Numerical Investigation of Pattern formation

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Abstract

I investigate pattern formation in a two-phase system on a two-dimensional manifold by numerically computing the minimizers of a Cahn-Hilliard-like model for micro-phase separation of diblock copolymers. The total energy of the system includes a short-range term - a Landau free energy and a long-range term - the Otha-Kawasaki functional. The short-range term favors large domains with minimum perimeter and the long-range inhibitory term favors small domains. The balance of these terms leads to minimizers with a variety of patterns, including single droplets, droplet assemblies, stripes, wriggled stripes and combinations thereof. For demonstration purposes, I focus on the triaxial ellipsoid, but our methods are general and can be applied to higher genus surfaces and surfaces with boundaries.

Introduction

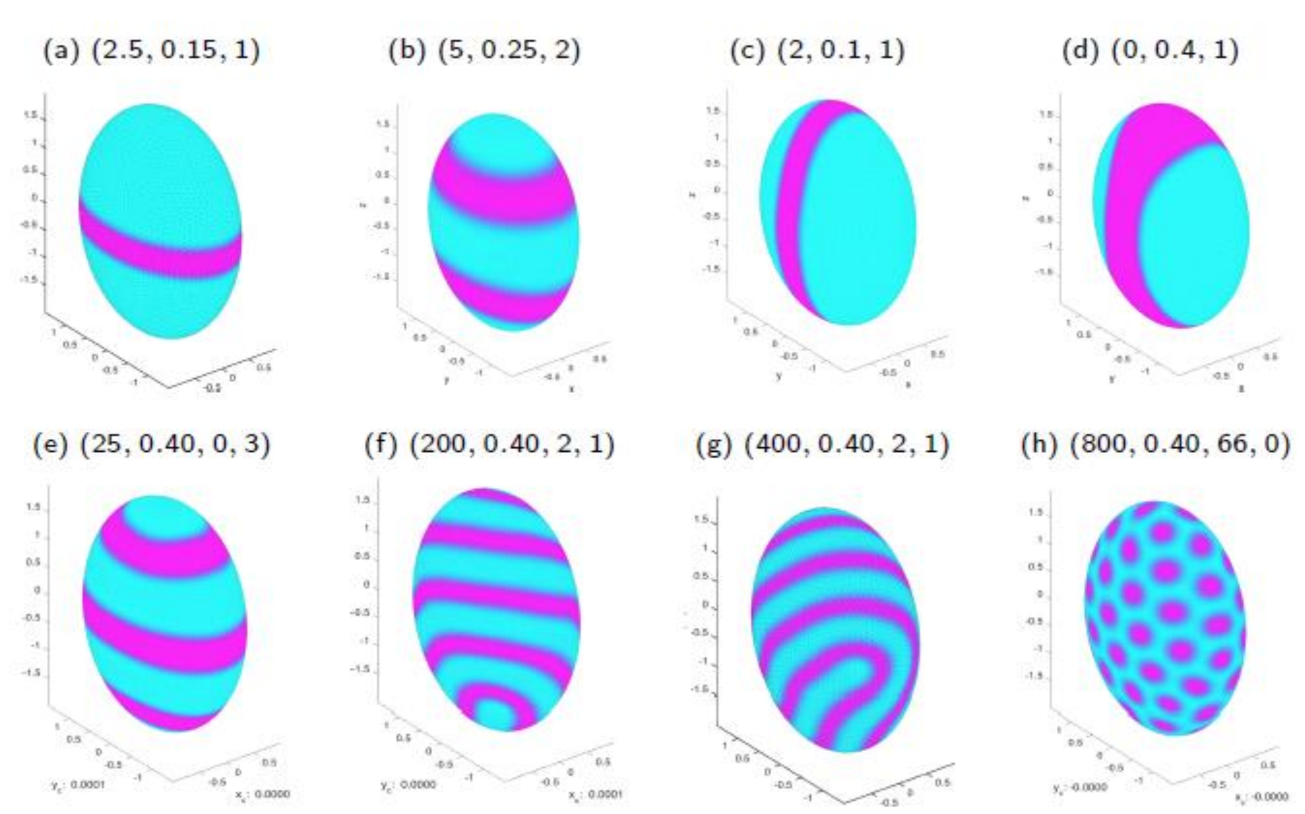
I'm studying a Cahn-Hilliard-like model

$$I_b(u) = \int_M \left\{ \tfrac{1}{2}\epsilon^2 |\nabla u|^2 + \beta u^2(1-u)^2 + \tfrac{1}{2}\epsilon\gamma \left((-\triangle)^{-1/2}(u-\omega) \right)^2 \right\} dS$$

The first two terms is named Landau free energy and the last term is a long range inhibitory interaction term named Otha-Kawasaki functional. My investigations are carried out by numerically computing the minimizers over a suitable class of admissible functions subject to $\int_M u dS = \omega$.

Solutions with stripes

Figure 3: Solutions with stripes $\beta = 1, 0 < \gamma < 800$. Sub-caption denotes $(\gamma, \omega, n_s, n_d)$.



Solutions with $0 \le \gamma < 7.5$

The minority constituent ($\phi = 1$) is with the dark (magenta) color, and the second constituent ($\phi = 0$) is with the light (cyan) color.

Figure 2: Small γ Case Study: $0 \le \gamma < 7.5, \omega = 0.25, \beta = 0.25$

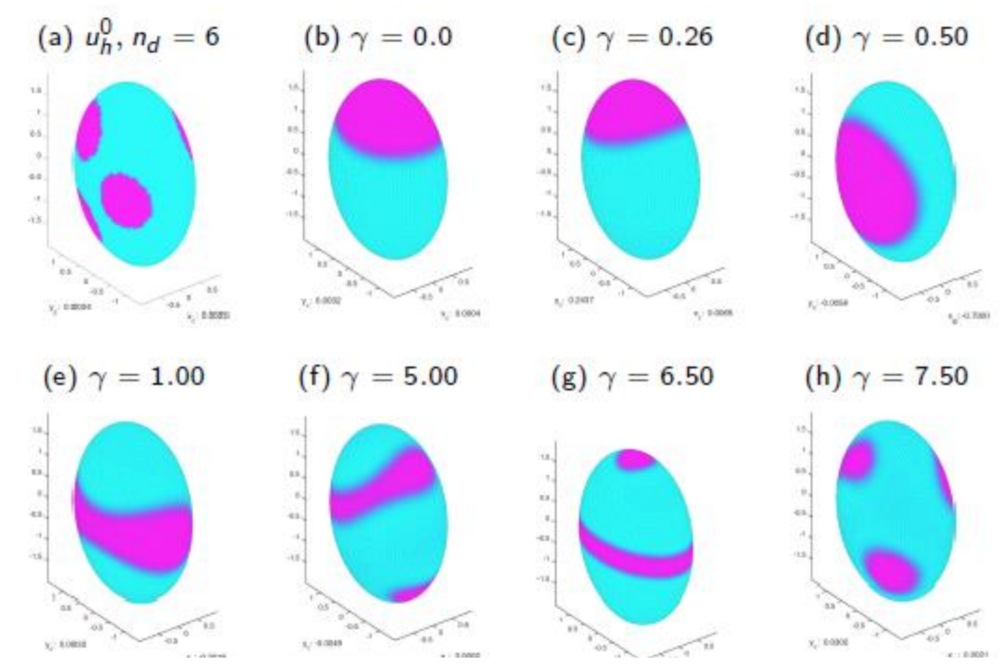


Table 1: Solutions with $0 \le \gamma < 7.5, \omega = 0.25, \beta = 0.25$

					Stability	
Fig. 2	γ	n_d	n_s	I_b^h	$\tilde{S}_h$	S_h
(b)	0.00	1	0	0.04187	s	u
(c)	0.26	1	0	0.07123	s	u
(d)	0.50	1	0	0.08609	s	u
(e)	1.00	1	0	0.11097	s	u
(f)	5.00	1	0	0.17979	s	u
(g)	6.50	2	1	0.19249	u	u
(h)	7.50	3	0	0.19159	u	u

Future Works

It may be of interest to investigate the remnant function and in particular determine where its maximum occurs so that we better understand the single droplet solution for small values of gamma as well as the transition to disk assemblies and stripe assemblies. Our solutions are in agreement with analytical results of models that predict a single droplet in the shape of a geodesic disk centered about a point of maximum Gauss curvature when gamma is zero or very close to zero.

Mesh refinement: $\gamma = 10$

Figure 6: Solutions for $\gamma = 10, \omega = 0.25$, and $h = 0.045$.

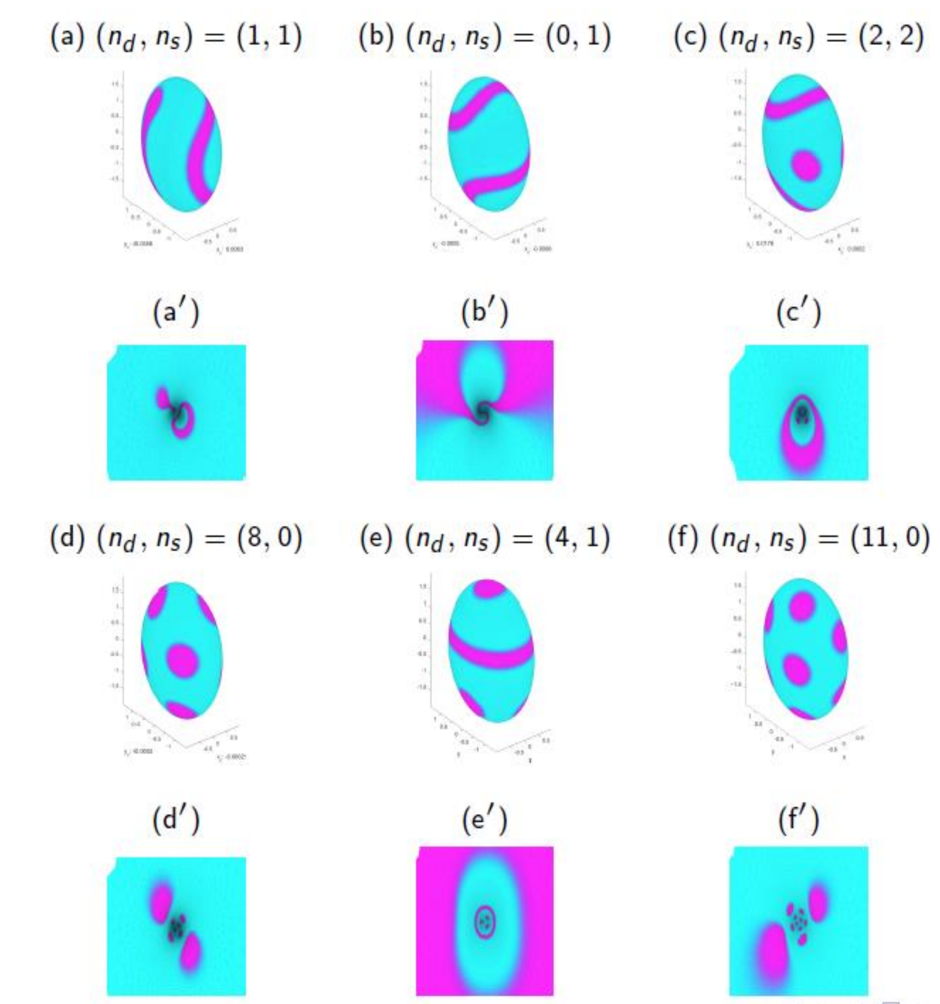


Table 3: Solutions with $h = 0.045, \gamma = 10, \omega = 0.25, \beta = 0.25$

				Stability	
Fig. 6	n_d	n_s	I_b^h	$\tilde{S}_h$	S_h
(a)	1	1	0.11920	s	u
(b)	0	1	0.12130	s	u
(c)	2	2	0.12098	s	u
(d)	8	0	0.11031	s	u
(e)	4	1	0.11546	s	u
-	5	0	0.11333	s	u
-	6	0	0.11182	s	u
-	7	0	0.11079	s	u
-	9	0	0.11124	s	u
-	10	0	0.11202	s	u
(f)	11	0	0.11377	s	u

Acknowledgements

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