Pólya Urn Models with Nonconstant Additions

By Sarah Yan Xu

B.S. in Mathematics, May 2011, Univeristy of Rochester, Rochester, New York

A Thesis submitted to:

The Faculty of
Columbian College of Arts and Sciences
of The George Washington University
in partial fulfillment of the requirements
for the degree of Master of Science

August 31, 2013

Thesis Directed by

Srinivasan Balaji
Professor of Statistics

Acknowledgements

I would like to express my deepest appreciation to all those who provided me the opportunity to complete this study. Without them, none of this would have been possible.
I would especially like to express my gratitude to my advisor, Prof. Balaji. Without his continuous optimism, enthusiasm, encouragement and support, this study would hardly have been completed.
I also would like to express my warmest gratitude to Prof. Mahmoud for introducing me to the interesting world of Maple.

Abstract

Pólya Urn Models with Nonconstant Additions

In this thesis, the discrete urn model with alternating Polya and Friedman addition matrices is considered. The urn has an initial composition of white and blue balls, and it is replenished by additional balls at odd times by Pólya addition matrix and at even times by Friedman addition matrix. The expectation and variance for the number of white balls at time n is obtained. We see that the "Friedman effect" is stronger than the "Pólya effect", and it is shown that the ratio of the number of white balls to the total number of draws approaches 1/2 when n gets large. A Central Limit Theorem is conjectured, and due to the difficulty in proving it theoretically, it is demonstrated using a detailed Maple simulation. Maple simulation also throws light on certain closely related problems with other alternating criteria. Some known results and methodologies in urn models are also discussed.

Contents

List of Figures

Chapter 1

Introduction

Urn models are stochastic processes that evolve over time by the drawing of a ball at random out of an urn (or multiple urns) at each discrete moment and adding or removing balls from the urn based on the associated replacement matrix. In the original Pólya Urn model, a ball is randomly drawn and put back into the urn after observing its color, together with a ball of the same color. In general, for urns containing balls of k different colors, the associated replacement matrix would be $((A_{ij}))$ where A_{ij} is the number of color j balls added, when color i ball is drawn, $i = 1, 2, \cdots, k$; $j = 1, 2, \cdots, k$. In general, the replacement matrix may also contain negative entries, which would result in removing balls from the urn. In such cases, we need to also make sure that the urn is tenable to avoid getting stuck. Mahmoud (2008) discusses the conditions for the urn's tenability in 2×2 urn models. In a replacement matrix of an urn model, the i^{th} row represents the action taken upon withdrawing color i ball each time, and the columns show the replacement actions for each color i ball drawn, A_{ij}.

The motivation behind this study is the application of these urn models as a powerful analysis tool in the real world. There are many instances where each colored ball can be used to represent the quantity of interest in the study. Debugging flaws in a system can be modeled as a Bernoulli removal process according to Siegrist (1987). Due to urn model characteristics, it is also heavily used in clinical trials. Wei and Durham (1978) explaines the application of random Bernoulli replacement in assignment for clinical trials using the play-the-winner model.

The main intent of this thesis is to study an alternating Pólya-Friedman urn and its asymptotics. In the literature, addition matrices with different conditions are considered but not the alternating case. This is very different from the random replacement matrix because here it deterministically alternates between the two matrices. Interestingly, it is known in the literature that the central limit theorem holds for the Friedman case and not for the Pólya case. Here we investigate the alternating case and find, using a computer simulation, that the central limit theorem holds asymptotically, which is the main contribution of this thesis. We also obtain the results of the expectation and variance of the number of white balls after n draws, along with a result on the convergence of the ratio of the number of white balls to the total number of draws. We have used Maple simulation to demonstrate the plausibility of the central limit theorem, as it seems intractable to do it theoretically. In Chapter 2, we highlight common methodologies that are used to derive results on moments, particularly, the expectation and variance of the total number of white balls. We discuss the probabilistic method in detail. Additional methods for multicolored urn models can be found in Mahmoud (2008). In Chapter 3, we discuss the alternating Pólya Friedman urn, and its results. In Chapter 4, we provide the conclusion and future work. The appendix contains the Maple code and simulations.

1.1 Notation

The Replacement Matrix defines the replacement actions for each possible draw. In our study, only balls of two colors are considered, say white and blue. The replacement matrix is a 2×2 matrix shown as following,

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

when a white ball is drawn, A_{11} white balls and A_{12} blue balls are added to the urn. When a blue ball is drawn, A_{21} white balls and A_{22} blue balls are added to the urn. In this thesis, most of the 2×2 urn models are balanced with constant additions at each draw, which is $A_{11} + A_{12} = A_{21} + A_{22} = K, K \in \mathbb{Z}$.

The random variables, w_n, b_n, are the number of white balls and blue balls after n^{th} draw respectively. And so, w_0, b_0 are the intial number of white balls and blue balls in the urn. The random variable, τ_n, represents the total number of balls in the urn after n^{th} draw. We start initially with τ_0 balls in the urn.

Also, x_n is the number of white ball draws after n draws.

Chapter 2

Methodology

There are multiple approaches to understanding the contents of an urn model. This section will introduce four distinct methods for doing so and will walk through each one using the Póyla-Eggenberger Urn model. The purpose is to provide the details of each method and to illustrate their pros and cons. We ultimately chose the Conditional Probability Approach for our main study because of its adaptability to our problem.

2.1 Conditional Probability Approach

Most urn models have the Markov property, which means that the future state of the system depends only on the current state. Because of this, the study uses the conditional probability approach. This method is flexible for modifying the recurrence equations of w_n, b_n when the replacement matrix changes. However, modifications done in this way would run into computing limitations due to the complexity of the recurrence equations for calculating the expectations and variances. There are some approaches available to help solve the equations in another way or to reduce the time of calculations, and these will be introduced in the results section. For now, we will take the special case Póyla-Eggenberger Urn model to demonstrate the conditional probability approach.

The special case of Póyla-Eggenberger Urn model, also known as Póyla urn with constant addition one, has the replacement matrix ,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Based on this replacement matrix, if a white ball is drawn, 1 white ball will be added to the urn along with the original white ball. If a blue ball is drawn, there is no change in the number of white balls in the urn, but the total number of balls increases by one. This urn scheme gives rise to the recurrence,

$$w_{n+1}|w_n = \begin{cases} w_n + 1, & \text{with probability: } \frac{w_n}{\tau_n}; \\ w_n, & \text{with probability: } 1 - \frac{w_n}{\tau_n}. \end{cases}$$

By the expectation definition, the conditional average number of white balls, w_{n+1} at time $n+1$ is,

$$E(w_{n+1|w_n}) = (w_n + 1) \cdot \frac{w_n}{\tau_n} + w_n \cdot \left(1 - \frac{w_n}{\tau_n}\right).$$

Taking the expectation again, we get,

$$E(w_{n+1}) = E(w_n) \cdot \left(\frac{1}{\tau_n} + 1\right).$$

Solving for the recurrence equations above yields

$$E(w_n) = \frac{w_0}{\tau_0} \cdot n + w_0. \tag{2.1}$$

The variance is developed using the same approach as the one we used for the expectation. According to the variance definition, $Var(w_n) = E(w_n^2) - [E(w_n)]^2$, we solve for $E(w_{n+1}^2)$ to get the variance. In view of the replacement scheme, we compute $E(w_{n+1}^2)$,

$$E(w_{n+1}^2|w_n) = (w_n + 1)^2 \cdot \frac{w_n}{\tau_n} + (w_n)^2 \cdot \left(1 - \frac{w_n}{\tau_n}\right).$$

After taking the expectation of both sides and working out the recurrence, we find that

$$Var(w_n) = \frac{w_0 b_0 (n + \tau_0)}{\tau_0^2 (\tau_0 + 1)}. \tag{2.2}$$

2.2 Exchangeability Approach

The exchangeability approach is another probabilistic method for finding the expectation and variance of an urn model, but it is defined with narrower assumptions, which limits its versatility as compared to the probabilistic method above. If the addition of the two ball colors depends on each other, the exchangeability cannot be applied. The Pólya-Eggenberger Urn is an example that can use this approach.

Exchangeability of two random variables, X and Y, is defined by the relation,

$$P(X = x, Y = y) = P(Y = x, X = y).$$

This equation states that the joint distribution of a series of random variables are the same regardless of the order of the random variables. For continuity, we use the Pólya-Eggenberger special case to illustrate this method.

According to Eggenberger and Pólya (1923), let x_n be the number of white balls draws from the Póyla addition matrix after n draws has probability,

$$\begin{aligned} &P(x_n = z) \\ &= \frac{w_0(w_0+1)\dots(w_0+(z-1)\cdot 1)b_0(b_0+1)\dots(b_0+(n-z-1)\cdot 1)}{\tau_0(\tau_0+1)\dots(\tau_0+(z-1)\cdot 1)(\tau_0+z\cdot 1)(\tau_0+(z+1)\cdot 1)\dots(\tau_0+(n-1)\cdot 1)} \cdot \begin{pmatrix} n \\ z \end{pmatrix}, \end{aligned} \tag{2.3}$$

where,

x_n is the number of white ball draws after n draws ,

w_n is the number of white balls by nth draw,

τ_0 is the intial number of balls in the urn,which is $w_0 + b_0$.

To calculate the probability of the state transition after n steps, we use the exchangeability, which is derived from the joint distribution of independent trials. The simplest path to start with is by having the first z draws be white balls and the rest $n-z$ draws be blue balls. The rationale of using exchangibility here is that each path has the same probability given z white balls and $n-z$ blue balls. The probability of drawing a white ball at draw n is $\frac{w_n}{\tau_n}$. The probability of drawing a blue at draw n is $\frac{b_n}{\tau_n}$. Here, τ_n is independent of each draw result because $\tau_n = \tau_0 + n$. Thus there is a constant addition to the urn regardless of the color drawn. The denominator of Equation 2.3 is the same for all paths. All other paths with z, white balls and $n-z$ blue balls after n draws, have the same probability fraction functions as the one for the simplest path after rearranging the bottom and the top of their probability fractions.

Mahmoud (2008) proves the expectation of w_n through deriving the average of the number of times of drawing a white ball x_n, which is given by:

$$E(x_n) = \sum_{z=0}^{\infty} z \cdot P(x_n = z).$$

Then, working through the algebra, we simplify the expectation to

$$E(x_n) = n \sum_{z=1}^{n} \binom{n-1}{z-1} \cdot \frac{\langle \frac{w_0}{1} \rangle_z \cdot \langle \frac{b_0}{1} \rangle_{n-z}}{\langle \frac{\tau_0}{1} \rangle_n} = \frac{w_0 n}{\tau_0} \sum_{z=1}^{n} \binom{n-1}{z-1} \cdot \frac{\langle \frac{w_0+1}{1} \rangle_{z-1} \cdot \langle \frac{b_0}{1} \rangle_{n-z}}{\langle \frac{\tau_0+1}{1} \rangle_{n-1}}.$$

Reducing this sum via standard combinatorial identities, he derives the expecation as

$$E(w_n) = \frac{w_0}{\tau_0} \cdot n + w_0,$$

and he showes that the variance can be calculated in the same way by changing

$$E(d_n) = \sum_{z=0}^{\infty} z^2 \cdot P(d_n = z)$$

Thus,

$$Var(w_n) = \frac{w_0 b_0 (n + \tau_0)}{\tau_0^2 (\tau_0 + 1)}.$$

2.3 Analytic Approach

The analytic approach is based on Theorem 2.1 and Proposition 1 described below,

Theorem 2.1 *(Flajolet-Dumas-Puyhaubert, 2006). Consider a balanced urn with constant K addition at each draw ($a+b=c+d=K$), and intially w_0 white balls and b_0 blue balls. It has a replacement matrix as below,*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then the generating function of histories is given by

$$H(x_0, y_0, z) = X(x_0, y_0, z)^{w_0} Y(x_0, y_0, z)^{b_0}, \tag{2.4}$$

where the total count of histories is $H(1,1,z)$; the pair of $(X(x_0, y_0, z), Y(x_0, y_0, z))$ is the pair solution of the differential system above with intial condition $X(x_0, y_0, 0) = x_0, Y(x_0, y_0, 0) = y_0,$ and $x_0 y_0 \neq 0$. This pair is

$$X'(t) = X^{a+1}(t)\ Y^b(t),$$

$$Y'(t) = X^c(t)\ Y^{d+1}(t).$$

Proposition 1 *(Flajolet-Dumas-Puyhaubert, 2006). Let W_n, B_n be the random variables reprenting the number of color 1 and color 2 in a balanced urn with constant addition respectively at time n. Then, we have:*

$$P(W_n = w_n, B_n = b_n) = \frac{[x^a y^b z^n] H(x, y, z)}{[z^n] H(1,1,z)}, \tag{2.5}$$

where z is one of the parameters defined in the trivariate exponential generating function. Here, it is used to extract the coefficient of each z^n to compute the number of histories with w_n and b_n after n draw.

The rationale behind the theorem and proposition is attributed to the combinatorial angle using generating functions. This approach is similar to exchangeability in terms of using combinatorics, but the assumptions and model structures are different. The basic structure of this method is that for a balanced urn model, every history has an equal probability because the total number of balls is the same at each time regardless of the last draw or any of the past histories (a property of a balanced urn with constant addition). The analytic method can be applied to many 2×2 balanced urns. It provides the exact probability distribution of the number of white balls in the urn at time n rather than the asymptotic solution. However, the drawback of this method is in the complexity of the calculations. This approach may be difficult to adapt for. Again, we use the example of the Pólya-Eggenberger Urn to demonstrate this method and to show the difference between the analytic method and others.

Mahmoud (2008) showes that by applying the analytic method to the model, we can find the probability of a state with $W_n = w_n, B_n = b_n$ at time n, with 1 additional ball at each draw.

It is easy to see that the differential equations for the Pólya-Eggenberger model are

$$X'(t) = X^{a+1}(t)\ Y^b(t) = X^2(t),$$

$$Y'(t) = X^c(t)\ Y^{d+1}(t) = Y^2(t).$$

This yields the solutions:

$$X(t) = \frac{X(0)}{1 - X(0)t},$$

$$Y(t) = \frac{Y(0)}{1 - Y(0)t}.$$

By Flajolet-Dumas-Puyhaubert's theorem, for a balanced urn with initial w_0 balls and b_0 blue balls, the moment generating function of the number of possible histories to get w_n white balls and b_n blue balls after n draws is given by:

$$H(x_0, y_0, z) = \left(\frac{x(0)}{1 - x(0)z}\right)^{w_0} \left(\frac{y(0)}{1 - y(0)z}\right)^{b_0} \text{ where } x_0 \cdot y_0 \neq 0.$$

Flajolet-Dumas-Puyhaubert (2006) also states that the generating function of the urn histories can be expressed as $H(z) = \frac{1}{(1-Kz)^{\tau_0/k}}$, where $k = a + b = c + d = 1$.
All together, we get the equality $H(1,1,z) = \frac{1}{(1-z)^{w_0+b_0}} = \sum(-1)^n \cdot \binom{-\tau_0}{n} \cdot z^n$.
Then, the total number of possible histories after n draws is given by

$$[z^n]H(1,1,z) = (-1)^n \binom{-\tau_0}{n} = \binom{n+\tau_0-1}{\tau_0-1}.$$

Similarly, we can find that

$$\begin{aligned}[x(0)^{w_n}y(0)^{b_n}z^n]H(x(0),y(0),z) &= [x(0)^{w_n}y(0)^{b_n}z^n]((x(0)^{w_0} \cdot \sum_{j=0}^{\infty}\binom{j+w_0-1}{w_0-1} \cdot (x(0)z)^j) \\ &\cdot (y(0)^{b_0} \cdot \sum_{k=0}^{\infty}\binom{k+b_0-1}{b_0-1} \cdot (y(0)z)^k)) \\ &= \binom{w_n-1}{w_0-1} \cdot \binom{b_n-1}{b_0-1}.\end{aligned}$$

Hence,

$$\begin{aligned}P(W_n = w_n) = P(W_n = w_n, B_n = b_n) &= \frac{h_n(w_n,b_n)}{h_n} \\ &= \frac{[x(0)^{w_n}y(0)^{b_n}z^n]H(x(0),y(0),z)}{[z^n]H(1,1,z)} \\ &= \frac{\binom{w_n-1}{w_0-1} \cdot \binom{n+\tau_0-w_n-1}{b_0-1}}{\binom{n+\tau_0-1}{\tau_0-1}}.\end{aligned}$$

2.4 Eigenvalue and Eigenvector Approach

The eigenvalue and eigenvector approach is based on Smythe (1996), which introduces a method using eigenvalues and eigenvectors of the replacement matrix for verifying asymptotic normality under certain conditions. The method states that when the eigenvalue of the replacement matrix of the urn model is less or equal to half of its maxium eigenvalue, there is a potential asymptotical normality for the normalized number of color i ball and color i ball draws after the n^{th} draw given the satisfication of all other hypotheses in Theorem 2.2.

Theorem 2.2 *(Smythe (1996)). Suppose $Z = (Z_1, Z_2)$ is an Extended Pólya Urn model satisfying the hypotheses that for a random matrix A:*

(a) constant row sums (the maximal eigenvalue of the replacement matrix, λ_1)

(b) finite second moments of each replacement matrix element, A_{ij}

(c) λ_1 is the maximal positive eigenvalue of multiplicity 1 with strictly positive left eigenvector v.

(d) $2\lambda < \lambda_1$

where, λ is any other eigenvalue of A.

Then,

$$\frac{w_n - \lambda_1 \cdot v_1 \cdot n}{n^{1/2}} \xrightarrow{D} N(0, \sigma^2), \tag{2.6}$$

where v_1 is the first component of the eigenvector corresponding to λ_1

This eigenvalue method takes a different approach than all other methods discussed. It estimates the urn scheme using the eigenvalue and eigenvector of the urn replacement matrix, and the Martingale Central Limit Theorem. It is also suitable for analyzing urns with more than 2 types of balls with additional initial assumptions as explained in Smythe (1996). Here we provide the example of this approach as applied to the Pólya-Eggenberger.

In the Pólya-Eggenberger case, the eigenvalue is

$$(\lambda_1 - 1) \cdot (\lambda_1 - 1) = 0 \Longrightarrow \lambda_1 = 1 \text{ with multiplicity of } 2,$$

which violate Theorem 2.2 Hypotheses *c*. Therefore, by the eigenvalue and eigenvector approach by R.T.Smythe, we cannot conclude that the normality of the Pólya-Eggenberger holds, which is consistent with Eggenberger and Pólya (1923) where the authors proved that the scaled number of white ball draws in the model is a beta distribution.

Chapter 3

Results

The main interest of the study is in understanding the contents of the Pólya/Friedman alternating urn model using the probabilistic approach described in the methodology. We apply these approachs to other alternating models and random replacement models, such as the Pólya/Ehrefest, Pólya($s = 1$)/Pólya($s = 2$), where s is the constant addition, and Bernoulli Replacement urns.

3.1 Alternating Pólya/Friedman Urn

We studied the contents of the Polya/Friedman alternating urn model using the probabilistic approach mentioned above. We calculate the expectation and variance using the conditional probability method. The limit of the variance over n^2 proves to be 0 as n goes to infinity. Consequently, we also show that the ratio of the white balls to the total number of draws converges to 1/2. We conjecture a Central Limit Theorem, but due to the complexity to obtain it theoretically, we instead ran simulations to test this alternating urn model, the results of which appear to be normal in distribution, matching the limit and Central Limit Theorem propositions. The model has the Pólya replacement matrix for odd draws and the simple Friedman model for even draws. The respective replacment matrices are shown below.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ for odd draws;}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ for even draws.}$$

In this section, we will determine the mean and variance of w_n. By Chebyshev's Inequality, we can capture some an important pattern of the number of white balls in this model. Together, our results of the alternating Pólya/Friedman Urn model is captured in the following theorem.

Theorem 3.1 *Take a balanced 2×2 urn initally containing 1 white and 1 blue ball with the Pólya addition matrix for odd draws, and the Friedman matrix for even draws. Let w_n be the number of white balls after the nth draw, then the ratio of white balls over number of draws is constant at n. Then we have*

$$\frac{w_n}{n} \xrightarrow{P} \frac{1}{2}. \tag{3.1}$$

Proof The total number of balls in the urn is incremented by 1 at each draw. The expectation and variance are computed by the conditional probability method described above. The number of white balls after $2n+1$ draws is derived from $E(w_{2n+1}|w_{2n})$ and $E(w_{2n}|w_{2n-1})$, which can be summarized as follows,

$$w_{2n+1}|w_{2n} = \begin{cases} w_{2n}+1, & \text{with probability: } \frac{w_{2n}}{\tau_{2n}}; \\ w_{2n}, & \text{with probability: } 1-\frac{w_{2n}}{\tau_{2n}}; \end{cases} \quad \text{at odd draw,}$$

$$w_{2n}|w_{2n-1} = \begin{cases} w_{2n-1}, & \text{with probability: } \frac{w_{2n-1}}{\tau_{2n-1}}; \\ w_{2n-1}+1, & \text{with probability: } 1-\frac{w_{2n-1}}{\tau_{2n-1}}; \end{cases} \quad \text{at even draw.}$$

By definition of expectation, we have

$$E(w_{2n+1}|w_{2n}) = \frac{w_{2n}}{\tau_{2n}} \cdot (w_{2n}+1) + (1-\frac{w_{2n-1}}{\tau_{2n-1}}) \cdot w_{2n} = \left(1+\frac{1}{\tau_{2n}}\right) \cdot w_{2n},$$

$$E(w_{2n}|w_{2n-1}) = \frac{w_{2n-1}}{\tau_{2n-1}} \cdot w_{2n-1} + \left(1-\frac{w_{2n-1}}{\tau_{2n-1}}\right) \cdot (1+w_{2n-1}) = \left(1-\frac{1}{\tau_{2n-1}}\right) \cdot w_{2n-1} + 1.$$

By taking the double expecatation above to remove the condition:

$$E(w_{2n+1}) = E(w_{2n}) \cdot \left(1 + \frac{1}{\tau_{2n}}\right),$$

$$E(w_{2n}) = E(w_{2n-1}) \cdot \left(1 - \frac{1}{\tau_{2n-1}}\right) + 1.$$

After working out the details, we get

$$E(w_n) = \begin{cases} (w_0 + \frac{n}{2}), \cdot(1 + \frac{1}{\tau_n}) & \text{when n is odd;} \\ (w_0 + \frac{n}{2}), & \text{when n is even.} \end{cases} \tag{3.2}$$

So the conditional second moment is,

$$E(w_{2n+1}^2 | w_{2n}) = \frac{w_{2n}}{\tau_{2n}} \cdot (w_{2n} + 1)^2 + (1 - \frac{w_{2n-1}}{\tau_{2n-1}}) \cdot (w_{2n})^2 = (1 + \frac{2}{\tau_{2n}}) \cdot (w_{2n})^2 + \frac{w_{2n}}{\tau_{2n}},$$

and we also have

$$\begin{aligned} E(w_{2n}^2 | w_{2n-1}) = & \frac{w_{2n-1}}{\tau_{2n-1}} \cdot (w_{2n-1})^2 + (1 - \frac{w_{2n-1}}{\tau_{2n-1}}) \cdot (1 + w_{2n-1})^2, \\ & = (1 - \frac{2}{\tau_{2n-1}}) \cdot (w_{2n-1})^2 + (2 - \frac{1}{\tau_{2n-1}}) \cdot (w_{2n-1}) + 1. \end{aligned}$$

By taking double expecatation, we remove the condition:

$$E(w_{2n+1}^2) = E(w_{2n}^2) \cdot \left(1 + \frac{2}{\tau_{2n}}\right) + \frac{E(w_{2n})}{\tau_{2n}},$$

$$E(w_{2n}^2) = E(w_{2n-1}^2) \cdot \left(1 - \frac{2}{\tau_{2n-1}}\right) + E(w_{2n-1}) \cdot \left(2 - \frac{1}{\tau_{2n-1}}\right) + 1.$$

The recursions for the variances listed above are significantly more complex than the recursions for the expectations due to the second power in the recurrence equations. In this study, Maple was used to compute the recurrence relations for variance. Here are the results for the recurrence relationship when $w_0 = 1, b_0 = 1$. The reason for this simplification is that the search for the optimal solution in Maple with more than two balls initially is too computationally complex and memory bound. The alternatives to this urn model with different initial conditions and different alternating rules are thus studied through the simulations.

We used Maple to solve the recurrence for the variance. The expressions for the recurrence for even and odd draws are very long, so only the leading terms in the final recurrences are shown:

$$E(w_n^2) \sim \begin{cases} \frac{2\cdot(\frac{-1}{2}+n)^5}{(\frac{-1}{2}+n)\cdot n\cdot(\frac{1}{2}+n)} \cdot \frac{2n+2}{2n+4} & \text{when n is even} \\ \\ \frac{-2\cdot(\frac{-1}{2}+n)^5}{(\frac{-1}{2}+n)\cdot n\cdot(\frac{1}{2}+n)} & \text{when n is odd} \end{cases}$$

The details of Maple coding can be found in Appendix. As $Var(X) = E(X^2) - (E(X))^2$, it can be shown that,

$$\sim n.$$

By taking the limit,

$$\lim \frac{Var(w_n)}{n^2} = 0, \qquad as\ n \to \infty. \tag{3.3}$$

By Chebyshev Inequality, we get

$$\begin{aligned} P[|w_n - E(w_n)| \geq \varepsilon \cdot E(w_n)] &= P[(w_n - E(w_n))^2 \geq \varepsilon^2 \cdot (E(w_n))^2] \\ &\leq \frac{Var(w_n)}{\varepsilon^2 \cdot (E(w_n))^2}. \end{aligned}$$

By Equation 3.3, we get,

$$\begin{aligned} \lim P[|w_n - E(w_n)| \geq \varepsilon \cdot E(w_n)] &= \lim P[|\frac{(w_n)}{E(w_n)} - 1| \geq \varepsilon] \\ &\leq \lim \frac{Var(w_n)}{\varepsilon^2 \cdot n^2} \\ &= 0. \end{aligned}$$

So, we have:

$$\frac{w_n}{E(w_n)} = \frac{w_n}{n/2 + w_0} \xrightarrow{P} 1,$$

$$\frac{w_n}{E(w_n)} = \frac{w_n}{(1 + 1/\tau_n)(n/2 + w_0)} \xrightarrow{P} 1,$$

and we conclude that,

$$\frac{w_n}{n} \xrightarrow{P} \frac{1}{2}.$$

3.1.1 Simulation

3.1.1.1 Simulation Results for Pólya/Friedman Model

Conjecture 3.1.1 *Take a balanced* 2×2 *urn initally containing 1 white and 1 blue ball with the Pólya addition matrix at odd draws, and the Friedman matrix at even draws. Let* w_n *be the number of white balls after* n *draws, then the Central Limit Theorem holds and the normalized number of white balls after the nth draw has a standard normal distribution,*

$$\frac{w_n - \frac{n}{2}}{\sqrt{n}} \xrightarrow{D} N(0, \sigma^2).$$

We could not prove the result theoretically as the calculation is complex. Hence, the simulation was conducted instead to verify the normality assumption. The simulation consists of 500 iterations with each simulation consisting of 1000 draws. The simulation plot takes the last draw of each simulation (there are 500 simulated data points), and then normalizes it, i.e. we compute

$$w_n^* = \frac{w_n - \frac{n}{2}}{\sqrt{n}}.$$

The x-axis shows the ratio $\frac{1}{2}$, and the y-axis shows the counts of each ratio in the 500 sample points. As we can see, Figure 3.1 suggests a normal distribution centered around 0, with constant variance, which gives credance our conjecture.

3.1.1.2 Simulation Results for Friedman/Pólya Model

Simulation Figure 3.1 veified the conjecture of normality of the Pólya/Friedman Model. Since it is versatile enough to apply to another similar model, we ran the simulation on the Friedman/Pólya urn model, with Friedman at odd draws and Pólya at even draws, as shown in Figure 3.2, which again suggested normality (although the actual proof requires

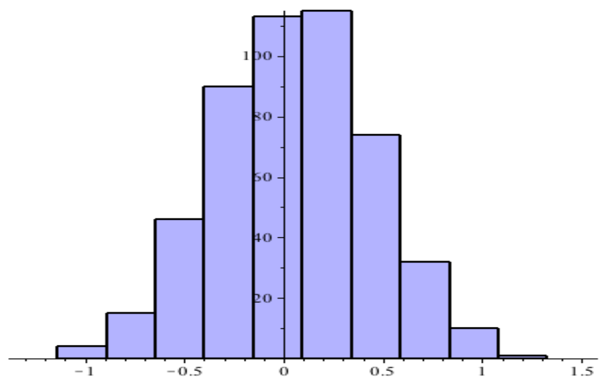

Figure 3.1: Simulation of Pólya/Friedman urn with (1,1)

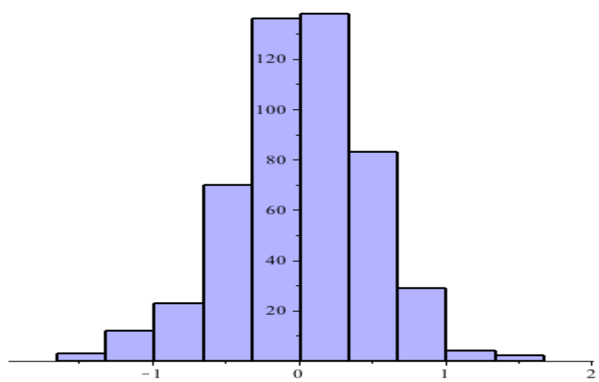

Figure 3.2: Simulation of Friedman/Pólya urn with (1,1)

future work).

3.1.1.3 Simulation Results for Friedman/Pólya Alternating in Every 5^{th} Draw

In another application, we alternated the Friedman urn to the Pólya urn only on every 5^{th} draw. Given that the intial state consists of 1 white ball and 1 blue ball, we believed that the normality would remain. In this case, the Friedman urn dominated the Pólya urn since the Pólya urn is only applied on every 5^{th} draw. Therefore, we assume that this special alternating urn will inheirt the nornamilty from the Friedman urn. We were able to verify this assumption using simulation although the theoritical proof requires further

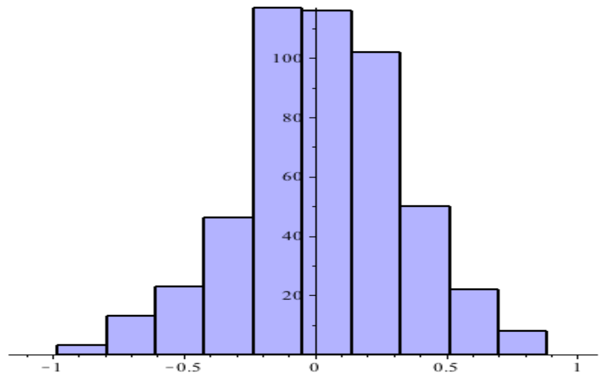

Figure 3.3: Simulation of Pólya/Friedman alternating in every 5^{th} draw with (1,1)

work. The simulation shows that the urn reacts to the alternation in a highly similar fashion to that of the Pólya/Friedman alternating every other time (the first model we stated in this section) as shown in Figure 3.3.

3.2 Alternating Pólya/Ehrenfest Urn

This model has two replacement matrices, the Pólya for odd draws and the Ehrenfest for even draws which are given by:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ for odd draws;}$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \text{ for even draws.}$$

By having the Ehrenfest urn for even draws, this model has a distinct characteristic for its replacement matrix in that there is no addition happening on the even draws. Thus this means that the total number of urn balls τ_{2n} is the same as τ_{2n-1}. This rationale is easy to see. The replacement matrix shows that when drawing a ball, the opposite colored ball is replaced into the Urn. In this case, exchangibility cannot be applied because the probability of drawing a white ball z times after n draws does depend on the draw indices due to the two replacement matrices. Therefore, instead of the combinatorics approach stated earlier, we use the more approachable conditional expectation method. Derived from the definition of the conditional expectation, we have:

$$w_{2n+1}|w_{2n} = \begin{cases} w_{2n}+1, & \text{with probability: } \frac{w_{2n}}{\tau_{2n}}; \\ w_{2n}, & \text{with probability: } 1-\frac{w_{2n}}{\tau_{2n}}; \end{cases} \quad \text{at odd draw,}$$

$$w_{2n}|w_{2n-1} = \begin{cases} w_{2n-1}-1, & \text{with probability: } \frac{w_{2n-1}}{\tau_{2n-1}}; \\ w_{2n-1}+1, & \text{with probability: } 1-\frac{w_{2n-1}}{\tau_{2n-1}}; \end{cases} \quad \text{at even draw.}$$

Then the expectation and variance for the even and odd draws are calculated as illustrated in the other examples. We find that

$$E(w_{2n+1}) = E(w_{2n}) \cdot \left(1+\frac{w_{2n}}{\tau_{2n}}\right),$$

$$E(w_{2n}) = E(w_{2n-1}) \cdot \left(1-\frac{2}{\tau_{2n-1}}\right)+1.$$

Due to the complexity of the recurrence calculation, only the expectation was studied, which gave the following result:

$$E(w_{2n+1}) = \frac{\tau_0+n+1}{(\tau_0+n)\cdot(\tau_0+n-1)} \cdot \left(E(w_0)\cdot(\tau_0-1)+n\tau_0+\frac{n^2-n}{2}\right).$$

3.3 Alternating Pólya/Pólya ($s = 2$) Urn

This model has two replacement matrices.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ for odd draws;}$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \text{ for even draws.}$$

In this case, the odd draw is the Pólya urn with $s = 2$.

$$w_{2n+1}|w_{2n} = \begin{cases} w_{2n}+1, & \text{with probability: } \frac{w_{2n}}{\tau_{2n}}; \\ w_{2n}, & \text{with probability: } 1-\frac{w_{2n}}{\tau_{2n}}; \end{cases} \quad \text{at odd draw,}$$

$$w_{2n}|w_{2n-1} = \begin{cases} w_{2n-1}+2, & \text{with probability: } \frac{w_{2n-1}}{\tau_{2n-1}}; \\ w_{2n-1}, & \text{with probability: } 1-\frac{w_{2n-1}}{\tau_{2n-1}}; \end{cases} \quad \text{at even draw.}$$

By changing the odd draw to a different scale of Pólya urn, we found that this alternating model has a higher expectation of white balls in the urn than the Pólya model with only $s = 1$, but smaller than the Pólya model with only $s = 2$, as shown in the expecation below,

$$E(w_n) = \begin{cases} (\frac{w_0}{\tau_0}) \cdot \left(\frac{3}{2}n+1\right) + w_0, & \text{when } n \text{ is odd;} \\ \left(\frac{\tau_0+\frac{3}{2}n}{\tau_0}\right) \cdot (w_0) & \text{when } n \text{ is even.} \end{cases}$$

3.4 Urn with Bernoulli Replacement

This section expands upon the urn with Bernoulli replacement models by Smythe (1996), who studies this model using the eigenvalue method. It was found that this model was asymptotically normal for the normalized number of white balls. The model is given by the following replacement matrix with random values, but constant addition. Let $B_1 \sim$ Ber(p_1),$B_2 \sim$ Ber(p_2) such that

$$\begin{pmatrix} B_1 & 1-B_1 \\ 1-B_2 & B_2 \end{pmatrix},$$

Smythe (1996) found that if $p_1 + p_2 = \frac{3}{2}$,

then,

$$\frac{w_n}{\sqrt{logn}} \xrightarrow{D} \text{Normal}$$

And, $\sqrt{n/(logn)}(x_n - \frac{q_2}{q_1+q_2})$ is asymptotically normal, where x_n is the number of times of drawing a white ball after n draws.

We expanded the model above by studying the following related replacement matrix, which is similiar to the urn above when $p_1 = p_2 = p$.

$$\begin{pmatrix} B_1 & 1 - B_1 \\ B_2 & 1 - B_2 \end{pmatrix}.$$

The white ball at w_{n+1} can be expressed as follows:

$$w_{n+1}|w_n = \begin{cases} w_n + B_1, & \text{with probability: } \frac{w_n}{\tau_n}; \\ w_n + B_2, & \text{with probability: } 1 - \frac{w_n}{\tau_n}, \end{cases}$$

$$E(w_{n+1}) = E(B_1 - B_2) \cdot E(\frac{w_n}{\tau_n}) + E(w_n) + E(B_2).$$

We study the special case here, where $p_1 = p_2 = p$. This special case is studied in the example above from Smythe (1996), which is proved to be asymptotically normal by the eigenvalue and eigenvector approach. Here, the model is analyzed by the conditional probability approach and the Central Limit Theorem. Since $B_1 \sim \text{Ber}(p_1)$,$B_2 \sim \text{Ber}(p_2)$, we know $E(B_1) = E(B_2) = p$. Hence, $E(w_{n+1}) = E(w_n) + p$.
By recurrence, we can find that,

$$E(w_n) = w_0 + np.$$

The variance is derived in a similar way,

$$E(w_{n+1}^2|w_n) = (2B_1 + 2B_2) \cdot \frac{w_n^2}{\tau_n} + (B_1^2 - B_2^2) \cdot \frac{w_n}{\tau_n} + (w_n^2 + 2B_2 w_n + B_2^2).$$

After solving the recurrence, the variance under $p_1 = p_2 = p$ is,

$$E(w_n^2) = w_0^2 + 2pnw_n + np + n^2p^2,$$

$$Var(w_n) = E(w_n^2) - [E(w_n)]^2 = np.$$

Both expectation and variance have order n, it can be shown that the central limit theorem holds here.

Chapter 4

Conclusion

Through the discussion of several 2×2 alternating balanced urns, we provide a technical guide for solving urn models with different charactertics. We have proven that the ratio of white balls to the number of draws in a Pólya/Friedman urn model approaches a constant with high probability at n, and furthermore we have empirically demonstrated that the scaled number of white balls is asymptotically normal by running simulations. Based on the simulation results for the two Friedman/Pólya urns, we see that the dominance of the Friedman in the alternating urn leads to the normality of the alternating urns. In other alternative Pólya cases, we see that mixing bigger additions for the even draws generate an urn scheme similar to the Pólya addition matrix urn model, but with larger expectations of w_n. The urn model with Bernoulli replacement discussed has random replacement rather than the deterministic replacement of the alternating models above. It has been proven through the probabilistic approach that normality is highly likely to hold, although the proof of central limit theorem requires future work.

The scope of this thesis focused only on unbiased experiments, meaning each ball was given the same probability (i.e. size). Further study would include the work by Fog (2013), which addressed biased urn theory. Future work could also involve the models with continuous time and higher dimensions. For example Balaji et al. (2006) studies the continuous case of the Ehrenfest model.

Together, these results provide a foundation for potential applications in the fields discussed earlier, which are exciting topics for further exploration.

Bibliography

S. Balaji, H. Mahmoud, and O. Watanabe. Distributions in the ehrenfest process. *Statistics & probability letters*, 76(7):666–674, 2006.

F. Eggenberger and G. Pólya. Uber die statistik verketteter vorgage. 3(4):279–289, 1923.

A. Fog. Biased urn theory. 2013.

H Mahmoud. *Pólya urn models*. CRC press, 2008.

K. Siegrist. An urn model with bernoulli removals and independent additions. *Stochastic Processes and their Applications*, 25(0):315 – 324, 1987.

R.T. Smythe. Central limit theorems for urn models. *Stochastic Processes and their Applications*, 65(1):115 – 137, 1996.

L. J. Wei and S. Durham. The randomized play-the-winner rule in medical trials. *Journal of the American Statistical Association*, 73(364), 1978.

Appendices

Maple

In this study, we used the Maple software to help us compute the recurrences of the variances as well as to study the limit of $\frac{Var(w_n)}{n^2}$ due to the complexity of the calculation. Typically, the 'rsolve' command is used to solve the recurrence. In our case, there are two different recurrence equations for the even and odd cases. 'rsolve' cannot solve two different recurrence equations simultaneously or recognize the even and odd relations separately. To handle this issue, the even and odd recurrence relations were solved seperately. Also, the rsolve function in Maple was lagged 2 instead of lagged 1. The other difficulty we faced was the lack of computing power for calculating $E(w_n)^2$ on the even draws. Instead of using 'rsolve' with the lag 2 recurrence function, the recurrence for odd draws is expressed as a function of even draws:

$$E(w_{2n+1}^2) = \{E(w_{2n}^2) - \frac{E(w_{2n})}{\tau_{2n}}\} \cdot \frac{2n+3}{2n+1}.$$

$E(w_{2n+1}^2)$ and $E(w_{2n})$ can be derived by Maple and plugged into the function above to solve the recurrence for even draws.

Maple Coding

a := 2; w0 := 1;

rsolve(EW(0) = w0, EW(1) = w0*(1+1/a), EW(j) = EW(j-2)*(1+1/(j-1+a))*(1-1/(j-2+a))+1+1/(j-1+a), EW);

rsolve(EW(0) = w0, EW(1) = w0*(1+1/a), EW(j) = EW(j-2)*(1+1/(j-2+a))*(1-1/(j-1+a))+1, EW);

rsolve(EWsq(0) = w0$^2, EWsq(1) = 2*w0^2+1/2, EWsq(j) = EWsq(j-2)*(1+2/(2+j-1))*(1-2/(2+j-2))+(1+(j-3)*(1/2))*(1+1/(2+j-3))*(2-1/(2+j-2))*(1+2/(2+j-1))+(1+(1/2)*j-1/2)/(2+j-1)+1+2/(2+j-1), EWsq$);

$EWsqodd := proc(j, w0) options operator, arrow;$

$simplify(-(3/2+(1/2)*j)*Psi(-1/2+(1/2)*j)/j-(1/2)*(-4*(-1/2+(1/2)*j)^5-20*(-1/2+(1/2)*j)^4-36*(-1/2+(1/2)*j)^3-21*(-1/2+(1/2)*j)^2+3/2+(5/2)*j+2*gamma*(-1/2+(1/2)*j)^3+6*gamma*(-1/2+(1/2)*j)^2+4*gamma*(-1/2+(1/2)*j)-2*w0^2*(-1/2+(1/2)*j)^3-6*w0^2*(-1/2+(1/2)*j)^2-4*w0^2*(-1/2+(1/2)*j))/((-1/2+(1/2)*j)*j*((1/2)*j+1/2)));$

$EWodd := proc(j) options operator, arrow; simplify(-(1/4)*(-4*w0+2-2*j)*(2+j)/(j+1));$

$limit((EWsqodd(j, 1)-EWodd(j)^2)/j^2, j = infinity);$

$EWeven := proc(j) options operator, arrow; simplify(w0 + (1/2) * j);$

$EWsqeven := proc(j) options operator, arrow; simplify(-(3/2+(1/2)*j)*Psi(-1/2+(1/2)*j)/j - (1/2)*(-4*(-1/2+(1/2)*j)^5 - 20*(-1/2+(1/2)*j)^4 - 38*(-1/2+(1/2)*j)^3 - 27*(-1/2+(1/2)*j)^2 + 7/2 + (1/2)*j + 2*gamma*(-1/2+(1/2)*j)^3 + 6*gamma*(-1/2+(1/2)*j)^2 + 4*gamma*(-1/2+(1/2)*j))/((-1/2+(1/2)*j)*j*((1/2)*j+1/2)) - j/(j+1))*(j+1)/(j+3);$

limit((EWsqeven(j)-EWeven(j)^2)$/j^2, j = infinity$);

Simulation

Simulation is used to verify the Martingale central limit theorem on the different alternating Pólya/Friedman urn models stated in Section 4.1. We used R and Maple for the computations. Here we give the Maple coding for the alternating Pólya/Friedman model since this provides a good template for other simulations.

Simulation Coding

```
replica := 500; n := 1000;
for r from 1 to replica do W := 1; B := 1; for sim from 1 to n do U := stats[random,uniform[0,1]](1); if sim mod 2 = 1 then if U < W/(W+B) then W := W+1 else B := B+1 fi else if U < W/(W+B) then B := B+1 else W := W+1; fi; fi; od; White[r] := evalf((W -n/2)/sqrt(n)); od:

with(stats[statplots]):

Whiteconverted := convert(White,'list');

histogram(Whiteconverted, area = count);
```