real-world data] (Hauke & Kossowski, 2011). Note instead that these two statistics are measuring different things: one is a linear relationship (Pearson) and the other is a monotonic relationship (Spearman) between random
variables. In our case we have an interesting problem with elements that lend themselves to using both correlations.

14.2. Figures & miscellaneous appendix
14.2.1. Simulations Matching graphs from Chapter 5

14.2.1.1. Matching Sequence: Government – Reform – Subsidy
14.2.1.2. Matching Sequence: Government – Subsidy – Reform

Figure 3g: GSR
14.2.1.3. Matching Sequence: Subsidy – Reform – Government

Figure 3e: SRG
14.2.1.4. Matching Sequence: Subsidy – Government – Reform

Figure 3f: SGR
14.2.1.5. Matching Sequence: Reform – Government – Subsidy

Figure 3d: RGS
14.2.2. Plots for correlation or anticorrelation between $x$ and $\gamma$

14.2.2.1. **Overall Picture**

14.2.2.2. **Zoomed in plots by correlation**

![Graphs showing correlation or anticorrelation between $x$ and $\gamma$.]
14.2.3. Distribution Assumption: Plots for Different Beta Distributions

14.2.3.1. $X = \text{Beta} (2, 8)$

14.2.3.2. $X = \text{Beta} (2, 2)$
14.2.3.3. \( X = \text{Beta} (8, 2) \)

14.2.3.4. \( X = \text{Beta} (0.5, 2) \)
14.2.3.5. \( X = \text{Beta}(0.5, 0.5) \)

14.2.3.6. \( X = \text{Beta}(2, 0.5) \)
14.2.4. A Note on the Efficiency of Subsidies & the Overall Normative Model

The Cordella normative model to evaluate the Cascade we simulated teaches us an important lesson about the interplay between different instruments used to close the investment gap. As we can see from the paper and our simulations, if reforms are efficient, they are always preferred to government funding or subsidies. This is because efficient reforms have minimal welfare costs, while still leveraging the private sector advantage. Subsidies are more costly than efficient reforms, and suffer from imperfect targeting, while government funding suffers from not leveraging \( X \), the private sector advantage (whatever they may be understood to be).

When reforms are less efficient, the welfare cost of reforms increases, and both subsidies and government funding soon surpass inefficient reforms. In that case, we are left with a competition between subsidies and government funding. Subsidies suffer from imperfect targeting, and even under perfect targeting have a certain welfare cost. On the other hand, government funding, again lacks the private sector advantage. As we saw in the model, under most conditions, subsidies end up being more efficient than government funding. Even the cases where technically “government first” strategy was better than “subsidies first,” reducing the subsidy threshold would have made “subsidies first” strategy succeed. Therefore, we conclude that under this model, if reforms are inefficient, subsidies first are almost always a better strategy than government first.

14.2.4.1. Shortcomings of Subsidies in the Original Model

We next consider whether subsidies as modeled are too idealistic when compared to real-world scenarios. First, we note that in the model, only projects that are not profitable without subsidies will take the subsidies. "quintessentially private" projects are out of the
scope of the analysis. However, in real-world it may be hard to target subsidies only to the projects that need them, and certain quintessentially private projects may end up receiving subsidies. Effectively, this would mean that some percent of subsidies would be simply lost, since the boundary between ‘quintessentially private’ and the ‘Cascade Space’ is likely not that clean.

Additionally, the model assumes 100% success rate among the projects that received the subsidies. However, the private sector does not have a way to perfectly predict whether it will be profitable, and it is safe to assume that some number of projects receiving subsidies may fail to reach positive income. This would also result in some fraction of allocated subsidies generating zero welfare gains.

We therefore extend our model to incorporate the fact that some fraction of the subsidies that are given out would generate zero welfare returns. As expected, we find that after a given ‘failure rate’ threshold, subsidies stop being advantageous, and GSR/GRS strategies prevail depending on the efficiency of reforms.

14.2.4.2. Model with Inefficiency of Subsidies

This modification presents a conceptually important modification of the model. With competition between the factors, and two parameters controlling efficiency of the two factors, it is possible to obtain nearly all possible solutions. Indeed, fully efficient reforms lead to “reforms first” solution, while fully efficient subsidies lead to “subsidies first” solution if reforms are not very efficient. Moreover, inefficient reforms and subsidies lead to “government first” solution, as expected. This creates a different regime in which “subsidies first”, “government first” and “reforms first” could all be the best strategy.
Unfortunately, we did not get access to data on subsidy efficiency to suggest a possible subsidy failure rate, nor were we able to re-create such a measure from the data given. Future researchers looking into this model might wish to consider the mathematical formulation and bring in observable on the trade-offs for reforms. This could likely be done if one restricts the space for projects to a given sector, say the energy sector, or subsidy reforms data might be more forthcoming. However, this is outside the scope of this project.

14.3. Data appendix

14.3.1. World Bank projects

For World Bank Group data, we have a comprehensive country tag for each project, and because of the large number of records, we also have multiple projects in each country. There is still an issue with regional projects as well here.
The histogram is quite interesting. First, we see a very strong right skew on the dataset, as expected, since this is a percentage return, so capped at a minimum of 100% on the left-hand side, but potentially extremely high on the right-hand side. In fact, we truncated the bins above 110% return to be able to see the data. Furthermore, the green shows that if we make the bin size small enough, there seems to be two histograms super imposed on one-another. In fact, this prompted us to investigate deconvolution as a means of analysis in the start of this research. Unfortunately, that chapter did not pan out. Finally, we note in the dotted blue circle the low outliers could also be truncated to better understand what is happening in the dashed green circle area.

14.3.2. International Finance Corporation projects

The map below shows an estimate of all countries where for which we have project data. 46 of the projects in our dataset unfortunately do not have a country in the current dataset, unfortunately. This precluded us from using country data in our analysis.
The histogram on the other hand is much more complete, has a similar right skew found in the government projects, and a few outliers, however not as many negative projects, or weird agglomerations in rounded numbers (all a lot less projects overall, so one might expect that this would be the case.
14.4. Code appendix

This section contains some of the python code used, which benefited from extensive consultations and web searches; however direct block quotes are expressly stated, and code cited to the best of the author’s ability, since it is never obvious. The reader should feel free to use this code as they see fit but note that it is not being maintained anymore.

13.1.3. Monte Carlo methods & normative model:

The code in this section defines the main replication exercise for the Cordella paper’s normative economic model using Monte Carlo method simulations.

```python
1. def simulator definition
2. class Simulator(object):
3.     def __init__(self, c = 1.5):
4.         self.c = c
5.     def generate(mytype, N=10000, **kwargs):
6.         if mytype.lower() == "uniform":
7.             self.generateSimple(N)
8.         if mytype.lower() == "corr":
9.             self.generateCorrelation(N, corr = kwargs["corr"])
10.        if mytype.lower() == "beta":
11.           self.generateBeta (N, **kwargs)
12. def generateSimple(self,N = 1000000):
13.     #generates a uniform distribution between 0 and c for x, and 0 to 1 for gamma
14.     self.x = np.random.random(N) * (self.c - 1)
15.     self.gamma = np.random.random(N)
16.     # if the fate is 0, a project is still available.
17.     # if the fate is more than 0, some funding was appleid to it
18.     self.fate = np.zeros(N)
19. def generateTriangularGamma(self, N=1000000):
20.     self.x = np.random.random(N) * (self.c - 1)
21.     self.gamma = np.random.triangular(0, 0.5, 1, N)
22.     self.fate = np.zeros(N)
```
# when funding is applied, we put welfare returns here
self.returns = np.zeros(N)

def generateCorrelation(self, N=1000000, corr=0.5):
    # this generates a multivariate normal. We generate more points than we need and then cut them
    newN = int(N / (self.c - 1) + 100 * np.sqrt(N) + 100)
    a = np.random.multivariate_normal((0,0), ((1,corr),(corr,1)),newN)

    # convert data to ranks
    x,y = [scipy.stats.rankdata(i)/len(i) for i in a.T]

    # cut it at c-1
    mask = x < (self.c - 1)
    self.name = "Correlated x and gamma \n corr={0:.2f}"
    x,y = x[mask], y[mask]

    # only retain up to N points
    x,y = x[:N], y[:N]
    self.x = x
    self.gamma = y
    self.fate = np.zeros(N)
    self.returns = np.zeros(N)

def generateuneven(self, N=100000, zero=1, xoffset=0.5, goffset=0.5):
    #This generates points with the PDF(x,gamma) = zero + xoffset*x+goffset*gamma
    self.name = "Linear distribution; \n PDF = \{0\} + \{1:.2f\} * x + \{2:.2f\} * gamma".format(zero, xoffset, goffset)
    assert zero + xoffset + goffset >= 0
    corners = [0, xoffset, goffset, goffset + xoffset]
    maxval = zero + max(corners)
    meanval = zero + np.mean(corners)
    toget = int(N * maxval / meanval + 0.1 + 50 * np.sqrt(N))
    x = np.random.random(toget)
    y = np.random.random(toget)
    z = np.random.random(toget) * maxval # a vector used for selection of
    x,y
    cutoff = z < zero + x * xoffset + y * goffset
    #selection procedure
    x = x[cutoff]
    y = y[cutoff]
    assert len(x) > N
    x = x[:N]
    y = y[:N]
    self.x = x
    self.gamma = y
    self.fate = np.zeros(N)
    self.returns = np.zeros(N)

def generateBeta(self, N=20000, xalpha=0.5, xbeta=0.5, galpha=0.5, gbeta=0.5):
    newN = int(1.3 * N / (self.c - 1) + 100 * np.sqrt(N) + 100)
    self.x = np.random.Beta (a=xalpha, b=xbeta, size=newN)
    self.x = self.x[self.x < (self.c -1)]
    if len(self.x) > N:
        self.x = self.x[:N]
    N = len(self.x)
self.gamma = np.random.Beta (a=galpha, b=gbeta, size=N)
self.fate = np.zeros(N)
self.returns = np.zeros(N)
self.name = {"Beta x=Beta ({0:.2f}, {1:.2f}) g=Beta ({2:.2f}, {3:.2f})".format(xalpha, xbeta, galpha, gbeta)}

def reform(self, alpha):
    ""
    Calculates the condition for reforms to be profitable. Then only selects projects that weren't funded yet sets their fate to 1 (code for reforms) And finally sets their welfare returns as per the formula found in Cordella's paper
    ""
    condition = self.x > self.c - 1 - alpha * self.gamma
    condition[self.fate != 0] = 0
    self.fate[condition] = 1

def subsidy_optimal(self):
    ""
    Uses dynamic programming to calculate most optimal subsidy
    ""
    c = self.c
    df = pd.DataFrame({"x":self.x, "gamma":self.gamma, "fate":self.fate}, index=range(len(self.x)))
    df = df.sort_values(by="x", ascending=False)  
    df = df[df["fate"] == 0]
    if len(df) == 0:
        return
    # we sort all the projects by X in descending order: first the projects that will get subsidies at the lowest s
    # later the projects that will get subsidies at increasing values of s
    # This is done because we'll be taking cumulative sum later, and subsidies are always applied to all projects at each given level of x regardless of their gamma
    # with X > X_cutoff = c - s + 1
    df = df.sort_values(by="x", ascending=False)  
    df = df[df["fate"] == 0]
    if len(df) == 0:
        return
    # Now we begin constructing the welfare
    # Welfare is (x + gamma + 1 - c + s*(1-c))
    # but s changes based on the cutoff we select so it will be added later
    df["welfare"] = df["x"] + df["gamma"] + 1 - c
    # Now we take cumulative sum of this welfare.
#Cumulative sum computes total welfare if projects up to this x were given subsidies
146. df["cumwelfare"] = np.cumsum(df["welfare"]).values

#But this is not a complete welfare yet because we don't have the s*(1-c) part because we don't have s yet
148. df["cumwelfare"] = df["welfare"][1:].values
149. # now we note that if projects up to X_cutoff received subsidies, then s = c - 1 - X_cutoff
150. # so we can just use a vector of s * (c-1) = (c - 1 - x) * (c-1) to calculate per-project subsidy
151. # but our cumwelfare vector has cumulative sums of welfare!
152. # So, if we are at position i in the vector X, then the cumulative sum has subsidies from (i+1) projects
153. # therefore, we must multiply by a (range() + 1) (python is 0-based so for 1 project we must multiply by 1)
154. df["cumwelfare"] = df["cumwelfare"] * (c - 1 - df["x"]) * (c - 1) * (np.arange(len(df)) + 1)

155.  # sometimes no values of s are good. Then there will be no positive values in cumulative welfare
156.  # so, we just quit
157.  if np.max(df["cumwelfare"]).values < 0:
158.    return
159.  # this is the index that maximizes cumwelfare vector. We will use it next
160.  maxind = np.argmax(df["cumwelfare"]).values
161.  # this is our optimal s. So, we can now use this s to find final welfare per project with optimal s
162.  s = c - 1 - df["x"][maxind].values
163.  df["welfareFinal"] = 1 + df["x"] + df["gamma"] - s * (c - 1) - c

164.  # now use index of our dataframe to recover positions of the finalized projects in the original vectors
165.  # then set fate for them and record their welfare returns
166.  self.fate[inds] = 2
167.  self.returns[inds] = df["welfareFinal"][maxind].values
168.  return df

169.  def subsidy_optimal_bad(self, s=0.1):
170.     
171.     Uses dynamic programming to calculate most optimal subsidy
172.     
173.     c = self.c
174.  
175.     #Here we are just putting X and gamma into a dataframe, and use range(len(x)) as an index
176.     #the latter is done so we can later use index to identify selected projects
177.     df = pd.DataFrame({"x":self.x, "gamma":self.gamma, "fate":self.fate}, index=range(len(self.x)))
178.  
179.     # we sort all the projects by X in descending order: first the projects that will get subsidies at the lowest s
# later the projects that will get subsidies at increasing value s of s
# This is done because we'll be taking cumulative sum later, and
# subsidies are always applied to projects
with X > X_cutoff = c - s + 1
    df = df.sort_values(by="x", ascending=False)
    df = df[df["fate"] == 0]

# sometimes there are no projects left anymore
if len(df) == 0:
    return

# Now we begin constructing the welfare
# Welfare is (x + gamma + 1 - c + s*(1-c))
# but s changes based on the cutoff we select so it will be added later
df["welfare"] = df["x"] + df["gamma"] + 1 - c

# Now we take cumulative sum of this welfare.
# Cumulative sum computes total welfare if projects up to this x were given subsidies

# But this is not a complete welfare yet because we don't have the s*(1-c) part
# because we don't have s yet
    df["cumwelfare"] = np.cumsum(df["welfare"]).values

# now we note that if projects up to X_cutoff received subsidies, then s = c - 1 - x_cutoff
# so, we can just use a vector of s * (c-1) = (c - 1 - x) * (c-1)
# to calculate per-project subsidy
# but our cumwelfare vector has cumulative sums of welfare!
# So, if we are at position i in the vector X, then the cumulative sum has subsidies from (i+1) projects
# therefore, we must multiply by a (range() + 1) (python is 0-based so for 1 project we must multiply by 1)
    df["cumwelfare"] = (df["cumwelfare"] + (c - 1 - df["x")]) * (np.arange(len(df)) + 1)) * (1 - s) - ((c - 1 - df["x")]) * (c) ) * (np.arange(len (df)) + 1)

# sometimes no values of s are good. Then there will be no positive values in cumulative welfare
# so we just quit
if np.max(df["cumwelfare"]).values) < 0:
    return

# this is the index that maximizes cumwelfare vector. We will use it next
maxind = np.argmax(df["cumwelfare"]).values
sub = c - 1 - df["x"]].values[maxind]
df["welfareFinal"] = (1 + df["x"] + df["gamma"] + sub -c) * (1-s) - sub * (c)

# this is our optimal s. So, we can now use this s to find final welfare per project with optimal s

# now use index of our dataframe to recover positions of the financed projects
# then set fate for them and record their welfare returns
inds = df.index.values[:maxind]
self.fate[inds] = 2
self.returns[inds] = df["welfareFinal"]["values][:maxind]
return df

def finance(self):
    
    Similar to the other, we find projects that are worth financing, and set fate and returns for them
    
    condition = self.gamma > self.c - 1
    condition[condition != 0] = 0
    self.fate[condition] = 3
    self.returns[condition] = self.gamma[condition] - self.c + 1

    def act(self, mystring, **kwargs):
        
        Based on string like SRG, apply sequentially the financing instruments
        
        for i in mystring:
            if i == "G":
                self.finance()
            elif i == "R":
                self.reform(kwargs["r"])
            elif i == "S":
                self.subsidy_optimal()
            elif i == "C":
                self.subsidy_optimal_bad(kwargs["s"])
            else:
                raise ValueError("Unknown letter: {0}".format(i))

    def getReturns(self):
        
        "Gets sum of the welfare returns for funded projects"
        return self.returns[self.fate != 0].sum()

    def reset(self):
        
        "Resets fate and returns, so that we can try different sequence for the same sequence"
        self.fate *= 0
        self.returns *= 0

13.1.4. Main Dot-Plot code

The code below creates the main comparison plots used to validate the simulation replication results in python (i.e. verify that the simulations have the same results as Cordella’s closed for solutions when the simulation uses the same assumptions as Cordella.

1. def plotFancy(df, values, tolerancePercent = 1, title="", ylabel="c"):
2.     """Makes a funky plot with dots inside squares"
3.     Accepts:
dataframe: as define in later sections.
values: as defined below
tolerancePercent: how many percent difference in welfare is considered equal
two welfares are equal if they are different by no more
than (max - min) of welfares * 0.01 * tolerancePercent at
that point

```python
plt.figure(figsize=(5,4))
plt.xlabel("r")
plt.ylabel(ylabel)
welfares = np.array([i["welfares"] for i in df["ret"].values])
welfareSTD = np.std(welfares, axis=1)
goodmask = np.max(welfares, axis=1) > 0.001 * np.max(welfares)
welfares = welfares + (np.array([2,3,4,5,0,1]) * 1e-4)[None,:]
swelfares = np.sort(welfares, axis=1)
bestWelfares = swelfares[:, -1] # this is the highest welfare
welfareDiff = np.abs(swelfares[:, -1] - swelfares[:, 0])
df = df.iloc[goodmask]
welfareInds = welfareInds[goodmask]
bestWelfares = bestWelfares[goodmask]
welfareDiff = welfareDiff[goodmask]
welfares = swelfares[goodmask]
sizes = [120,30,8]
markers = ["s","o","o"]
for i,(m,s) in enumerate(zip(markers,sizes)):
    if i == 0:
        mask = np.ones(len(df), dtype = np.bool)
    else:
        mask = np.abs(bestWelfares - swelfares[:, -1-i]) < (welfareDiff * tolerancePercent * 0.01)
    plt.scatter(df["r"].values[mask], df["c"].values[mask], c = welfareInds[:, -1-1][mask],
            cmap = truncate_colormap(plt.get_cmap("Set1"), maxval=0.66), vmin=0, vmax=5, linewidth=0,
            s=s, marker=m)
plt.title(title)
plt.xlim([-0.05,1.05])
plt.ylim([0.95,2.05])
plt.gca().set_aspect('equal', 'box')
ax = plt.colorbar()
ax.set_ticks(range(6))
ax.set_ticklabels(values)
plt.tight_layout()
return welfares
```

13.1.5. Creative Common license for code

The code in this thesis is protected under a creative commons license, in particular
‘Attribution-Noncommercial’ see below.
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This work is copyrighted with a creative commons license where not directly specified on a ‘copyleft’ basis. See below. However, there is an application for a copyright patent pending that was submitted August 8\textsuperscript{th} 2018. Once the copyright claim has been processed by the US Government, the other licenses herein will no longer be valid.