Analysis of the Long Range Interaction in the Ternary System

by Lu Xie

B.S. in Mathematics and Applied mathematics, June 2007, Heifei University of Science and Technology, China
M.A. in Mathematics, January 2010, George Washington University

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Xiaofeng Ren
Professor of Mathematics

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Lu Xie

Dissertation Research Committee:

Xiaofeng Ren, Professor of Mathematics, Dissertation Director

Frank Baginski, Professor of Mathematics, Committee Member

Svetlana Roudenko, Professor of Mathematics, Committee Member

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Abstract

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In this dissertation, we show that a core-shell structure is a stable solution of the free energy problem for a particular ternary system. The free energy of this ternary system includes an interface energy and a long range interaction energy. This model is derived from a density functional theory for triblock copolymers proposed by Nakazawa and Ohta. We prove the existence of a core-shell solution by reducing the free energy problem to a finite dimensional problem. The finite dimensional problem is then solved explicitly. Finally, we demonstrate that a stability criterion on the parameters determines whether or not the core-shell solution is stable.

Contents

Abstract **iv**

Contents **v**

List of Figures **vi**

List of Symbols **vii**

Chapter 1 Introduction **1**

1.1 Main result 7

1.2 Outline 8

Chapter 2 Exact core-shells as approximate solutions **10**

Chapter 3 Perturbed core-shells **20**

Chapter 4 Linear analysis **27**

Chapter 5 Reduction to two dimensions **38**

Chapter 6 Existence and stability **44**

Chapter 7 Conclusion and discussion **50**

References **51**

Appendix A Derivation of the Euler Lagrange Equation **54**

Index **62**

List of Figures

1.1 Blue spotted grouper. Image credit: By Adrian Pingstone [Public domain], via Wikimedia Commons. 3
1.2 Leopard. Image credit: By U.S. Fish and Wildlife Service [Public domain], via Wikimedia Commons. 3
1.3 Three dimensional co-shell pattern 5
1.4 Core Shell. 5
1.5 An equal area double-bubble. 7

List of Symbols

$(-\Delta)^{-\frac{1}{2}}$: The positive square root of the inverse of $-\Delta$ with the Neumann boundary condition.

χ_Ω: The characteristic function of Ω.

$\langle \cdot, \cdot \rangle$: The inner product in $L^2(S^1)$.

$\mathbb{R}^n$: The real coordinate space of n dimensions.

$\mathbb{S}^2$: The set of 2 by 2 symmetric matrices.

$\overline{v(x)} = 0$: $\int_D v(x)dx = 0$ for $x \in D$.

$\partial\Omega \cap D$: The boundary of Ω in D.

∂D: The boundary of Ω in $\mathbb{R}^n$.

D: An open and bounded set in $\mathbb{R}^n$.

$H^k(S^1)$: The Sobolev $W^{k,2}$ space where $k \geq 1$ and k is an integer.

S^1: The interval $[0, 2\pi]$.

$|\cdot|$: Lebesgue measure of a set.

$||\cdot||_{H^k}$: The $W^{k,2}$ norm.

$||\cdot||_{L^2}$: The L^2 norm.

$||\cdot||_{L^\infty}$: The L^∞ norm.

Chapter 1

Introduction

Different patterns may form as orderly outcomes of self-organization principles in multi-constituent systems. Examples include morphological phases in block copolymers, animal coats and skin pigmentation.

Binary systems, a special case of multi-constituent systems, have been studied extensively. Let D be an open and bounded set in $\mathbb{R}^n$. The set D is partitioned into two subsets: Ω, occupied by one constituent, and $D\backslash\Omega$, occupied by the other constituent. These sets may or may not be connected, but here we consider the two constituents to be completely separated.

The free energy of the binary system is composed of two parts. First, there is the energy coming from the interface separating Ω and $D\backslash\Omega$. In other words, this energy is the perimeter of Ω in D. We denote this perimeter by $\mathcal{P}_D(\Omega)$, and will give the precise definition later in this chapter. If Ω is bounded by a smooth boundary $\partial\Omega$, then $\mathcal{P}_D(\Omega)$ is the length of $\partial\Omega \cap D$ when $D \subset \mathbb{R}^2$, and, is the area of $\partial\Omega \cap D$ when $D \subset \mathbb{R}^3$. Note that ∂D stands for the boundary of Ω in $\mathbb{R}^n$ not the boundary of Ω in D. The latter is $\partial\Omega \cap D$.

The second part of the free energy in the self-assembling system is a longer ranging, inhibitory term. It is not difficult to imagine that two different constituents will have a long range interaction before the patterns form. One example concerning the Ohta-Kawasaki density functional theory for diblock copolymers may be found in [8] where the long range term is $\int_D |(-\Delta)^{-1/2}(\chi_\Omega - w)|^2 dx$. The function χ_Ω is the characteristic function of Ω, that

is, $\chi_\Omega = 1$ if $x \in \Omega$ and $\chi_\Omega = 0$ if $x \in D\backslash\Omega$. The positive constant $w \in (0,1)$ satisfies $|\Omega| = w|D|$, where $|\Omega|$ is the Lebesgue measure of Ω and $|D|$, the Lebesgue measure of D.

The free energy $\mathcal{J}_B$ of the binary system combines these two terms:

$$\mathcal{J}_B(\Omega) = \frac{1}{n-1}\mathcal{P}_D(\Omega) + \frac{\gamma}{2}\int_D |(-\Delta)^{-1/2}(\chi_\Omega - w)|^2 dx. \tag{1.1}$$

The positive constant γ balances the short range interaction $\mathcal{P}_D(\Omega)$ with the long range interaction in $\mathcal{J}_B$. Both γ and w are parameters of the energy function. The operator $(-\Delta)^{-1/2}$ is the positive square root of the inverse of $-\Delta$ with the Neumann boundary condition. A critical point of the function $\mathcal{J}_B$ satisfies the Euler-Lagrange equation

$$\mathcal{H}(\partial\Omega \cap D) + \gamma(-\Delta)^{-1}(\chi_\Omega - w) = \lambda. \tag{1.2}$$

In (1.2) $\mathcal{H}(\partial\Omega \cap D)$ is the mean curvature of $\partial\Omega \cap D$, and λ is a Lagrange multiplier associated with the constraint $|\Omega| = w|D|$. The coefficients $\frac{1}{n-1}$ and $\frac{\gamma}{2}$ in (1.1) simplify the form of (1.2). Equation (1.2) holds on the part of the boundary of Ω that is inside D, namely $\partial\Omega \cap D$. If $\partial\Omega$ intersects ∂D, then the two hyper-surfaces must meet perpendicularly. By minimizing $\mathcal{J}_B$, or more generally by solving the equation (1.2), one discovers patterns that match many observed in nature.

This problem is solved completely in one-dimension [12]. In particular, there are countably-many solutions to (1.2), and every solution is a local minimizer of the energy functional $\mathcal{J}_B$. Depending on w and γ, there are either two or four global minimizers among these local minimizers. In higher dimensions, people have found several types of solutions [4, 5, 9, 10, 13, 16, 17, 11, 18, 19]. Further studies are needed to address the global minimizers of the problem. One can see progresses from [1, 6, 21].

Figure 1.1: Blue spotted grouper. Image credit: By Adrian Pingstone [Public domain], via Wikimedia Commons.

Figure 1.2: Leopard. Image credit: By U.S. Fish and Wildlife Service [Public domain], via Wikimedia Commons.

In this dissertation we study the ternary case. A ternary system has three constituents that occupy disjoint subsets Ω_1, Ω_2 and $\Omega_3 = D\backslash(\Omega_1 \cup \Omega_2)$ of D, and D is, again, an open and bounded subset of $\mathbb{R}^n$.

As in a binary system, the free energy of a ternary system is composed of two parts. One part of the free energy is the size of the interfaces separating the three domains Ω_1, Ω_2 and Ω_3. There are three types of interfaces: $\partial\Omega_1\backslash\partial\Omega_2$, the interfaces separating Ω_1 from Ω_3; $\partial\Omega_2\backslash\partial\Omega_1$, the interfaces separating Ω_2 from Ω_3; and $\partial\Omega_1 \cap \partial\Omega_2$, the interfaces separating Ω_1 from Ω_2.

The other part of the energy is, as before, the long range interaction term. We use $\int_D((-\Delta)^{-1/2}(\chi_{\Omega_i} - w_i))((-\Delta)^{-1/2}(\chi_{\Omega_j} - w_j))dx$ to model the interaction between Ω_i and Ω_j. The overall free energy of the ternary system takes the form

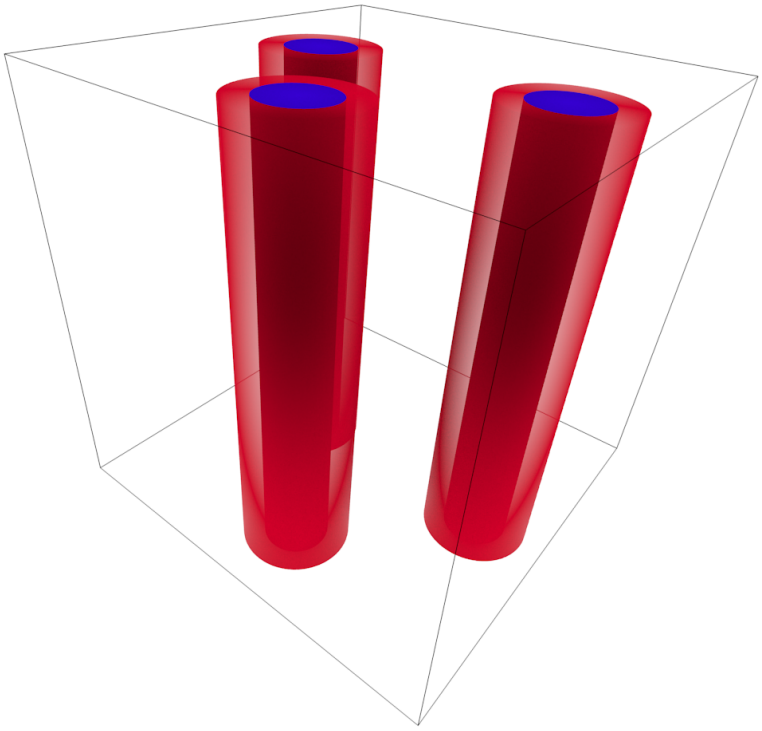
$$\mathcal{J}_T(\Omega) = \frac{1}{2(n-1)} \sum_{i=1}^{3} \mathcal{P}_D(\Omega_i) + \sum_{i,j=1}^{2} \int_D \frac{\gamma_{ij}}{2}((-\Delta)^{-1/2}(\chi_{\Omega_i} - w_i))((-\Delta)^{-1/2}(\chi_{\Omega_j} - w_j))dx. \tag{1.3}$$

This model was derived from the density functional theory for triblock copolymers proposed by Nakazawa and Ohta [7]. In this work, we consider only the situation where the three constituents are totally separated from each other: there is no mixture of the three constituents. Figure 5 in [2] illustrates many possible patterns in the ternary system.

The one dimensional case of this problem has also been studied. The cyclic lamellar phase is shown to exist in [15]; a diblock copolymer-homopolymer blend problem is investigated in [3, 22, 23].

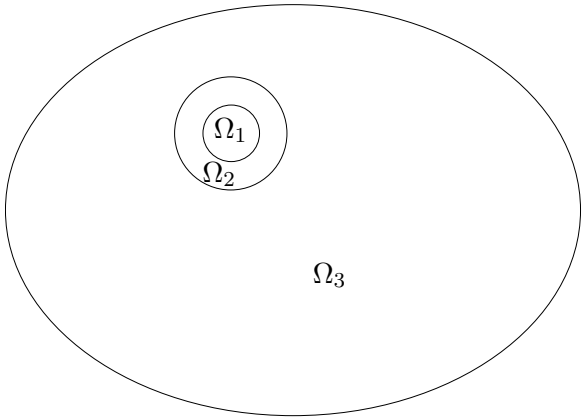
In two dimensions, Ren and Wei have recently found a double bubble solution [20]. In this work, we will investigate Case (b) in Figure 5 of [2], a core-shell pattern. Figure 1.3 is reproduced from the case (b) of Figure 5 of [2]. We will prove that in two dimensions a core-shell solution exists as a critical point of $\mathcal{J}_T$.

Figure 1.3: Three dimensional co-shell pattern



A single core-shell pattern is a cylinder inside another cylinder. In nature, there appear to be approximate core-shell patterns. Figures 1.1 and 1.2 show animals with core-shell patterns observed in nature. Figure 1.3 is a picture of multiple core-shells. Figure 1.4 is a cross-section of a single core-shell pattern.

Figure 1.4: Core Shell.



We consider (1.3) on a bounded, smooth, open subset D of $\mathbb{R}^2$. Let w_1 and w_2 be two positive numbers such that $w_1 + w_2 < 1$. For two measurable sets Ω_1 and Ω_2 satisfying $|\Omega_1| = w_1|D|$, $|\Omega_2| = w_2|D|$, and Ω_1 is enclosed by Ω_2, set $\Omega_3 = D\backslash(\Omega_1 \cup \Omega_2)$.

Both w_1 and w_2 will be small in this work, but their ratio $q = w_1/w_2$ will be fixed. We let $|w| = \sqrt{w_1^2 + w_2^2}$. The main result in this work holds when $|w|$ is small. Since the fraction $q = \frac{w_1}{w_2}$ is held fixed, $|w|$ being small is equivalent to w_1 being small or w_2 being small.

The energy of the system is

$$(1.4)\quad J(\Omega_1, \Omega_2) = \frac{1}{2}\sum_{1}^{3} \mathcal{P}_D(\Omega_i) + \sum_{i,j=1}^{2} \frac{\gamma_{ij}}{2} \int_D ((-\Delta)^{-\frac{1}{2}}(\chi_{\Omega_i} - w_i))((-\Delta)^{-\frac{1}{2}}(\chi_{\Omega_j} - w_j))dx.$$

The interaction matrix $\gamma = [\gamma_{ij}]$ is symmetric. It balances the long range interaction strength between the three constituents.

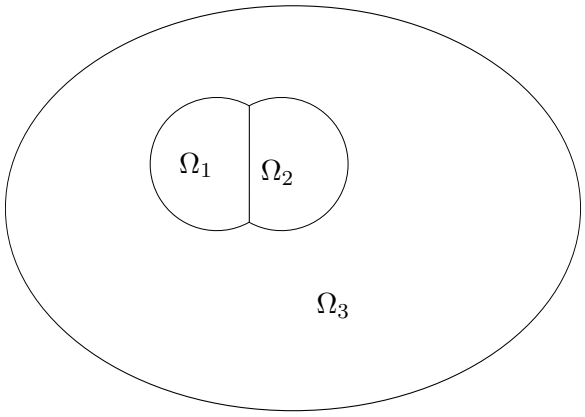
We now denote $\mathcal{J}_T$ from (1.3) by J for simplicity.

In (1.4), $\mathcal{P}_D(\Omega_i)$ is the perimeter of Ω in D. We define the perimeter in the following way: for any measurable subset E of D,

$$(1.5)\qquad \mathcal{P}_D(E) = \sup\{\int_E \text{div } g(x)dx : g \in C_0^1(D, \mathbb{R}^2), |g(x)| \leq 1 \ \forall x \in D\}$$

where div $g(x)$ is the divergence of a C^1 vector field g on D with compact support and $|g(x)|$ stands for the Euclidean norm of the vector $g(x) \in \mathbb{R}^2$. A subset E in D is said to have a finite perimeter if and only if χ_E, the characteristic function of E is a function of bounded variation on D. In this work, $\mathcal{P}_D(E)$ is just the size of the interfaces separating the three domains; that is, the first term of (1.4) is the combined length of $\partial\Omega_1\backslash\partial\Omega_2$, $\partial\Omega_2\backslash\partial\Omega_1$, and $\partial\Omega_1 \cap \partial\Omega_2$. In the core-shell case, $\partial\Omega_1\backslash\partial\Omega_2$ is absent. However in Figure 1.5, all three types of boundaries are present. Figure 1.5 is a 2-dimensional equal area double bubble. The perimeter here is the sum of the lengths of the the two arcs and the middle line segment.

Figure 1.5: An equal area double-bubble.



1.1 Main result

Let us denote the Green's function of $-\Delta$ by G. It is a sum of two terms:

$$G(x,y) = \frac{1}{2\pi}\log\frac{1}{|x-y|} + R(x,y). \tag{1.6}$$

The regular part of $G(x,y)$ is $R(x,y)$. The Green's function satisfies the equation

$$-\Delta_x G(x,y) = \delta(x-y) - \frac{1}{|D|}, \partial_{\nu(x)} G(x,y) = 0 \text{ on } \partial D, \overline{G(\cdot,y)} = 0 \ \forall y \in D. \tag{1.7}$$

In (1.7), Δ_x is the Laplacian with respect to the x-variable of G, $\nu(x)$ is the outward normal direction at $x \in \partial D$, and $\partial_{\nu(x)}$ is the normal derivative there with respect to the x variable.

Let $\Gamma = [\Gamma_{ij}]$, satisfying $\gamma = (\frac{w_1|D|}{\pi})^{-3/2}\Gamma$, i.e $\begin{bmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{21} & \gamma_{22}\end{bmatrix} = (\frac{w_1|D|}{\pi})^{-3/2}\begin{bmatrix}\Gamma_{11} & \Gamma_{12}\\ \Gamma_{21} & \Gamma_{22}\end{bmatrix}$. Let

(1.8)

$$M_n = \begin{bmatrix} (n^2-1) - \frac{2\Gamma_{12}-\Gamma_{11}-\Gamma_{22}}{2n} - \frac{\Gamma_{11}-\Gamma_{12}}{2} & \frac{\Gamma_{12}-\Gamma_{22}}{2n}(\frac{1}{1+q})^{n/2} \\ \frac{\Gamma_{12}-\Gamma_{22}}{2n}(\frac{1}{1+q})^{n/2} & \frac{1}{(1+q)^{3/2}}(n^2-1) + \frac{\Gamma_{22}}{2n} - \frac{\Gamma_{22}}{2}(1-\frac{1}{1+q}) - \frac{\Gamma_{12}}{2}\frac{1}{1+q} \end{bmatrix}$$

where $q = \frac{w_2}{w_1}$. The symbol q will denote a fixed positive constant throughout the dissertation.

Let $\mathbb{S}^2$ be the set of 2 by 2 symmetric matrices. Let

$$\mathcal{C}_q = \big\{\Gamma \in \mathbb{S}^2 : \Gamma > 0\big\} \cap \{\Gamma \in \mathbb{S}^2 : \Gamma_{22} > \Gamma_{12}\} \cap \{\Gamma \in \mathbb{S}^2 : M_n > 0, \forall n \geq 2\}. \tag{1.9}$$

A proof will be given later to show that $\mathcal{C}_q \neq \emptyset$.

A main result of this dissertation is the following theorem.

1.10 Theorem. *For every $q > 0$ and every $\Gamma \in \mathcal{C}_q$, there exists $\delta > 0$ such that if $|w| < \delta$ and $\gamma = (\frac{w_1|D|}{\pi})^{-3/2}\Gamma$, then there is a core-shell solution to the ternary system.*

Remark:

1. We construct the solution in two steps. The construction is stable in each step. Therefore, the solution is stable in some sense.

2. The solution is a core-shell near a minimum of the function $x \to R(x, x)$.

1.2 Outline

Write $B_k = \{x \in \mathbb{R}^2 : |x - \xi| < r_k\}$. The proof of the theorem starts by finding a family of approximate solutions, the pair $(B_1, B_2 \backslash B_1)$. Both r_1 and r_2 are fixed. Additionally, $\pi r_1^2 = w_1|D|$, and $\pi r_2^2 = w_2|D|$.

We then perturb $(B_1, B_2 \backslash B_1)$ by a pair of 2π-periodic functions $\phi = (\phi_1, \phi_2)$ where $r_1^2 + 2\phi_1$ is the perturbed inner radius squared and $r_2^2 + 2\phi_2$ is the perturbed outer radius squared. The perturbed B_1 is the set

$$E_1^\phi = \{\xi + \alpha e^{i\theta} : \theta \in [0, 2\pi], \alpha \in (0, \sqrt{r_1^2 + 2\phi_1(\theta)})\}.$$

The perturbed $B_2 \backslash B_1$ is the set

$$E_2^\phi = \{\xi + \alpha e^{i\theta} : \theta \in [0, 2\pi], \alpha \in (\sqrt{r_1^2 + 2\phi_1(\theta)}, \sqrt{r_2^2 + 2\phi_2(\theta)})\}.$$

We perturbed the squares of the radii instead of radii so that the constraints $|E_1^\phi| = w_1|D|$ and $|E_2^\phi| = w_2|D|$ become simple linear constraints on ϕ_1 and ϕ_2 of the form

$$\int_0^{2\pi} \phi_1(\theta)d\theta = 0,$$

$$\int_0^{2\pi} \phi_2(\theta)d\theta = 0.$$

In terms of ϕ, the Euler-Lagrange equation of (1.4) is written as $\mathcal{S}(\phi) = 0$ where $\mathcal{S}$ is a two component, nonlinear, integro-differential operator from a functional space $\mathcal{X}$ to another functional space $\mathcal{Y}$.

There is a subset $\mathcal{X}_*$ of $\mathcal{X}$ which, roughly speaking, ignores the effect of the translation of the center ξ. There is also a corresponding subset $\mathcal{Y}_*$ of $\mathcal{Y}$. Given ξ, we look for $\varphi = \varphi(\theta, \xi)$ in $\mathcal{X}_*$ that solves the equation up to a translation of ξ, i.e., $\Pi\mathcal{S}(\varphi) = 0$ where Π is the projection operator from $\mathcal{Y}$ to $\mathcal{Y}_*$.

Finally we study the dependence of $J(\varphi(\cdot, \xi))$ on ξ. We show that under certain conditions there exists ζ such that $J(\varphi(\cdot, \xi))$ is minimized at $\xi = \zeta$. It turns out that at this minimum, $\mathcal{S}(\varphi(\cdot, \zeta)) = 0$. Stability of this solution will also be proved.

In this dissertation, the interval $[0, 2\pi]$ with 0 and 2π identified is denoted by S^1. The L^2 space on S^1 is $L^2(S^1)$. The inner product in $L^2(S^1)$ is denoted by $< \cdot, \cdot >$. The L^2 norm is denoted by $|| \cdot ||_{L^2}$, and the L^∞ norm by $|| \cdot ||_{L^\infty}$. The Sobolev $W^{k,2}$ space is denoted by $H^k(S^1)$ where $k \geq 1$ is an integer. The $W^{k,2}$ norm is denoted by $|| \cdot ||_{H^k}$.

The point $(\cos\theta, \sin\theta)$ is often written as $e^{i\theta}$ for a simpler notation even though no complex structure is assumed on $\mathbb{R}^2$. For example, the expression $e^{i\theta} \cdot x$ is simply the inner product of two real vectors $(\cos\theta, \sin\theta)$ and x.

Chapter 2

Exact core-shells as approximate solutions

Let U be a neighborhood of the set

$$\{z \in D : R(z,z) = \min_{x \in D} R(x,x)\} \tag{2.1}$$

where $R(x,x)$ is defined in (1.6). The set defined in (2.1) is compact since $R(x,x) \to \infty$ as $x \to \partial D$. We can choose U so that the closure of U in $\mathbb{R}^2$ is compact and is contained in D.

Let B_1 and B_2 both be centered at $\xi \in U$ and have inner radius r_1 and r_2 respectively. The exact core-shell is $(B_1, B_2 \backslash B_1)$.

The energy functional is the following

$$J(\Omega_1, \Omega_2) = \frac{1}{2} \sum_{1}^{3} \mathcal{P}_D(\Omega_i) + \sum_{i,j=1}^{2} \frac{\gamma_{ij}}{2} \int_D ((-\Delta)^{-\frac{1}{2}}(\chi_{\Omega_i} - w_i))((-\Delta)^{-\frac{1}{2}}(\chi_{\Omega_j} - w_j))dx \tag{2.2}$$

where $w_1, w_2 \in (0,1)$ and $w_1 + w_2 < 1$.

The Euler-Lagrange equations of J are

$$\begin{aligned}(2.3)\qquad &\mathcal{H}_1(\phi_1)(\theta)+(\gamma_{12}-\gamma_{22})(-\Delta)^{-1}(\chi_{\Omega_2}-w_2)(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta})\\ &+(\gamma_{11}-\gamma_{12})(-\Delta)^{-1}(\chi_{\Omega_1}-w_1)(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta})=\lambda_1\end{aligned}$$

$$\begin{aligned}(2.4)\qquad &\mathcal{H}_2(\phi_2)(\theta)+\gamma_{12}(-\Delta)^{-1}(\chi_{\Omega_1}-w_1)(\xi+\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta})\\ &+\gamma_{22}(-\Delta)^{-1}(\chi_{\Omega_2}-w_2)(\xi+\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta})=\lambda_2\end{aligned}$$

where $\mathcal{H}_1$ is the curvature of $\partial\Omega_1$ and $\mathcal{H}_2$ is the curvature of $\partial\Omega_2\backslash\partial\Omega_1$. See Appendix A for the derivation of the preceding formulas.

We plug the exact core-shell into the left side of the Euler Lagrange equation and see, as an approximate solution, how much error the exact core-shell has.

2.5 Lemma. *If $(\Omega_1,\Omega_2)=(B_1,B_2\backslash B_1)$, the left side of the Euler Lagrange equation is*

$$\begin{aligned}&\frac{1}{r_1}+(\gamma_{12}-\gamma_{22})(-\frac{r_2^2\log r_2-r_1^2\log r_1}{2}+\frac{w_2|D|}{4\pi}+w_2|D|R(\xi,\xi)+\frac{\pi(r_2^4-r_1^4)}{8|D|}\\ &+w_2|D|\bigtriangledown R(\xi,\xi)r_1e^{i\theta})\\ &+(\gamma_{11}-\gamma_{12})(-\frac{w_1|D|}{2\pi}\log r_1+w_1|D|R(\xi,\xi)+\frac{\pi r_1^4}{8|D|}+w_1|D|\bigtriangledown R(\xi,\xi)r_1e^{i\theta})+O(|w|^{\frac{1}{2}})\end{aligned}$$

on the inner circle of the exact core-shell $(B_1,B_2\backslash B_1)$, and is

$$\begin{aligned}&\frac{1}{r_2}+\gamma_{12}(-\frac{w_1|D|}{2\pi}\log r_2+w_1|D|R(\xi,\xi)+\frac{\pi r_1^4}{8|D|}+w_1|D|\bigtriangledown R(\xi,\xi)r_2e^{i\theta})\\ &+\gamma_{22}(-\frac{w_2|D|}{2\pi}\log r_2+w_2|D|R(\xi,\xi)+\frac{\pi(r_2^4-r_1^4)}{8|D|}+w_2|D|\bigtriangledown R(\xi,\xi)r_2e^{i\theta})+O(|w|^{\frac{1}{2}})\end{aligned}$$

on the outer circle of the exact core-shell $(B_1,B_2\backslash B_1)$.

Proof. Let v_1 and v_2 satisfy

$$-\Delta v_k = \chi_{B_k} - \frac{\pi r_k^2}{|D|} \text{ in } D, \partial_\nu v_k = 0 \text{ on } \partial D, \overline{v_k} = 0, \tag{2.6}$$

where $k = 1, 2$.

Define

$$P_k(x) = \begin{cases} -\frac{|x|^2}{4} + \frac{r_k^2}{4} - \frac{r_k^2}{2}\log r_k, \text{if} |x| < r_k \\ -\frac{r_k^2}{2}\log|x|, \text{if} |x| \geq r_k \end{cases}.$$

Then $-\Delta P_k(\cdot - \xi) = \chi_{B_k}$. Write $v_k = P_k(\cdot - \xi) + Q_k(\cdot, \xi)$. Clearly

$$\begin{cases} -\Delta Q_k(x,\xi) = -\frac{\pi r_k^2}{|D|} \\ \partial_\nu Q_k(x,\xi) = \partial_\nu \frac{r_k^2}{2}\log|x-\xi| \text{ on } \partial D \\ \overline{Q_k(\cdot,\xi)} = -\overline{P_k(|\cdot - \xi|)} \end{cases}.$$

Here the Laplacian Δ and the outward normal derivative ∂_ν are taken with respect to x.

Note from (1.7) it follows that $Q_k(x,\xi)$ and $\pi r_k^2 R(x,\xi)$ satisfy the same equation and the same boundary condition. Therefore they can differ only by a constant. The constant is $\overline{Q_k(\cdot,\xi)} - \pi r_k^2 \overline{R(\cdot,\xi)}$. But $\overline{v_k} = \overline{G(\cdot,\xi)} = 0$ implies that the constant is also equal to

$$-\frac{r_k^2}{2}\overline{\log|\cdot - \xi|} - \overline{P_k(\cdot - \xi)} = \frac{\pi r_k^4}{8|D|}.$$

Hence

$$Q_k(x,\xi) = \pi r_k^2 R(x,\xi) + \frac{\pi r_k^4}{8|D|}.$$

Direct calculations show that,

$$
\begin{aligned}
&(-\Delta)^{-1}(\chi_{B_2\setminus B_1} - w_2)(\xi + r_1 e^{i\theta}) \\
&= v_2(\xi + r_1 e^{i\theta}) - v_1(\xi + r_1 e^{i\theta}) \\
&= P_2(r_1 e^{i\theta}) + Q_2(\xi + r_1 e^{i\theta}, \xi) - P_1(r_1 e^{i\theta}) - Q_1(\xi + r_1 e^{i\theta}, \xi) \\
&= -\frac{r_2^2 \log r_2 - r_1^2 \log r_1}{2} + \frac{r_2^2 - r_1^2}{4} + \pi(r_2^2 - r_1^2) R(\xi + r_1 e^{i\theta}, \xi) + \frac{\pi(r_2^4 - r_1^4)}{8|D|} \\
&= -\frac{r_2^2 \log r_2 - r_1^2 \log r_1}{2} + \frac{w_2|D|}{4\pi} + w_2|D| R(\xi + r_1 e^{i\theta}, \xi) + \frac{\pi(r_2^4 - r_1^4)}{8|D|} \\
&= -\frac{r_2^2 \log r_2 - r_1^2 \log r_1}{2} + \frac{w_2|D|}{4\pi} + w_2|D|(R(\xi,\xi) + \bigtriangledown R(\xi,\xi) r_1 e^{i\theta} + O(r_1^2)) + \frac{\pi(r_2^4 - r_1^4)}{8|D|} \\
&= -\frac{r_2^2 \log r_2 - r_1^2 \log r_1}{2} + \frac{w_2|D|}{4\pi} + w_2|D| R(\xi,\xi) + \frac{\pi(r_2^4 - r_1^4)}{8|D|} + w_2|D| \bigtriangledown R(\xi,\xi) r_1 e^{i\theta} \\
&\quad + O(|w|^2)
\end{aligned}
$$

$$
\begin{aligned}
&(-\Delta)^{-1}(\chi_{B_2\setminus B_1} - w_2)(\xi + r_2 e^{i\theta}) \\
&= v_2(\xi + r_2 e^{i\theta}) - v_1(\xi + r_2 e^{i\theta}) \\
&= P_2(r_2 e^{i\theta}) + Q_2(\xi + r_2 e^{i\theta}, \xi) - P_1(r_2 e^{i\theta}) - Q_1(\xi + r_2 e^{i\theta}, \xi) \\
&= -\frac{r_2^2 - r_1^2}{2} \log r_2 + \pi(r_2^2 - r_1^2) R(\xi + r_2 e^{i\theta}, \xi) + \frac{\pi(r_2^4 - r_1^4)}{8|D|} \\
&= -\frac{w_2|D|}{2\pi} \log r_2 + w_2|D| R(\xi + r_2 e^{i\theta}, \xi) + \frac{\pi(r_2^4 - r_1^4)}{8|D|} \\
&= -\frac{w_2|D|}{2\pi} \log r_2 + w_2|D|(R(\xi,\xi) + \bigtriangledown R(\xi,\xi) r_2 e^{i\theta} + O(r_2^2)) + \frac{\pi(r_2^4 - r_1^4)}{8|D|} \\
&= -\frac{w_2|D|}{2\pi} \log r_2 + w_2|D| R(\xi,\xi) + \frac{\pi(r_2^4 - r_1^4)}{8|D|} + w_2|D| \bigtriangledown R(\xi,\xi) r_2 e^{i\theta} + O(|w|^2)
\end{aligned}
$$

$$
\begin{aligned}
&(-\Delta)^{-1}(\chi_{B_1} - w_1)(\xi + r_1 e^{i\theta}) = v_1(\xi + r_1 e^{i\theta}) \\
&= P_1(r_1 e^{i\theta}) + Q_1(\xi + r_1 e^{i\theta}, \xi) \\
&= -\frac{r_1^2}{2}\log|r_1| + \pi r_1^2 R(\xi + r_1 e^{i\theta}, \xi) + \frac{\pi r_1^4}{8|D|} \\
&= -\frac{w_1|D|}{2\pi}\log r_1 + w_1|D|(R(\xi,\xi) + \bigtriangledown R(\xi,\xi) r_1 e^{i\theta} + O(r_1^2)) + \frac{\pi r_1^4}{8|D|} \\
&= -\frac{w_1|D|}{2\pi}\log r_1 + w_1|D|R(\xi,\xi) + \frac{\pi r_1^4}{8|D|} + w_1|D| \bigtriangledown R(\xi,\xi) r_1 e^{i\theta} + O(|w|^2)
\end{aligned}
$$

$$
\begin{aligned}
&(-\Delta)^{-1}(\chi_{B_1} - w_1)(\xi + r_2 e^{i\theta}) = v_1(\xi + r_2 e^{i\theta}) \\
&= P_1(r_2 e^{i\theta}) + Q_1(\xi + r_2 e^{i\theta}, \xi) \\
&= -\frac{r_1^2}{2}\log|r_2| + \pi r_1^2 R(\xi + r_2 e^{i\theta}, \xi) + \frac{\pi r_1^4}{8|D|} \\
&= -\frac{w_1|D|}{2\pi}\log r_2 + w_1|D|(R(\xi,\xi) + \bigtriangledown R(\xi,\xi) r_2 e^{i\theta} + O(r_2^2)) + \frac{\pi r_1^4}{8|D|} \\
&= -\frac{w_1|D|}{2\pi}\log r_2 + w_1|D|R(\xi,\xi) + \frac{\pi r_1^4}{8|D|} + w_1|D| \bigtriangledown R(\xi,\xi) r_2 e^{i\theta} + O(|w|^2).
\end{aligned}
$$

Therefore, at each $\xi + r_1 e^{i\theta}$, a point on the inner circle

$$\frac{1}{r_1}+(\gamma_{12}-\gamma_{22})(-\Delta)^{-1}(\chi_{B_2\setminus B_1}-w_2)(\xi+r_1e^{i\theta})+(\gamma_{11}-\gamma_{12})(-\Delta)^{-1}(\chi_{B_1}-w_1)(\xi+r_1e^{i\theta})$$
$$=\frac{1}{r_1}+(\gamma_{12}-\gamma_{22})(-\frac{r_2^2\log r_2-r_1^2\log r_1}{2}+\frac{w_2|D|}{4\pi}+w_2|D|R(\xi,\xi)+\frac{\pi(r_2^4-r_1^4)}{8|D|}$$
$$+w_2|D|\bigtriangledown R(\xi,\xi)r_1e^{i\theta}+O(|w|^2))$$
$$+(\gamma_{11}-\gamma_{12})(-\frac{w_1|D|}{2\pi}\log r_1+w_1|D|R(\xi,\xi)+\frac{\pi r_1^4}{8|D|}+w_1|D|\bigtriangledown R(\xi,\xi)r_1e^{i\theta}+O(|w|^2))$$
$$=\frac{1}{r_1}+(\gamma_{12}-\gamma_{22})(-\frac{r_2^2\log r_2-r_1^2\log r_1}{2}+\frac{w_2|D|}{4\pi}+w_2|D|R(\xi,\xi)+\frac{\pi(r_2^4-r_1^4)}{8|D|}$$
$$+w_2|D|\bigtriangledown R(\xi,\xi)r_1e^{i\theta})$$
$$+(\gamma_{11}-\gamma_{12})(-\frac{w_1|D|}{2\pi}\log r_1+w_1|D|R(\xi,\xi)+\frac{\pi r_1^4}{8|D|}+w_1|D|\bigtriangledown R(\xi,\xi)r_1e^{i\theta})$$
$$+(\gamma_{12}-\gamma_{22})O(|w|^2)+(\gamma_{11}-\gamma_{12})O(|w|^2)$$
$$=\frac{1}{r_1}+(\gamma_{12}-\gamma_{22})(-\frac{r_2^2\log r_2-r_1^2\log r_1}{2}+\frac{w_2|D|}{4\pi}+w_2|D|R(\xi,\xi)+\frac{\pi(r_2^4-r_1^4)}{8|D|}$$
$$+w_2|D|\bigtriangledown R(\xi,\xi)r_1e^{i\theta})$$
$$+(\gamma_{11}-\gamma_{12})(-\frac{w_1|D|}{2\pi}\log r_1+w_1|D|R(\xi,\xi)+\frac{\pi r_1^4}{8|D|}+w_1|D|\bigtriangledown R(\xi,\xi)r_1e^{i\theta})+O(|w|^{\frac{1}{2}}).$$

At each $\xi+r_2e^{i\theta}$ on the outer circle

$$\frac{1}{r_2}+\gamma_{12}(-\Delta)^{-1}(\chi_{B_1}-w_1)(\xi+r_2e^{i\theta})+\gamma_{22}(-\Delta)^{-1}(\chi_{B_2\setminus B_1}-w_2)(\xi+r_2e^{i\theta})$$
$$=\frac{1}{r_2}+\gamma_{12}(-\frac{w_1|D|}{2\pi}\log r_2+w_1|D|R(\xi,\xi)+\frac{\pi r_1^4}{8|D|}+w_1|D|\bigtriangledown R(\xi,\xi)r_2e^{i\theta}+O(|w|^2))$$
$$+\gamma_{22}(-\frac{w_2|D|}{2\pi}\log r_2+w_2|D|R(\xi,\xi)+\frac{\pi(r_2^4-r_1^4)}{8|D|}+w_2|D|\bigtriangledown R(\xi,\xi)r_2e^{i\theta}+O(|w|^2))$$
$$=\frac{1}{r_2}+\gamma_{12}(-\frac{w_1|D|}{2\pi}\log r_2+w_1|D|R(\xi,\xi)+\frac{\pi r_1^4}{8|D|}+w_1|D|\bigtriangledown R(\xi,\xi)r_2e^{i\theta})$$
$$+\gamma_{22}(-\frac{w_2|D|}{2\pi}\log r_2+w_2|D|R(\xi,\xi)+\frac{\pi(r_2^4-r_1^4)}{8|D|}+w_2|D|\bigtriangledown R(\xi,\xi)r_2e^{i\theta})$$
$$+(\gamma_{12}+\gamma_{22})O(|w|^2)$$
$$=\frac{1}{r_2}+\gamma_{12}(-\frac{w_1|D|}{2\pi}\log r_2+w_1|D|R(\xi,\xi)+\frac{\pi r_1^4}{8|D|}+w_1|D|\bigtriangledown R(\xi,\xi)r_2e^{i\theta})$$
$$+\gamma_{22}(-\frac{w_2|D|}{2\pi}\log r_2+w_2|D|R(\xi,\xi)+\frac{\pi(r_2^4-r_1^4)}{8|D|}+w_2|D|\bigtriangledown R(\xi,\xi)r_2e^{i\theta})+O(|w|^{\frac{1}{2}}).$$

This proves the lemma.

□

The value of J at the exact core-shell $(B_1, B_2\backslash B_1)$ can be calculated as well.

2.7 Lemma. *The value of J at the exact core-shell is*

$$\begin{aligned}
J(B_1, B_2\backslash B_1) =& 2\pi(r_1+r_2) \\
&+ \frac{\gamma_{11}}{2}[-\frac{\pi r_1^4 \log r_1}{2} + \frac{\pi r_1^4}{8} + \frac{\pi r_1^6}{4|D|} + (w_1|D|)^2 R(\xi,\xi)] \\
&+ \gamma_{12}[-\frac{\pi r_1^4}{8} + \pi r_1^2(\frac{r_2^2}{4} - \frac{r_2^2}{2}\log r_2) + \frac{\pi^2 r_2^2 r_1^4}{8|D|} + \frac{\pi^2 r_1^2 r_2^4}{8|D|} + \frac{\pi r_1^4 \log r_1}{2} \\
&- \frac{\pi r_1^4}{8} - \frac{\pi^2 r_1^6}{4|D|}] + (w_1|D|)(w_2|D|)R(\xi,\xi) \\
&+ \frac{\gamma_{22}}{2}[-\frac{\pi r_1^4 \log r_1}{2} - \frac{\pi r_2^4 \log r_2}{2} + \pi r_1^2 r_2^2 \log r_2 - \frac{w_2|D| r_1^2}{4} + \frac{w_2^2|D|^2}{8\pi} \\
&+ w_2|D|^2 R(\xi,\xi) + \frac{w_2^2|D|}{4}(r_1^2+r_2^2)].
\end{aligned}$$

Proof. The local part of $J(B_1, B_2\backslash B_1)$ is just the arclength

$$2\pi(r_1+r_2) \tag{2.8}$$

The nonlocal part of $J(B_1, B_2\backslash B_1)$ is

$$\begin{aligned}
J_{nonlocal} =& \frac{\gamma_{11}}{2}\int_D |\ (-\Delta)^{-\frac{1}{2}}(\chi_{B_1} - w_1)\ |^2\ dx \\
&+ \gamma_{12}\int_D (-\Delta)^{-\frac{1}{2}}(\chi_{B_1} - w_1)(-\Delta)^{-\frac{1}{2}}(\chi_{B_2\backslash B_1} - w_2)dx \\
&+ \frac{\gamma_{22}}{2}\int_D |\ (-\Delta)^{-\frac{1}{2}}(\chi_{B_2\backslash B_1} - w_2)\ |^2\ dx \\
\equiv& I + II + III.
\end{aligned} \tag{2.9}$$

$$
\begin{aligned}
III &= \frac{\gamma_{22}}{2}\int_D \mid (-\Delta)^{-\frac{1}{2}}(\chi_{B_2\setminus B_1} - w_2) \mid^2 dx \\
&= \frac{\gamma_{22}}{2}\int_D (\chi_{B_2\setminus B_1} - w_2)(v_2(x) - v_1(x))dx \\
&= \frac{\gamma_{22}}{2}\int_{B_2\setminus B_1} (v_2(x) - v_1(x))dx \\
&= \frac{\gamma_{22}}{2}[\int_{B_1} v_1 + \int_{B_2} v_2 - \int_{B_1} v_2 - \int_{B_2} v_1].
\end{aligned} \tag{2.10}
$$

From the definition of P_k one finds that

$$
\begin{aligned}
\int_{B_k} P_k(x)dx &= \int_{B_k} -\frac{|x|^2}{4} + \frac{r_k^2}{2} - \frac{r_k^2}{2}\log r_k dx \\
&= \int_{B_k} -\frac{|x|^2}{4}dx + \pi r_k^2(\frac{r_k^2}{4} - \frac{r_k^2}{2}\log r_k) \\
&= \int_0^{r_k} -\frac{r^2}{4}2\pi r dr + \pi r_k^2(\frac{r_k^2}{4} - \frac{r_k^2}{2}\log r_k) \\
&= -\frac{\pi r_k^4}{8} + \frac{\pi r_k^4}{4} - \frac{\pi r_k^4}{2}\log r_k \\
&= -\frac{\pi r_k^4}{2}\log r_k + \frac{\pi r_k^4}{8}.
\end{aligned} \tag{2.11}
$$

For the integral of Q_k, note that, since $\Delta Q_k(\cdot,\xi) = \frac{\pi r_k^2}{|D|}$, $Q_k(x,\xi) - \frac{\pi r_k^2}{4|D|}|x-\xi|^2$ is harmonic in x. By the Mean Value Theorem for harmonic functions

$$
\begin{aligned}
\int_{B_k} Q_k(x,\xi)dx &= \int_{B_k}(Q_k(x,\xi) - \frac{\pi r_k^2}{4|D|}|x-\xi|^2)dx + \int_{B_k}\frac{\pi r_k^2}{4|D|}|x-\xi|^2 dx \\
&= \pi r_k^2(Q_k(\xi,\xi) - \frac{\pi r_k^2}{4|D|}\cdot 0) + \frac{\pi^2 r_k^6}{8|D|} \\
&= \pi r_k^2(\pi r_k^2 R(\xi,\xi) + \frac{\pi r_k^4}{8|D|} + \frac{\pi^2 r_k^6}{8|D|} \\
&= \pi^2 r_k^4 R(\xi,\xi) + \frac{\pi^2 r_k^6}{4|D|}.
\end{aligned} \tag{2.12}
$$

Therefore, from (2.11) and (2.12), it follows

$$
\int_{B_1} v_1 = -\frac{\pi r_1^4 \log r_1}{2} + \frac{\pi r_1^4}{8} + \pi^2 r_1^4 R(\xi,\xi) + \frac{\pi^2 r_1^6}{4|D|}, \tag{2.13}
$$

$$\int_{B_2} v_2 = -\frac{\pi r_2^4 \log r_2}{2} + \frac{\pi r_2^4}{8} + \pi^2 r_2^4 R(\xi,\xi) + \frac{\pi^2 r_2^6}{4|D|}. \tag{2.14}$$

Next note that

$$\begin{aligned} -\int_{B_1} v_2 dx &= -\int_D \chi_{B_1} v_2 = -\int_D (\chi_{B_1} - \frac{\pi r_1^2}{|D|}) v_2 dx \\ &= \int_D \Delta v_1 v_2 = -\int_D \bigtriangledown v_1 \bigtriangledown v_2 dx = \int_D v_1 \Delta v_2 dx = -\int_{B_2} v_1 dx. \end{aligned}$$

Therefore,

$$\begin{aligned} -\int_{B_2} v_1 dx &= -\int_{B_1} v_2 dx \\ &= -\int_{B_1} P_2 dx - \int_{B_1} Q_2 dx \\ &= \frac{\pi r_1^4}{8} - \pi r_1^2 (\frac{r_2^2}{4} - \frac{r_2^2}{2} \log r_2) - \int_{B_1} (Q_2(x) - \frac{\pi r_2^2}{4|D|} |x-\xi|^2) dx - \int_{B_1} \frac{\pi r_2^2}{4|D|} |x-\xi|^2 dx \\ &= \frac{\pi r_1^4}{8} - \pi r_1^2 (\frac{r_2^2}{4} - \frac{r_2^2}{2} \log r_2) - \pi r_1^2 Q_2(\xi) - \frac{\pi^2 r_2^2 r_1^4}{8|D|} \\ &= \frac{\pi r_1^4}{8} - \pi r_1^2 (\frac{r_2^2}{4} - \frac{r_2^2}{2} \log r_2) - \pi^2 r_1^2 r_2^2 R(\xi,\xi) - \frac{\pi^2 r_2^2 r_1^4}{8|D|} - \frac{\pi^2 r_1^2 r_2^4}{8|D|}. \end{aligned}$$

We sum all these identities and use the fact that $\pi r_2^2 - \pi r_1^2 = w_2|D|$, to deduce

$$
\begin{aligned}
III =&\frac{\gamma_{22}}{2}\int_D \mid (-\Delta)^{-\frac{1}{2}}(\chi_{B_2\backslash B_1}-w_2)\mid^2 dx\\
=&\frac{\gamma_{22}}{2}[\int_{B_1} v_1+\int_{B_2} v_2-\int_{B_1} v_2-\int_{B_2} v_1]\\
=&\frac{\gamma_{22}}{2}[-\frac{\pi r_1^4\log r_1}{2}+\frac{\pi r_1^4}{8}+\pi^2 r_1^4 R(\xi,\xi)+\frac{\pi^2 r_1^6}{4|D|}-\frac{\pi r_2^4\log r_2}{2}\\
&+\frac{\pi r_2^4}{8}+\pi^2 r_2^4 R(\xi,\xi)+\frac{\pi^2 r_2^6}{4|D|}+\frac{\pi r_1^4}{4}-2\pi r_1^2(\frac{r_2^2}{4}-\frac{r_2^2}{2}\log r_2)\\
&-2\pi^2 r_1^2 r_2^2 R(\xi,\xi)-\frac{\pi^2 r_2^2 r_1^4}{4|D|}-\frac{\pi^2 r_1^2 r_2^4}{4|D|}]\\
=&\frac{\gamma_{22}}{2}[-\frac{\pi r_1^4\log r_1}{2}-\frac{\pi r_2^4\log r_2}{2}+\pi r_1^2 r_2^2\log r_2-\frac{w_2|D|r_1^2}{4}+\frac{w_2^2|D|^2}{8\pi}\\
&+w_2|D|^2 R(\xi,\xi)+\frac{w_2^2|D|}{4}(r_1^2+r_2^2)].
\end{aligned}
\tag{2.15}
$$

Similarly we can get

$$
\begin{aligned}
I =&\frac{\gamma_{11}}{2}\int_D \mid (-\Delta)^{-\frac{1}{2}}(\chi_{B_1}-w_1)\mid^2 dx\\
=&\frac{\gamma_{11}}{2}\int_{B_1}(P_1+Q_1)\\
=&\frac{\gamma_{11}}{2}[-\frac{\pi r_1^4\log r_1}{2}+\frac{\pi r_1^4}{8}+\frac{\pi r_1^6}{4|D|}+\pi^2 r_1^4 R(\xi,\xi)]
\end{aligned}
\tag{2.16}
$$

and

$$
\begin{aligned}
II =&\gamma_{12}\int_D (-\Delta)^{-\frac{1}{2}}(\chi_{B_1}-w_1)(-\Delta)^{-\frac{1}{2}}(\chi_{B_2\backslash B_1}-w_2)dx\\
=&\gamma_{12}[\int_{B_2} v_1-\int_{B_1} v_1]\\
=&\gamma_{12}[-\frac{\pi r_1^4}{8}+\pi r_1^2(\frac{r_2^2}{4}-\frac{r_2^2}{2}\log r_2)+\pi^2 r_1^2 r_2^2 R(\xi,\xi)\\
&+\frac{\pi^2 r_2^2 r_1^4}{8|D|}+\frac{\pi^2 r_1^2 r_2^4}{8|D|}+\frac{\pi r_1^4\log r_1}{2}-\frac{\pi r_1^4}{8}\\
&-\pi^2 r_1^4 R(\xi,\xi)-\frac{\pi^2 r_1^6}{4|D|}].
\end{aligned}
\tag{2.17}
$$

Adding (2.8),(2.16),(2.17) and (2.15) up we can get the value of J at the exact core-shell $(B_1, B_2\backslash B_1)$. □

Chapter 3

Perturbed core-shells

We perturb the exact core-shell $(B_1, B_2 \backslash B_1)$ by a pair of 2π-periodic functions $\phi = (\phi_1, \phi_2)$ where $r_1^2 + 2\phi_1$ is the perturbed inner radius squared and $r_2^2 + 2\phi_2$ is the perturbed outer radius squared. The perturbed B_1 is the set

$$E_1^\phi = \{\xi + \alpha e^{i\theta} : \theta \in [0, 2\pi], \alpha \in (0, \sqrt{r_1^2 + 2\phi_1(\theta)})\}.$$

The perturbed $B_2 \backslash B_1$ is the set

$$E_2^\phi = \{\xi + \alpha e^{i\theta} : \theta \in [0, 2\pi], \alpha \in (\sqrt{r_1^2 + 2\phi_1(\theta)}, \sqrt{r_2^2 + 2\phi_2(\theta)})\}$$

and the boundaries of the perturbed core-shell are two curves parametrized by $\theta : \xi + \sqrt{r_1^2 + 2\phi_1(\theta)} e^{i\theta}$, the perturbed inner circle, and $\xi + \sqrt{r_2^2 + 2\phi_2(\theta)} e^{i\theta}$, the perturbed outer circle. We will restrict the size of $\phi_1(\theta)$, $\phi_2(\theta)$ so that $r_1^2 + 2\phi_1$, $r_2^2 + 2\phi_2$ are always positive. Moreover it is always assumed that $\phi \perp \left\{[1, 0]^T, [0, 1]^T\right\}$, i.e.

$$\int_0^{2\pi} \phi_1(\theta) d\theta = 0, \text{and} \int_0^{2\pi} \phi_2(\theta) d\theta = 0. \tag{3.1}$$

This ensures that the area of E_1^ϕ remains $w_1|D|$ and the area of E_2^ϕ remains $w_2|D|$:

$$|E_1^\phi| = \int_0^{2\pi} \int_0^{\sqrt{r_1^2 + 2\phi_1(\theta)}} r dr d\theta = \int_0^{2\pi} \frac{r_1^2 + 2\phi_1(\theta)}{2} d\theta = \pi r_1^2 = w_1|D|$$

$$|E_2^\phi| = \int_0^{2\pi} \int_{\sqrt{r_1^2+2\phi_1(\theta)}}^{\sqrt{r_2^2+2\phi_1(\theta)}} r dr d\theta = \int_0^{2\pi} \frac{r_2^2+2\phi_2(\theta)-r_1^2-2\phi_1(\theta)}{2} d\theta = \pi r_2^2 - \pi r_1^2 = w_2|D|.$$

The local part of the energy of (E_1^ϕ, E_2^ϕ) can be expressed as

$$\begin{aligned}\frac{1}{2}(\mathcal{P}_D(E_1^\phi)+\mathcal{P}_D(E_2^\phi)+\mathcal{P}_D(D\backslash(E_1^\phi\cup E_2^\phi)) = &\int_0^{2\pi}\sqrt{r_1^2+2\phi_1(\theta)+\frac{(\phi_1'(\theta))^2}{r_1^2+2\phi_1(\theta)}}d\theta \\ &+\int_0^{2\pi}\sqrt{r_2^2+2\phi_2(\theta)+\frac{(\phi_2'(\theta))^2}{r_2^2+2\phi_2(\theta)}}d\theta.\end{aligned} \tag{3.2}$$

The variation of (3.2) gives us two quasi-linear operators

$$\mathcal{H}_k(\phi_k)(\theta) = \frac{r_k^2+2\phi_k(\theta)+\frac{3(\phi_k'(\theta))^2}{r_k^2+2\phi_k(\theta)}-\phi_k''(\theta)}{(r_k^2+2\phi_k(\theta)+\frac{(\phi_k'(\theta))^2}{r_k'+2\phi_k(\theta)})^{\frac{3}{2}}}, \quad k=1,2.$$

Note that $\mathcal{H}_2$ gives the curvature of the perturbed outer boundary viewed from E_2^ϕ. However $\mathcal{H}_1$ is the curvature of the perturbed inner boundary viewed from E_1^ϕ.

The nonlocal part of J may be written in terms of ϕ as

(3.3)

$$\begin{aligned}J_{nonlocal} =&\frac{\gamma_{11}}{2}\int_D |\ (-\Delta)^{-\frac{1}{2}}(\chi_{\Omega_1}-w_1)\ |^2\ dx \\ &+\gamma_{12}\int_D (-\Delta)^{-\frac{1}{2}}(\chi_{\Omega_1}-w_1)(-\Delta)^{-\frac{1}{2}}(\chi_{\Omega_2}-w_2)dx \\ &+\frac{\gamma_{22}}{2}\int_D |\ (-\Delta)^{-\frac{1}{2}}(\chi_{\Omega_2}-w_2)\ |^2\ dx \\ =&\frac{\gamma_{11}}{2}\int_0^{2\pi}d\theta\int_0^{\sqrt{r_1^2+2\phi_1(\theta)}}dr\int_0^{2\pi}d\omega\int_0^{\sqrt{r_1^2+2\phi_1(\omega)}}dtG(\xi+re^{i\theta},\xi+te^{i\omega})rt \\ &+\gamma_{12}\int_0^{2\pi}d\theta\int_0^{\sqrt{r_1^2+2\phi_1(\theta)}}dr\int_0^{2\pi}d\omega\int_{\sqrt{r_1^2+2\phi_1(\omega)}}^{\sqrt{r_2^2+2\phi_2(\omega)}}dtG(\xi+re^{i\theta},\xi+te^{iw})rt \\ &+\frac{\gamma_{22}}{2}\int_0^{2\pi}d\theta\int_{\sqrt{r_1^2+2\phi_1(\theta)}}^{\sqrt{r_2^2+2\phi_2(\theta)}}dr\int_0^{2\pi}d\omega\int_{\sqrt{r_1^2+2\phi_1(\omega)}}^{\sqrt{r_2^2+2\phi_2(\omega)}}dtG(\xi+re^{i\theta},\xi+te^{i\omega})rt.\end{aligned}$$

The variation of (3.3) with respect to ϕ_1 is

$$
\begin{aligned}
&(\gamma_{12}-\gamma_{22})(-\Delta)^{-1}(\chi_{\Omega_2}-w_2)(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta})\\
&+(\gamma_{11}-\gamma_{12})(-\Delta)^{-1}(\chi_{\Omega_1}-w_1)(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta})\\
=&\gamma_{12}\int_{\Omega_2}G(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},y)dy\\
&-\gamma_{12}\int_{\Omega_1}G(\xi+\sqrt{r_1^2+2\phi_1(\omega)}e^{i\omega},y)dy\\
&+\gamma_{11}\int_{\Omega_1}G(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},y)dy\\
&-\gamma_{22}\int_{\Omega_2}G(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},y)dy.
\end{aligned}
\tag{3.4}
$$

The variation of (3.3) with respect to ϕ_2 is

$$
\begin{aligned}
&\gamma_{12}(-\Delta)^{-1}(\chi_{\Omega_1}-w_1)(\xi+\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta})\\
&+\gamma_{22}(-\Delta)^{-1}(\chi_{\Omega_2}-w_2)(\xi+\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta})\\
=&\gamma_{12}\int_{\Omega_1}G(\xi+\sqrt{r_2^2+2\phi_2(\omega)}e^{i\omega},y)dy\\
&+\gamma_{22}\int_{\Omega_2}G(\xi+\sqrt{r_2^2+2\phi_2(\omega)}e^{i\omega},y)dy.
\end{aligned}
\tag{3.5}
$$

Under the constraint (3.1) the Euler Lagrange equations of J can be written as

$$
\begin{aligned}
&\mathcal{H}_1(\phi_1)(\theta)+(\gamma_{12}-\gamma_{22})(-\Delta)^{-1}(\chi_{\Omega_2}-w_2)(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta})\\
&+(\gamma_{11}-\gamma_{12})(-\Delta)^{-1}(\chi_{\Omega_1}-w_1)(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta})=\lambda_1
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
&\mathcal{H}_2(\phi_2)(\theta)+\gamma_{12}(-\Delta)^{-1}(\chi_{\Omega_1}-w_1)(\xi+\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta})\\
&+\gamma_{22}(-\Delta)^{-1}(\chi_{\Omega_2}-w_2)(\xi+\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta})=\lambda_2
\end{aligned}
\tag{3.7}
$$

in terms of ϕ_1 and ϕ_2. λ_1 and λ_2 are the Lagrange multipliers of the constraint (3.1).

Let us define

$$\begin{aligned}\mathcal{A}_1(\phi) = &-\frac{\gamma_{12}-\gamma_{22}}{2\pi}\int_0^{2\pi}\int_{\sqrt{r_1^2+2\phi_1(\theta)}}^{\sqrt{r_2^2+2\phi_2(\theta)}} \log\mid \sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta} - te^{i\omega}\mid tdtd\omega \\ &-\frac{\gamma_{11}-\gamma_{12}}{2\pi}\int_0^{2\pi}\int_0^{\sqrt{r_1^2+2\phi_1(\theta)}} \log\mid \sqrt{r_1^2+2\phi_1(\theta)}e^{i\omega} - te^{i\omega}\mid tdtd\omega\end{aligned} \tag{3.8}$$

$$\begin{aligned}\mathcal{B}_1(\phi) =&(\gamma_{12}-\gamma_{22})\int_0^{2\pi}\int_{\sqrt{r_1^2+2\phi_1(\theta)}}^{\sqrt{r_2^2+2\phi_2(\theta)}} R(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},\xi+te^{i\omega})tdtd\omega \\ &+(\gamma_{11}-\gamma_{22})\int_0^{2\pi}\int_0^{\sqrt{r_1^2+2\phi_1(\theta)}} R(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},\xi+te^{i\omega})tdtd\omega\end{aligned} \tag{3.9}$$

$$\begin{aligned}\mathcal{A}_2(\phi) = &-\frac{\gamma_{22}}{2\pi}\int_0^{2\pi}\int_{\sqrt{r_1^2+2\phi_1(\theta)}}^{\sqrt{r_2^2+2\phi_2(\theta)}} \log\mid \sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta} - te^{i\omega}\mid tdtd\omega \\ &-\frac{\gamma_{12}}{2\pi}\int_0^{2\pi}\int_0^{\sqrt{r_1^2+2\phi_1(\theta)}} \log\mid \sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta} - te^{i\omega}\mid tdtd\omega\end{aligned} \tag{3.10}$$

$$\begin{aligned}\mathcal{B}_2(\phi) =&\gamma_{22}\int_0^{2\pi}\int_{\sqrt{r_1^2+2\phi_1(\theta)}}^{\sqrt{r_2^2+2\phi_2(\theta)}} R(\xi+\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta},\xi+te^{i\omega})tdtd\omega \\ &+\gamma_{12}\int_0^{2\pi}\int_0^{\sqrt{r_1^2+2\phi_1(\theta)}} R(\xi+\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta},\xi+te^{i\omega})tdtd\omega,\end{aligned} \tag{3.11}$$

so that (3.6) and (3.7) become

$$\begin{aligned}\mathcal{H}_1(\phi_1)+\mathcal{A}_1(\phi)+\mathcal{B}_1(\phi) &= \lambda_1 \\ \mathcal{H}_2(\phi_2)+\mathcal{A}_2(\phi)+\mathcal{B}_2(\phi) &= \lambda_2.\end{aligned} \tag{3.12}$$

Note that the operators $\mathcal{H}_k$ and $\mathcal{A}_k$ are independent of ξ while the operators $\mathcal{B}_k$ do depend on ξ.

Let $\mathcal{S}=(\mathcal{S}_1,\mathcal{S}_2)$ be the operator that appears on the left side of equation (3.12) projected to $\left\{[1,0]^T,[0,1]^T\right\}$, i.e.

$$\mathcal{S}_k(\phi) = \mathcal{H}_k(\phi_k) + \mathcal{A}_k(\phi) + \mathcal{B}_k(\phi) - \lambda_k \tag{3.13}$$

for $k = 1, 2$. Here $\lambda_k(\phi)$ is so chosen that $\mathcal{S}(\phi) \perp \{[1,0],[0,1]\}$, i.e.

$$\int_0^{2\pi} \mathcal{S}_1(\phi) d\theta = 0, \int_0^{2\pi} \mathcal{S}_2(\phi) d\theta = 0.$$

Now (E_1^ϕ, E_2^ϕ) is a solution of the Euler Lagrange equation(and of course (3.12)) if and only if

$$\mathcal{S}(\phi) = 0. \tag{3.14}$$

The operator $\mathcal{S} = (\mathcal{S}_1, \mathcal{S}_2)$ maps from

$$\mathcal{X} = \left\{ \phi = \begin{bmatrix} \phi_1(\theta) \\ \phi_2(\theta) \end{bmatrix} : \phi_k \in H^2(S^1), k = 1, 2, \phi \perp \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \right\} \tag{3.15}$$

to

$$\mathcal{Y} = \left\{ q = \begin{bmatrix} q_1(\theta) \\ q_2(\theta) \end{bmatrix} : q_k \in L^2(S^1), k = 1, 2, q \perp \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \right\}. \tag{3.16}$$

The Frechet derivative of $\mathcal{S}$ is given by

$$\mathcal{H}_k'(\phi_k)(u_k) = \mathcal{H}_{k,\phi_k}(\phi) u_k + \mathcal{H}_{k,\phi_k'}(\phi) u_k' + \mathcal{H}_{k,\phi_k''}(\phi) u_k'' \tag{3.17}$$

(3.18)

$$\begin{aligned}\mathcal{A}_1^{'}(\phi)(u)(\theta) =&\frac{2\gamma_{12}-\gamma_{11}-\gamma_{22}}{2\pi}\int_0^{2\pi} u_1(\omega)\log\mid \sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta}-\sqrt{r_1^2+2\phi_1(\omega)}e^{i\omega}\mid d\omega\\ &+\frac{\gamma_{12}-\gamma_{22}}{2\pi}\frac{u_1(\theta)}{\sqrt{r_1^2+2\phi_1(\theta)}}\int_{\Omega_2}\frac{(\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta}-y)\cdot e^{i\theta}}{\mid \sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta}-y\mid^2}dy\\ &-\frac{\gamma_{11}-\gamma_{12}}{2\pi}\frac{u_1(\theta)}{\sqrt{r_1^2+2\phi_1(\theta)}}\int_{\Omega_1}\frac{(\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta}-y)\cdot e^{i\theta}}{\mid \sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta}-y\mid^2}dy\\ &-\frac{\gamma_{12}-\gamma_{22}}{2\pi}\int_0^{2\pi} u_2(\omega)\log\mid \sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta}-\sqrt{r_2^2+2\phi_2(\omega)}e^{i\omega}\mid d\omega\end{aligned}$$

(3.19)

$$\begin{aligned}\mathcal{B}_1^{'}(\phi)(u)(\theta) =&(\gamma_{11}-\gamma_{12})\int_0^{2\pi} u_1(\omega)R(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},\xi+\sqrt{r_1^2+2\phi_1(\omega)}e^{i\omega})d\omega\\ &-(\gamma_{12}-\gamma_{22})\frac{u_1(\theta)}{\sqrt{r_1^2+2\phi_1(\theta)}}\int_{\Omega_2}\bigtriangledown R(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},y)\cdot e^{i\theta}dy\\ &+(\gamma_{11}-\gamma_{22})\frac{u_1(\theta)}{\sqrt{r_1^2+2\phi_1(\theta)}}\int_{\Omega_1}\bigtriangledown R(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},y)\cdot e^{i\theta}dy\\ &+(\gamma_{12}-\gamma_{22})\int_0^{2\pi} u_2(\omega)R(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},\xi+\sqrt{r_2^2+2\phi_2(\theta)}e^{i\omega})d\omega\end{aligned}$$

(3.20)

$$\begin{aligned}\mathcal{A}_2^{'}(\phi)(u)(\theta) =&\frac{\gamma_{22}-\gamma_{12}}{2\pi}\int_0^{2\pi} u_1(\omega)\log\mid \sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta}-\sqrt{r_1^2+2\phi_1(\omega)}e^{i\omega}\mid d\omega\\ &-\frac{\gamma_{22}}{2\pi}\frac{u_2(\theta)}{\sqrt{r_2^2+2\phi_2(\theta)}}\int_{\Omega_2}\frac{(\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta}-y)\cdot e^{i\theta}}{\mid \sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta}-y\mid^2}dy\\ &-\frac{\gamma_{12}}{2\pi}\frac{u_2(\theta)}{\sqrt{r_2^2+2\phi_2(\theta)}}\int_{\Omega_1}\frac{(\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta}-y)\cdot e^{i\theta}}{\mid \sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta}-y\mid^2}dy\\ &-\frac{\gamma_{22}}{2\pi}\int_0^{2\pi} u_2(\omega)\log\mid \sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta}-\sqrt{r_2^2+2\phi_2(\omega)}e^{i\omega}\mid d\omega\end{aligned}$$

(3.21)

$$\begin{aligned}\mathcal{B}_2'(\phi)(u)(\theta) =&(\gamma_{12}-\gamma_{22})\int_0^{2\pi} u_1(\omega)R(\xi+\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta},\xi+\sqrt{r_1^2+2\phi_1(\omega)}e^{i\omega})d\omega\\ &+\gamma_{22}\frac{u_2(\theta)}{\sqrt{r_2^2+2\phi_2(\theta)}}\int_{\Omega_2}\bigtriangledown R(\xi+\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta},y)\cdot e^{i\theta}dy\\ &+\gamma_{12}\frac{u_2(\theta)}{\sqrt{r_2^2+2\phi_2(\theta)}}\int_{\Omega_1}\bigtriangledown R(\xi+\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta},y)\cdot e^{i\theta}dy\\ &+\gamma_{22}\int_0^{2\pi} u_2(\omega)R(\xi+\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta},\xi+\sqrt{r_2^2+2\phi_2(\theta)}e^{i\omega})d\omega.\end{aligned}$$

Chapter 4

Linear analysis

Let $\mathcal{L}$ be the linearized operator of $\mathcal{S}$ at $\phi = 0$, i.e. at the exact core-shell $(B_1, B_2 \backslash B_1)$:

$$\mathcal{L} = \mathcal{S}^{'}(0) \tag{4.1}$$

Going back to the expressions of the Frechet derivatives (3.17)-(3.21)

$$\mathcal{H}_k^{'}(0)(u_k) = -\frac{1}{r_k^3}(u_k^{''} + u_k) \tag{4.2}$$

$$\begin{aligned} \mathcal{A}_1^{'}(0)(u)(\theta) =& \frac{2\gamma_{12} - \gamma_{11} - \gamma_{22}}{2\pi} \int_0^{2\pi} u_1(\omega) \log \mid r_1 e^{i\theta} - r_1 e^{i\omega} \mid d\omega \\ & - \frac{\gamma_{11} - \gamma_{12}}{2\pi} \pi u_1(\theta) \\ & + \frac{\gamma_{22} - \gamma_{12}}{2\pi} \int_0^{2\pi} u_2(\omega) \log \mid r_1 e^{i\theta} - r_2 e^{i\omega} \mid d\omega \end{aligned} \tag{4.3}$$

$$
\begin{aligned}
\mathcal{B}_1^{'}(0)(u)(\theta) =&(\gamma_{11}-\gamma_{12})\int_0^{2\pi} u_1(\omega)R(\xi+r_1e^{i\theta},\xi+r_1e^{i\omega})d\omega \\
&-(\gamma_{12}-\gamma_{22})\frac{u_1(\theta)}{r_1}\int_{\Omega_2}\bigtriangledown R(\xi+r_1e^{i\theta},y)\cdot e^{i\theta}dy \\
&+(\gamma_{11}-\gamma_{22})\frac{u_1(\theta)}{r_1}\int_{\Omega_1}\bigtriangledown R(\xi+r_1e^{i\theta},y)\cdot e^{i\theta}dy \\
&+(\gamma_{12}-\gamma_{22})\int_0^{2\pi} u_2(\omega)R(\xi+r_1e^{i\theta},\xi+r_2e^{i\omega})d\omega
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
\mathcal{A}_2^{'}(0)(u)(\theta) =&\frac{\gamma_{22}-\gamma_{12}}{2\pi}\int_0^{2\pi} u_1(\omega)\log\mid r_2^2e^{i\theta}-r_1e^{i\omega}\mid d\omega \\
&-\frac{\gamma_{22}}{2\pi}u_2(\theta)\pi(1-\frac{r_1^2}{r_2^2}) \\
&-\frac{\gamma_{12}}{2\pi}u_2(\theta)\pi\frac{r_1^2}{r_2^2} \\
&-\frac{\gamma_{22}}{2\pi}\int_0^{2\pi} u_2(\omega)\log\mid r_2e^{i\theta}-r_2e^{i\omega}\mid d\omega
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
\mathcal{B}_2^{'}(0)(u)(\theta) =&(\gamma_{12}-\gamma_{22})\int_0^{2\pi} u_1(\omega)R(\xi+r_2e^{i\theta},\xi+r_1e^{i\omega})d\omega \\
&+\gamma_{22}\frac{u_2(\theta)}{r_2}\int_{\Omega_2}\bigtriangledown R(\xi+r_2e^{i\theta},y)\cdot e^{i\theta}dy \\
&+\gamma_{12}\frac{u_2(\theta)}{r_2}\int_{\Omega_1}\bigtriangledown R(\xi+r_2e^{i\theta},y)\cdot e^{i\theta}dy \\
&+\gamma_{22}\int_0^{2\pi} u_2(\omega)R(\xi+r_2e^{i\theta},\xi+r_2e^{i\omega})d\omega.
\end{aligned} \tag{4.6}
$$

Let us separate $\mathcal{L}$ to a dominant part $\mathcal{L}_1$ and a minor part $\mathcal{L}_2$. We define $\mathcal{L}_{1,k}$ the k-th component of $\mathcal{L}_1$ to be

$$
\begin{aligned}
\mathcal{L}_{1,1}(u)(\theta) = & -\frac{1}{r_1^3}(u_1''(\theta)+u_1(\theta)) \\
& + \frac{2\gamma_{12}-\gamma_{11}-\gamma_{22}}{2\pi}\int_0^{2\pi} u_1(\omega)\log|r_1e^{i\theta}-r_1e^{i\omega}|d\omega \\
& - \frac{\gamma_{12}-\gamma_{22}}{2\pi}\int_0^{2\pi} u_2(\omega)\log|r_1e^{i\theta}-r_2e^{i\omega}|d\omega \\
& - \frac{\gamma_{11}-\gamma_{12}}{2}u_1(\theta)+l_1(u)
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
\mathcal{L}_{1,2}(u)(\theta) = & -\frac{1}{r_2^3}(u_2''(\theta)+u_2(\theta)) \\
& + \frac{\gamma_{22}-\gamma_{12}}{2\pi}\int_0^{2\pi} u_1(\omega)\log|r_2e^{i\theta}-r_1e^{i\omega}|d\omega \\
& - \frac{\gamma_{22}}{2\pi}\int_0^{2\pi} u_2(\omega)\log|r_2e^{i\theta}-r_2e^{i\omega}|d\omega \\
& - \frac{\gamma_{22}}{2}u_2(\theta)(1-\frac{r_1^2}{r_2^2}) - \frac{\gamma_{12}}{2}u_2(\theta)\frac{r_1^2}{r_2^2}+l_2(u).
\end{aligned} \tag{4.8}
$$

The real valued linear operator l_i is so chosen that $\mathcal{L}_1$ maps from $\mathcal{X}$ to $\mathcal{Y}$(See (3.15) and (3.16) for their definitions). The rest of $\mathcal{L}$ is denoted by $\mathcal{L}_2$.

We are more interested in the operator $\Pi\mathcal{L}$ and $\Pi\mathcal{L}_1$ where Π is the orthogonal projection operator from $\mathcal{Y}$ to

$$
\mathcal{Y}_* = \left\{ q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \in \mathcal{Y} : q \perp \cos\theta \begin{bmatrix} r \\ 1 \end{bmatrix}, q \perp \sin\theta \begin{bmatrix} r \\ 1 \end{bmatrix}, q \perp \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \right\} \tag{4.9}
$$

where $r = r_1/r_2$.

The operator $\Pi\mathcal{L}$ is defined on

$$
\mathcal{X}_* = \left\{ u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathcal{X} : u \perp \cos\theta \begin{bmatrix} r \\ 1 \end{bmatrix}, u \perp \sin\theta \begin{bmatrix} r \\ 1 \end{bmatrix}, u \perp \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \right\} \subset \mathcal{Y}_*. \tag{4.10}
$$

Since every element of $\mathcal{X}$ (and $\mathcal{Y}$) is perpendicular to $\{[1,0]^\perp, [0,1]^\perp\}$, if $u = (u_1, u_2) \in \mathcal{X}_*$

(or $\mathcal{Y}*$) , it must be perpendicular to any constant vector, i.e.

$$\int_0^{2\pi} u_k(\theta)d\theta = 0, \quad k = 1, 2. \tag{4.11}$$

Consequently,

$$l_k(u) = 0, \text{ if } u \in \mathcal{X}_*, k = 1, 2. \tag{4.12}$$

Calculation gives us:

$$\begin{aligned}\mathcal{L}_{1,1}(v_1\cos(n\theta), v_2\cos(n\theta)) = &-\frac{1}{r_1^3}(1-n^2)v_1\cos(n\theta)\\ &-\frac{2\gamma_{12}-\gamma_{11}-\gamma_{22}}{2n}v_1\cos(n\theta)\\ &-\frac{\gamma_{11}-\gamma_{12}}{2}v_1\cos(n\theta)\\ &+\frac{\gamma_{12}-\gamma_{22}}{2n}(\frac{r_1}{r_2})^n v_2\cos(n\theta)\end{aligned} \tag{4.13}$$

$$\begin{aligned}\mathcal{L}_{1,2}(v_1\cos(n\theta), v_2\cos(n\theta)) = &-\frac{1}{r_2^3}(1-n^2)v_2\cos(n\theta)\\ &-\frac{\gamma_{22}-\gamma_{12}}{2n}(\frac{r_1}{r_2})^n v_1\cos(n\theta)\\ &+\frac{\gamma_{22}}{2n}v_2\cos(n\theta)-\frac{\gamma_{22}}{2}(1-\frac{r_1^2}{r_2^2})v_2\cos(n\theta)\\ &-\frac{\gamma_{12}}{2}(\frac{r_1^2}{r_2^2})v_2\cos(n\theta).\end{aligned} \tag{4.14}$$

We define

$$\tilde{M}_n = \begin{bmatrix} \frac{1}{r_1^3}(n^2-1) - \frac{2\gamma_{12}-\gamma_{11}-\gamma_{22}}{2n} - \frac{\gamma_{11}-\gamma_{12}}{2} & \frac{\gamma_{12}-\gamma_{22}}{2n}(\frac{r_1}{r_2})^n \\ \frac{\gamma_{12}-\gamma_{22}}{2n}(\frac{r_1}{r_2})^n & \frac{1}{r_2^3}(n^2-1) + \frac{\gamma_{22}}{2n} - \frac{\gamma_{22}}{2}(1-\frac{r_1^2}{r_2^2}) - \frac{\gamma_{12}}{2}\frac{r_1^2}{r_2^2} \end{bmatrix}. \tag{4.15}$$

Let $\frac{r_1}{(\frac{\omega_1|D|}{\pi})^{\frac{1}{2}}} \equiv R_1 \equiv 1$, $R_2 = (1+q)^{\frac{1}{2}}$, where $q = \frac{\omega_2}{\omega_1}$ fixed. We can now write

$\Gamma_{ij} = (\frac{\omega_1|D|}{\pi})^{\frac{3}{2}}\gamma_{ij}$.

After rescaling, $\tilde{M}_n = (\frac{\omega_1|D|}{\pi})^{-\frac{3}{2}}M_n$, where

(4.16)

$$M_n = \begin{bmatrix} (n^2-1) - \frac{2\Gamma_{12}-\Gamma_{11}-\Gamma_{22}}{2n} - \frac{\Gamma_{11}-\Gamma_{12}}{2} & \frac{\Gamma_{12}-\Gamma_{22}}{2n}(\frac{1}{1+q})^{n/2} \\ \frac{\Gamma_{12}-\Gamma_{22}}{2n}(\frac{1}{1+q})^{n/2} & \frac{1}{(1+q)^{3/2}}(n^2-1) + \frac{\Gamma_{22}}{2n} - \frac{\Gamma_{22}}{2}(1-\frac{1}{1+q}) - \frac{\Gamma_{12}}{2}\frac{1}{1+q} \end{bmatrix}.$$

For $n \geq 2$ case, the following result can be proved.

4.17 Lemma. $\left\{\Gamma \in \mathbb{S}^2 : \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12} & \Gamma_{22} \end{bmatrix} > 0\right\} \cap \{\Gamma \in \mathbb{S}^2 : \Gamma_{22} > \Gamma_{12}\} \cap \{\Gamma \in \mathbb{S}^2 : M_n > 0, \forall n \geq 2\} \neq \emptyset$.

Proof. For each $n \geq 2$, let $P_n = \{(\Gamma_{11}, \Gamma_{12}, \Gamma_{22}) \in R^3 | M_n > 0\}$. It is easy to see that $\tilde{0} = (0,0,0) \in P_n$. Hence there exists a neighborhood $B_{\delta(n)}(\tilde{0})$ such that $B_{\delta(n)}(\tilde{0}) \subset P_n$. Note that for large enough n, M_n is positive definite for all $\Gamma \in B_1(\tilde{0})$. Therefore $\exists N_0 \in N$ such that for $n \geq N_0$, $B_1(\tilde{0}) \subset P_n$. For $n < N_0$, $B_{\delta(n)}(\tilde{0}) \subset P_n$. Take B_0 to be the smallest of $\{B_1(\tilde{0}), B_{\delta(1)}(\tilde{0}), B_{\delta(2)}(\tilde{0}), \cdots, B_{\delta(N_0-1)}(\tilde{0})\}$. The condition $\Gamma_{22} > \Gamma_{12}$ implies that half of the ball B_0 is in the intersection. Then the intersection of the ball B_0, half space $\{\Gamma \in \mathbb{S}^2 : \Gamma_{22} > \Gamma_{12}\}$, and the cone $\left\{\Gamma \in \mathbb{S}^2 : \Gamma > 0\right\}$ is nonempty. □

From now on, we will assume that $(\Gamma_{11}, \Gamma_{12}, \Gamma_{22})$ is in the intersection set of the preceding lemma.

4.18 Lemma. *When $n = 1$,*

$$\tilde{M}_1 = \begin{bmatrix} \frac{\gamma_{22}-\gamma_{12}}{2} & \frac{\gamma_{12}-\gamma_{22}}{2}r \\ \frac{\gamma_{12}-\gamma_{22}}{2}r & \frac{\gamma_{22}-\gamma_{12}}{2}r^2 \end{bmatrix}$$

has two eigenvalues. One is $\tilde{\lambda}_{1,1} = 0$, with eigenvectors

$$\cos\theta \begin{bmatrix} r \\ 1 \end{bmatrix}, \sin\theta \begin{bmatrix} r \\ 1 \end{bmatrix}.$$

However they are perpendicular to $\mathcal{X}_*$ and therefore discarded.

The second vector is $\tilde{\lambda}_{1,2} = \frac{\gamma_{22}-\gamma_{12}}{2}(1+r^2)$ with eigenvectors

$$\cos\theta \begin{bmatrix} 1 \\ \\ -r \end{bmatrix}, \sin\theta \begin{bmatrix} 1 \\ \\ -r \end{bmatrix}$$

which are in $\mathcal{X}_*$.

It is easy to see that for $n = 1$ case, the eigenvalue for M_n satisfies

$$\lambda_{1,2} = (\frac{\omega_1|D|}{\pi})^{\frac{3}{2}}\tilde{\lambda}_{1,2} = (\frac{\omega_1|D|}{\pi})^{\frac{3}{2}}\frac{\gamma_{22}-\gamma_{12}}{2}(1+r^2) = \frac{\Gamma_{22}-\Gamma_{12}}{2}(1+r^2).$$

One can see $\Gamma_{22} > \Gamma_{12}$ implies that $\lambda_{1,2} > 0$.

4.19 Lemma. *There exists a real number* $s > 0$ *such that for all* $n \geq 2$*, we have* $\frac{\lambda_{n,k}}{n^2} \geq s$ *for* $k = 1, 2$*. Moreover, let* $\tilde{\lambda}_{n,1}$,$\tilde{\lambda}_{n,2}$ *be the two eigenvalues of* $\tilde{M}_n$*, we can conclude that*

$$\frac{\tilde{\lambda}_{n,k}}{n^2} = (\frac{w_1|D|}{\pi})^{-3/2}\lambda_{n,k} \geq (\frac{w_1|D|}{\pi})^{-3/2}s. \tag{4.20}$$

Proof. For $n \geq 2$, denote the (1,1) entry of M_n by c_1, (2,2) entry by c_2, and (1,2) and (2,1) entry by d. Then

$$\det(\lambda I - M_n) = \lambda^2 - (c_1 + c_2)\lambda + c_1c_2 - d^2. \tag{4.21}$$

Let $\lambda_{n,1}$,$\lambda_{n,2}$ be the two eigenvalues of M_n. They are necessarily positive by Lemma 4.17 and our choice of Γ. We derive from (4.21) that

$$\lambda_{n,1} = \frac{c_1 + c_2 + \sqrt{(c_1 - c_2)^2 + 4d^2}}{2}, \tag{4.22}$$

$$\lambda_{n,2} = \frac{c_1 + c_2 - \sqrt{(c_1 - c_2)^2 + 4d^2}}{2}. \tag{4.23}$$

It is obvious that

$$\lim_{n\to\infty}\frac{\lambda_{n,1}}{n^2}=\lim_{n\to\infty}\frac{c_1+c_2+\sqrt{(c_1-c_2)^2+4d^2}}{2n^2}=\frac{1}{2}(1+\frac{1}{(1+q)^{3/2}})$$

and

$$\lim_{n\to\infty}\frac{\lambda_{n,2}}{n^2}=\lim_{n\to\infty}\frac{c_1+c_2-\sqrt{(c_1-c_2)^2+4d^2}}{2n^2}=\frac{1}{2}(1-\frac{1}{(1+q)^{3/2}}).$$

Both limits are positive values since $q=\frac{w_2}{w_1}$ is positve.

Hence there exists a real valued number $s>0$ such that $\frac{\lambda_{n,k}}{n^2}\geq s$ for $k=1,2$.

Let $\tilde{\lambda}_{n,1}$, $\tilde{\lambda}_{n,2}$ be the two eigenvalues of $\tilde{M}_n$, because of the rescaling relations we can conclude

$$\frac{\tilde{\lambda}_{n,k}}{n^2}=(\frac{w_1|D|}{\pi})^{-3/2}\lambda_{n,k}\geq(\frac{w_1|D|}{\pi})^{-3/2}s, \tag{4.24}$$

for $k=1,2$.

□

After the preceding work, we can prove the following result.

4.25 Lemma. *1.* $||u||_{H^2}\leq C|w|^{3/2}||\mathcal{L}_1(u)||_{L^2}$ *for all* $u\in\mathcal{X}_*$.

2. $||u||^2_{H^1}\leq C|w|^{3/2}<\mathcal{L}_1(u),u>$ *for all* $u\in\mathcal{X}_*$.

Proof. For any $u\in\mathcal{X}_*$, we can write

$$u=\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}\sum_{|n|=2}^{\infty}C_{n,1}e^{in\theta}\\\sum_{|n|=2}^{\infty}C_{n,2}e^{in\theta}\end{bmatrix}+C_1e^{i\theta}\begin{bmatrix}1\\-r\end{bmatrix}+C_{-1}e^{-i\theta}\begin{bmatrix}1\\-r\end{bmatrix}$$

where $C_{n,k}=\frac{1}{2\pi}\int_{-\pi}^{\pi}u_k(x)e^{-inx}dx$ for $k=1,2$ and $|n|\geq 2$.

$$u''=\begin{bmatrix}u_1''\\u_2''\end{bmatrix}=\begin{bmatrix}-\sum_{|n|=2}^{\infty}n^2C_{n,1}e^{in\theta}\\-\sum_{|n|=2}^{\infty}n^2C_{n,2}e^{in\theta}\end{bmatrix}-C_1e^{i\theta}\begin{bmatrix}1\\-r\end{bmatrix}-C_{-1}e^{-i\theta}\begin{bmatrix}1\\-r\end{bmatrix}.$$

$$||u^{''}||^2_{L^2} = \sum_{|n|=2}^{\infty} |C_{n,1}|^2 n^4 + \sum_{|n|=2}^{\infty} |C_{n,2}|^2 n^4 + C_1^2(1+r^2)2\pi + C_{-1}^2(1+r^2)2\pi.$$

On the other hand,

$$\mathcal{L}_1(u) = \begin{bmatrix} \sum_{|n|=2}^{\infty} \tilde{\lambda}_{n,1} C_{n,1} e^{in\theta} \\ \sum_{|n|=2}^{\infty} \tilde{\lambda}_{n,2} C_{n,2} e^{in\theta} \end{bmatrix} + \tilde{\lambda}_{1,2}(C_1 e^{i\theta} \begin{bmatrix} 1 \\ -r \end{bmatrix} + C_{-1} e^{-i\theta} \begin{bmatrix} 1 \\ -r \end{bmatrix}).$$

$$||\mathcal{L}_1(u)||^2_{L^2} = \sum_{|n|=2}^{\infty} |C_{n,1}|^2 \tilde{\lambda}^2_{n,1} + \sum_{|n|=2}^{\infty} |C_{n,2}|^2 \tilde{\lambda}^2_{n,2} + \lambda^2_{12}(C_1^2(1+r^2)2\pi + C_{-1}^2(1+r^2)2\pi).$$

By (4.20), we conclude there is a constant C such that

$||u^{''}||_{L^2} \leq C|w|^{3/2}||\mathcal{L}_1(u)||_{L^2}$ for all $u \in \mathcal{X}_*$.

Similarly $||u||_{L^2}$ and $||u^{'}||_{L^2}$ are both bounded by $C||\mathcal{L}_1(u)||_{L^2}$ for some constant C.

Hence the first part of the lemma is proved.

For the second part, note that

$$||u^{'}||^2_{L^2} = \sum_{|n|=2}^{\infty} |C_{n,1}|^2 n^2 + \sum_{|n|=2}^{\infty} |C_{n,2}|^2 n^2 + C_1^2(1+r^2)2\pi + C_{-1}^2(1+r^2)2\pi$$

and

$$< \mathcal{L}_1(u), u >= \sum_{|n|=2}^{\infty} |C_{n,1}|^2 w_1^{-3/2} \lambda_{n,1} + \sum_{|n|=2}^{\infty} |C_{n,2}|^2 w_1^{-3/2} \lambda_{n,2} + \lambda^2_{12}(C_1^2(1+r^2)2\pi + C_{-1}^2(1+r^2)2\pi)$$

for some constant C.

By (4.20)

$||u||^2_{H^1} \leq C|w|^{3/2} < \mathcal{L}_1(u), u >$ for all $u \in \mathcal{X}_*$. This proves the lemma.

□

The second part $\mathcal{L}_2$ in $\mathcal{L}$ is a minor part.

4.26 Lemma. *There exists $C > 0$ independent of ξ, r_1,r_2 such that $||\mathcal{L}_2(u)||_{L^2} \leq C|w|^{-1}||u||_{L^2}$ for all $u \in \mathcal{X}_*$.*

Proof. Recall $\mathcal{L}_2$ from (4.7) and (4.8):

$$\begin{aligned}\mathcal{L}_{2,1}(u)(\theta) =&(\gamma_{11}-\gamma_{12})\int_0^{2\pi} u_1(\omega)R(\xi+r_1e^{i\theta},\xi+r_1e^{i\omega})d\omega\\ &-(\gamma_{12}-\gamma_{22})\frac{u_1(\theta)}{r_1}\int_{\Omega_2}\bigtriangledown R(\xi+r_1e^{i\theta},y)\cdot e^{i\theta}dy\\ &+(\gamma_{11}-\gamma_{22})\frac{u_1(\theta)}{r_1}\int_{\Omega_1}\bigtriangledown R(\xi+r_1e^{i\theta},y)\cdot e^{i\theta}dy\\ &+(\gamma_{12}-\gamma_{22})\int_0^{2\pi} u_2(\omega)R(\xi+r_1e^{i\theta},\xi+r_2e^{i\omega})d\omega\\ &+l_3(u)\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{2,2}(u)(\theta) =&(\gamma_{12}-\gamma_{22})\int_0^{2\pi} u_1(\omega)R(\xi+r_2e^{i\theta},\xi+r_1e^{i\omega})d\omega\\ &+\gamma_{22}\frac{u_2(\theta)}{r_2}\int_{\Omega_2}\bigtriangledown R(\xi+r_2e^{i\theta},y)\cdot e^{i\theta}dy\\ &+\gamma_{12}\frac{u_2(\theta)}{r_2}\int_{\Omega_1}\bigtriangledown R(\xi+r_2e^{i\theta},y)\cdot e^{i\theta}dy\\ &+\gamma_{22}\int_0^{2\pi} u_2(\omega)R(\xi+r_2e^{i\theta},\xi+r_2e^{i\omega})d\omega\\ &+l_4(u).\end{aligned}$$

Because

$$R(\xi+r_ke^{i\theta},\xi+r_le^{i\theta})-R(\xi,\xi)=O(|w|^{1/2})$$

and $\int_0^{2\pi} u_k(\omega)d\omega = 0$, we obtain that for $s,t=1,2$

$$
\begin{aligned}
&||\gamma_{st}\int_0^{2\pi} u_j(\omega)R(\xi+r_ke^{i\theta},\xi+r_le^{i\theta})d\omega||_{L^2}\\
&=||\gamma_{st}\int_0^{2\pi} u_j(\omega)(R(\xi+r_ke^{i\theta},\xi+r_le^{i\theta})-R(\xi,\xi))d\omega||_{L^2}\\
&\le C\gamma_{st}|w|^{1/2}||u_j||_{L_2}\\
&\le C|w|^{-1}||u_j||_{L^2}.
\end{aligned}
$$

Since the area of the exact core-shell $(B_1, B_2\backslash B_1)$ is of order $O(|w|)$,

$$
||\gamma_{st}u_k(\theta)\int_{\Omega_i}\bigtriangledown R(\xi+r_ke^{i\theta},y)\cdot e^{i\theta}dy||_{L^2}\le C|w|^{-1}||u_k||_{L^2}.
$$

The condition

$$
\overline{-\mathcal{L}_{2,1}(u)}=0 \text{ and } \overline{-\mathcal{L}_{2,2}(u)}=0
$$

implies that

$$
|l_3(u)|\le C|w|^{-1}||u||_{L^2} \text{ and } |l_4(u)|\le C|w|^{-1}||u||_{L^2}.
$$

Hence the lemma is proved. □

4.27 Lemma. *1. For $u\in\mathcal{X}_*$,$||u||_{H^2}\le C|w|^{3/2}||\Pi\mathcal{L}(u)||_{L^2}$.*

2. $||u||^2_{H^1}\le C|w|^{3/2}<\Pi\mathcal{L}(u),u>$.

3. $\Pi\mathcal{L}$: $\mathcal{X}_*\to\mathcal{Y}_*$ is one-to-one and on to.

Proof. When w_1 is small, by Lemma (4.25) Part 1 and Lemma (4.26),

$$
\begin{aligned}
||\Pi\mathcal{L}(u)||_{L^2}&\ge||\Pi\mathcal{L}_1(u)||_{L^2}-||\Pi\mathcal{L}_2(u)||_{L^2}\\
&\ge C|w|^{-3/2}||u||_{L^2}-C|w|^{-1}||u||_{L^2}\ge C|w|^{-3/2}||u||_{L^2},
\end{aligned}
$$

proving part 1.

It follows from Lemma (4.25) Part 2 and Lemma (4.26) that

$$< \Pi\mathcal{L}(u), u >=$$
$$< \Pi\mathcal{L}_1(u), u > + < \Pi\mathcal{L}_2(u), u >\geq C|w|^{-3/2}||u||^2_{H_1} - C|w|^{-1}||u||^2_{L^2} \geq C|w|^{-3/2}||u||^2_{H_1}$$

when $|w|$ is sufficiently small.

Part 1 of Lemma 4.27 assures that $\Pi\mathcal{L}$ is one-to-one from $\mathcal{X}_*$ to $\mathcal{Y}_*$. Since $\Pi\mathcal{L}$ is self-adjoint and hence closed, it also assures that the range of $\Pi\mathcal{L}$ is closed. The Closed Range Theorem (see Yosida [24, Page 205],e.g.) then implies that $\Pi\mathcal{L}$ is onto. □

4.28 Lemma. *Assume that $||\phi||_{H_2} \leq c|w|$ where c is sufficiently small. The following estimates hold for $u = (u_1, u_2) \in \mathcal{X}$, $v = (v_1, v_2) \in \mathcal{X}$.*

1. $||\mathcal{H}_k^{''}(\phi_k)(u_k, v_k)||_{L^2} \leq C|w|^{-5/2}||u_k||_{H^2}||v_k||_{H^2}$.
2. $||\mathcal{A}^{''}(\phi)(u, v)||_{L^2} \leq C|w|^{-5/2}||u||_{H^1}||v||_{H^1}$.
3. $||\mathcal{B}^{''}(\phi)(u, v)||_{L^2} \leq C|w|^{-2}||u||_{H^1}||v||_{H^1}$.
4. $|\lambda_k^{''}(\phi)(u, v)| \leq C|w|^{-5/2}||u||_{H^2}||v||_{H^2}$.

In summary $|\mathcal{S}^{''}(\phi)(u, v)||_{L^2} \leq C|w|^{-5/2}||u||_{H^2}||v||_{H^2}$.

Note that by taking c small, we keep $r_k^2 + 2\phi_k$ positive, so (E_1^ϕ, E_2^ϕ) is a perturbation of the exact core-shell $(B_1, B_2\backslash B_1)$. The proof of this lemma is similar to that of [11, Lemma 3.2] and [17, Lemma 6.1]. We omit the details.

Chapter 5

Reduction to two dimensions

In this section, it will be proved that, for each ξ, there exists a pair of functions $\varphi(\cdot,\xi) = (\varphi_1(\cdot,\xi), \varphi_2(\cdot,\xi)) \in \mathcal{X}_*$ such that

$$\mathcal{S}(\varphi)(\theta) = A_1 \cos\theta \begin{bmatrix} r \\ 1 \end{bmatrix} + A_2 \sin\theta \begin{bmatrix} r \\ 1 \end{bmatrix} \tag{5.1}$$

for some real numbers A_1, A_2. Note that φ is sought in $\mathcal{X}_*$.

The equation (5.1) is written as

$$\Pi\mathcal{S}(\varphi(\cdot,\xi)) = 0 \tag{5.2}$$

where Π is the orthogonal projection operator from $\mathcal{Y}$ to $\mathcal{Y}_*$.

Recall $\mathcal{L}$, the linearized operator of $\mathcal{S}$ at $\phi = 0$, i.e. $\mathcal{L} = \mathcal{S}'(0)(u)$. Expand $\mathcal{S}(\phi)$ as

$$\mathcal{S}(\phi) = \mathcal{S}(0) + \mathcal{L}(\phi) + \mathcal{N}(\phi) \tag{5.3}$$

where $\mathcal{N}$ is a higher order term defined by (5.3). Rewrite (5.2) in a fixed point form:

$$\phi = -(\Pi\mathcal{L})^{-1}(\Pi\mathcal{S}(0) + \Pi\mathcal{N}(\phi)). \tag{5.4}$$

5.5 Lemma. *There is $\varphi = \varphi(\cdot, \xi)$ such that for every $\xi \in U_1$, $\varphi(\cdot, \xi) \in \mathbb{X}_*$ solves (5.4) and $||\varphi(\cdot, \xi)||_{H_2} \leq c|w|^2$ where c is a sufficiently large constant independent of $|w|$ and ξ.*

Proof. To use the Contraction Mapping Principle in the fixed point setting (5.4), let

$$\mathcal{T}(\phi) = -(\Pi\mathcal{L})^{-1}(\Pi\mathcal{S}(0) + \Pi\mathcal{N}(\phi)) \tag{5.6}$$

be the operator defined on

$$D(\mathcal{T}) = \{\phi \in \mathcal{X}_* : ||\phi||_{H^2} \leq c|w|^2\} \tag{5.7}$$

where the constant c is sufficiently large and will be made more precise later. We know from Lemma (2.5) that $\mathcal{S}(0)$ is a sum of a θ independent part, an $e^{i\theta}$ part and a quantity of order $O(w_2^{1/2})$. After one applies Π the θ independent part and $e^{i\theta}$ part become 0 and we have

$$||\Pi\mathcal{S}(0)||_{L^2} = O(|w|^{1/2}). \tag{5.8}$$

From Lemma (4.25) we deduce that

$$||(\Pi L)^{-1}\Pi\mathcal{S}(0)||_{H^2} \leq C|w|^2. \tag{5.9}$$

Lemma (4.28) implies that

$$||\mathcal{N}(\phi)||_{L^2} \leq C|w|^{-5/2}||\phi||_{H^2}^2 \tag{5.10}$$

and consequently

$$||(\Pi\mathcal{L})^{-1}\Pi\mathcal{N}(\phi)||_{H^2} \leq C|w|^{-1}||\phi||_{H^2}^2. \tag{5.11}$$

Using (5.6),(5.7), (5.9),and (5.11) we find that

$$||\mathcal{T}(\phi)||_{H^2} \leq C|w|^2 + C|w|^{-1}c^2|w|^4 \leq c|w|^2$$

if c is large enough and $|w|$ sufficiently small. Therefore $\mathcal{T}$ is map from $D(\mathcal{T})$ to itself. Finally we show that $\mathcal{T}$ is a contraction. Let ϕ_1, $\phi_2 \in D(\mathcal{T})$. First note that

$$\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2) = (\Pi\mathcal{L})^{-1}(-\Pi)(\mathcal{N}(\phi_1) - \mathcal{N}(\phi_2)). \tag{5.12}$$

Because

$$\mathcal{N}(\phi_1) - \mathcal{N}(\phi_2) = \mathcal{S}(\phi_1) - \mathcal{S}(\phi_2) - \mathcal{S}^{'}(0)(\phi_1 - \phi_2),$$

we deduce, with the help of Lemma (4.28) and (5.7),that

$$\begin{aligned}
||\mathcal{N}(\phi_1) - \mathcal{N}(\phi_2)||_{L^2} &\leq ||\mathcal{S}^{'}(\phi_2)(\phi_1 - \phi_2) - \mathcal{S}^{'}(0)(\phi_1 - \phi_2)||_{L^2} + C|w|^{-5/2}||\phi_1 - \phi_2||_{H^2}^2 \\
&\leq C|w|^{-5/2}||\phi_2||_{H^2}||\phi_2 - \phi_2||_{H^2} + C|w|^{-5/2}||\phi_1 - \phi_2||_{H^2}^2 \\
&\leq C|w|^{-5/2}(||\phi_2||_{H^2} + ||\phi_1||_{H_2}^2)||\phi_2 - \phi_2||_{H^2} \\
&\leq C|w|^{-1/2}||\phi_2 - \phi_2||_{H^2}.
\end{aligned}$$

Then Lemma (4.27) implies that

$$||\mathcal{T}(\phi_1) - \mathcal{T}(\phi_2)||_{H^2} \leq C|w|||\phi_1 - \phi_2||_{H_2}. \tag{5.13}$$

Therefore $\mathcal{T}$ is a contraction mapping in $D(\mathcal{T})$, if $|w|$ is sufficiently small. There is a fixed point φ,we write as $\varphi = \varphi(\cdot, \xi)$. Being in $D(\mathcal{T})$, $||\varphi||_{H^2} = O(|w|^2)$, which is smaller than the radius r_1^2 and r_2^2. Hence $(E_1^{\varphi}, E_2^{\varphi})$ is a perturbed exact core-shell. □

We state a result regarding the linearization of $\mathcal{S}$ at $\varphi(\cdot, \xi)$. Denote this linearized operator by $\tilde{\mathcal{L}}$, i.e.

$$\tilde{\mathcal{L}} = \mathcal{S}^{'}(\varphi). \tag{5.14}$$

We have the following analogy of Lemma (4.27).

5.15 Lemma. *1. There exists $C > 0$ such that for all $u \in \mathcal{X}_*$*

$$||u||_{H^2} \leq C|w|^{3/2}||\Pi\tilde{\mathcal{L}}(u)||_{L^2}.$$

2. There exists $C > 0$ such that for all $u \in \mathcal{X}_*$

$$||u||_{H^1}^2 \leq C|w|^{3/2} < \Pi\tilde{\mathcal{L}}(u), u > .$$

Proof. By Lemma (4.27), Lemma (4.28) and the fact $||\varphi||_{H^2} = O(|w|^2)$,we deduce

$$\begin{aligned} ||\Pi\tilde{\mathcal{L}}(u)||_{L^2} &\geq ||\Pi\mathcal{L}(u)||_{L^2} - ||\Pi(\tilde{\mathcal{L}} - \mathcal{L}(u)||_{L^2} \\ &\geq C|w|^{-3/2}||u||_{H^2} - C|w|^{-5/2}||\varphi||_{H^2}||u||_{H^2} \\ &\geq C|w|^{-3/2}||u||_{H^2} - C|w|^{-1/2}||u||_{H^2} \geq C|w|^{-3/2}||u||_{H^2} \end{aligned}$$

when $|w|$ is small.

Lemma 4.27 and Lemma 4.28 imply that

$$\begin{aligned}
< \Pi\tilde{\mathcal{L}}(u), u > &=< \Pi\mathcal{L}(u), u > - < \Pi(\tilde{\mathcal{L}} - \mathcal{L})(u), u > \\
&\geq C|w|^{-3/2}||u||^2_{H^1} - C|w|^{-5/3}||\varphi||_{H^2}||u||^2_{H^1} \\
&\geq C|w|^{-3/2}||u||^2_{H^1} - C|w|^{1/3}||u||^2_{H^1} \\
&\geq C|w|^{-3/2}||u||^2_{H^1}.
\end{aligned}$$

This proves the second part of the lemma. □

One consequence of this lemma is an estimate of $\frac{\partial\varphi}{\partial\xi_j}$.

5.16 Lemma. *The fixed point φ satisfties*

$$||\frac{\partial\varphi}{\partial\xi_j}||_{H^2} = O(|w|).$$

Proof. We prove this lemma by the Implicit Function Theorem. Differentiating $\Pi\mathcal{S}(\varphi)$ with respect to ξ_j finds that

$$\begin{aligned}
0 &= \frac{\partial\Pi\mathcal{S}_k(\varphi)}{\partial\xi_j} \\
&= \Pi\tilde{\mathcal{L}}_k(\frac{\partial\varphi}{\partial\xi_j}) + \Pi\gamma_{12}\int_{\Omega_2}[\frac{\partial R(\xi + \sqrt{r_k^2 + 2\varphi_k(\theta)}e^{i\theta}, y)}{\partial x_j} + \frac{\partial R(\xi + \sqrt{r_k^2 + 2\varphi_k(\theta)}e^{i\theta}, y)}{\partial y_j}]dy \\
&+ (\gamma_{11} + \gamma_{12} - \gamma_{22})\int_{\Omega_1}[\frac{\partial R(\xi + \sqrt{r_k^2 + 2\varphi_k(\theta)}e^{i\theta}, y)}{\partial x_j} + \frac{\partial R(\xi + \sqrt{r_k^2 + 2\varphi_k(\theta)}e^{i\theta}, y)}{\partial y_j}]dy
\end{aligned}$$

where $R = R(x, y)$.

It is clear that

$$||\int_{\Omega_i}[\frac{\partial R(\xi + \sqrt{r_k^2 + 2\varphi_k(\theta)}e^{i\theta}, y)}{\partial x_j} + \frac{\partial R(\xi + \sqrt{r_k^2 + 2\varphi_k(\theta)}e^{i\theta}, y)}{\partial y_j}]dy||_{L^2} = O(|w|).$$

With the help of Lemma (5.15) we deduce that

$$||\frac{\partial \varphi}{\partial \xi_j}||_{H^2} \leq C|w|^{3/2}\gamma_{st}|w| = C|w|.$$

□

Chapter 6

Existence and stability

From Lemma (5.5) we know that for every $\xi \in U$ there exists $\varphi(\cdot,\xi) \in \mathcal{X}_*$ such that $\Pi\mathcal{S}(\varphi(\cdot,\xi)) = 0$. In this section, we find particular ξ denoted by ζ such that $\mathcal{S}(\varphi(\cdot,\zeta)) = 0$.

6.1 Lemma. $J(E_1^{\varphi(\cdot,\xi)}, E_2^{\varphi(\cdot,\xi)}) = J(B_1, B_2\backslash B_1) + O(|w|^{5/2})$.

Proof. Expanding $J(E_1^{\varphi(\cdot,\xi)}, E_2^{\varphi(\cdot,\xi)})$ yields

(6.2)

$$J(E_1^{\varphi(\cdot,\xi)}, E_2^{\varphi(\cdot,\xi)}) = J(B_1, B_2\backslash B_1) + \sum_{k=1}^{2}\int_0^{2\pi} \mathcal{S}_k(0)\varphi_k d\theta + \frac{1}{2}\sum_{k=1}^{2}\int_0^{2\pi} \mathcal{L}_k(\varphi)\varphi_k d\theta + O(|w|^{7/2}).$$

The error term in (6.2) is obtained from Lemma 4.28 and the fact $||\varphi||_{H^2} = O(|w|^2)$.

On the other hand $\Pi\mathcal{S}(\varphi) = 0$ implies that

$$\Pi(\mathcal{S}(0) + \mathcal{L}(\varphi) + \mathcal{N}(\varphi)) = 0$$

where $\mathcal{N}$ is given in (5.3). We multiply the last equation by φ and integrate to derive, again with the help of Lemma (4.28),

$$\sum_{k=1}^{2}\int_0^{2\pi} \mathcal{S}(0)\varphi_k d\theta + \sum_{k=1}^{2}\int_0^{2\pi} \mathcal{L}_k(\varphi)\varphi_k d\theta = O(|w|^{7/2}).$$

We can now rewrite (6.2) as

$$J(E_1^{\varphi(\cdot,\xi)}, E_2^{\varphi(\cdot,\xi)}) = J(B_1, B_2\backslash B_1) + \frac{1}{2}\sum_{k=1}^{2}\int_0^{2\pi} \mathcal{S}(0)\varphi_k d\theta + O(|w|^{7/2}).$$

Lemma (2.5) and the fact $||\varphi||_{H^2} = O(|w|^2)$ implies that

$$J(E_1^{\varphi(\cdot,\xi)}, E_2^{\varphi(\cdot,\xi)}) = J(B_1, B_2\backslash B_1) + O(|w|^{7/2}) + O(|w|^{5/2}) = J(E_1^{\varphi(\cdot,\xi)}, E_2^{\varphi(\cdot,\xi)}) + O(|w|^{5/2}).$$

When we use Lemma (2.5), note that $\mathcal{S}(0)$ is a sum of a θ independent part, an $e^{i\theta}$ part and a quantity of order $O(|w|^{1/2})$, and that $\varphi_k \perp \{[1,0]^{\perp}, [0,1]^{\perp}\}$. This proves the lemma. □

6.3 Lemma. *As a function of ξ, $J(E_1^{\varphi(\cdot,\xi)}, E_2^{\varphi(\cdot,\xi)})$ is locally minimized at some ζ, when $|w|$ is small. As $|w| \to 0$,*

$$\zeta \to \zeta_0,$$

where $R(\zeta_0, \zeta_0) = min_{x\in D} R(x,x)$.

Proof. If we consider $J(E_1^{\varphi(\cdot,\xi)}, E_2^{\varphi(\cdot,\xi)})$ as a function of ξ, then Lemma 2.5 and Lemma 6.1 imply that

$$\begin{aligned}
J(E_1^{\varphi(\cdot,\xi)}, E_2^{\varphi(\cdot,\xi)}) =& 2\pi(r_1+r_2) \\
&+ \frac{\gamma_{11}}{2}[-\frac{\pi r_1^4 \log r_1}{2} + \frac{\pi r_1^4}{8} + \frac{\pi r_1^6}{4|D|} + \pi_2 r_1^4 R(\xi,\xi)] \\
&+ \gamma_{12}[-\frac{\pi r_1^4}{8} + \pi r_1^2(\frac{r_2^2}{4} - \frac{r_2^2}{2}\log r_2) + \pi r_1^2 r_2^2 R(\xi,\xi) + \frac{\pi^2 r_2^2 r_1^4}{8|D|} \\
&+ \frac{\pi^2 r_1^2 r_2^4}{8|D|} + \frac{\pi r_1^4 \log r_1}{2} - \frac{\pi r_1^4}{8} - \pi_2 r_1^4 R(\xi,\xi) - \frac{\pi^2 r_1^6}{4|D|}] \\
&+ \frac{\gamma_{22}}{2}[-\frac{\pi r_1^4 \log r_1}{2} - \frac{\pi r_2^4 \log r_2}{2} + \pi r_1^2 r_2^2 \log r_2 - \frac{w_2|D| r_1^2}{4} + \frac{w_2^2|D|^2}{8\pi} \\
&+ w_2|D|^2 R(\xi,\xi) + \frac{w_2^2|D|}{4}(r_1^2+r_2^2)] + O(|w|^{5/2}).
\end{aligned}$$

The coefficient in front of $R(\xi,\xi)$ is

$$\frac{\gamma_{22}}{2}(\pi r_2^2 - \pi r_1^2)^2 + \frac{\gamma_{11}}{2}\pi^2 r_1^4 + \gamma_{12}\pi^2 r_1^2 r_2^2 - \gamma_{12}\pi^2 r_1^4 \tag{6.4}$$

which is equal to

$$\pi^2[\frac{(r_1^2-r_2^2)^2\gamma_{22}}{2}+\frac{r_1^4}{2}\gamma_{11}+r_1^2r_2^2\gamma_{12}-r_1^4\gamma_{12}].$$

We can write the above quantity as

$$\frac{\pi^2}{2}[r_1^2,r_2^2-r_1^2]\begin{bmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{12} & \gamma_{22}\end{bmatrix}\begin{bmatrix} r_1^2 \\ r_2^2-r_1^2\end{bmatrix}.$$

Since $R_j=(\frac{w_1|D|}{\pi})^{-1/2}r_j, j=1,2; \Gamma_{ij}=(\frac{\omega_1|D|}{\pi})^{\frac{3}{2}}\gamma_{ij}$,

we can write (6.4), the coefficient in front of $R(\xi,\xi)$ as

$$(\frac{w_1|D|}{\pi})^{-1/2}\frac{\pi^2}{2}[R_1^2,R_2^2-R_1^2]\begin{bmatrix}\Gamma_{11} & \Gamma_{12}\\ \Gamma_{12} & \Gamma_{22}\end{bmatrix}\begin{bmatrix} R_1^2 \\ R_2^2-R_1^2\end{bmatrix}.$$

This coefficient is greater than zero because the matrix $\begin{bmatrix}\Gamma_{11} & \Gamma_{12}\\ \Gamma_{12} & \Gamma_{22}\end{bmatrix}$ is definite positive.

Since $R(\xi,\xi)\to+\infty$ if $\xi\to\partial D$, $\min J(E_1^{\varphi(\cdot,\xi)},E_2^{\varphi(\cdot,\xi)})$ as a function of ξ is obtained inside of D at some ζ when $|w|$ is small. Moreover as $|w|\to 0$, if $\zeta\to\zeta_0$, possibly along a subsequence, ζ_0 must be a minimun of $R(\xi,\xi)$.

□

We prove the existence of a solution in the next lemma. It uses a tricky re-parametrization technique.

6.5 Lemma. *At* $\xi=\zeta$, $\mathcal{S}(\varphi)=\begin{bmatrix}0\\ 0\end{bmatrix}$.

Proof. For $\xi=(\xi_1,\xi_2)$ near ζ we re-parametrize near the boundaries of $(E_1^{\varphi(\cdot,\xi)},E_2^{\varphi(\cdot,\xi)})$. Let ζ be the center of a new polar coordinates, $r_1^2+2\psi_1$, $r_2^2+2\psi_2$ the new inner and outer radii square and η the new angle. A point on the boundaries of $(E_1^{\varphi(\cdot,\xi)},E_2^{\varphi(\cdot,\xi)})$ is described as $\zeta+\sqrt{r_1^2+2\psi_1}e^{i\eta}$ or $\zeta+\sqrt{r_2^2+2\psi_2}e^{i\eta}$. It is related to the old polar coordinates via

$$\zeta+\sqrt{r_k^2+2\psi_k}e^{i\eta}=\xi+\sqrt{r_k^2+2\varphi_k}e^{i\theta} \tag{6.6}$$

for $k = 1, 2$.

In the new coordinates $(E_1^{\varphi(\cdot,\xi)}, E_2^{\varphi(\cdot,\xi)})$ becomes $(E_1^{\psi(\cdot,\xi)}, E_2^{\psi(\cdot,\xi)})$. It is viewed as a perturbation of the exact core-shell centered at ζ with radii r_1, r_2. The perturbation is described by $\psi = (\psi_1, \psi_2)$ which is a pair of functions of η. It also depends on ξ and r.

The main effect of the new coordinates is to "freeze" the center. The center of the new polar system is ζ which is fixed while the center of the old polar system is ξ which varies in U.

We now consider the derivative of $J(E_1^{\varphi(\cdot,\xi)}, E_2^{\varphi(\cdot,\xi)}) = J(E_1^{\psi(\cdot,\xi)}, E_2^{\psi(\cdot,\xi)})$ with respect to ξ. On the one hand, at $\xi = \zeta$,

$$\frac{\partial J(E_1^{\psi(\cdot,\xi)}, E_2^{\psi(\cdot,\xi)})}{\partial \xi_j}|_{\xi=\zeta} = \frac{\partial J(E_1^{\varphi(\cdot,\xi)}, E_2^{\varphi(\cdot,\xi)})}{\partial \xi_j}|_{\xi=\zeta} = 0,\ j = 1, 2, \tag{6.7}$$

since ζ is a minimum.

On the other hand calculations show that

$$\frac{\partial J(E_1^{\psi(\cdot,\xi)}, E_2^{\psi(\cdot,\xi)})}{\partial \xi_j} = \sum_{l=1}^{2} \int_0^{2\pi} \mathcal{S}_l(\psi(\cdot,\xi))(\eta) \frac{\partial \psi_l}{\partial \xi_j} d\eta. \tag{6.8}$$

We emphasize that (6.8) is obtained under the re-parametrized coordinates, in which the dependence of $J(E_1^{\varphi(\cdot,\xi)}, E_2^{\varphi(\cdot,\xi)})$ on ξ is only reflected in the dependence of ψ on ξ. Had we calculated in the original coordinates, ξ would have appeared also in the nonlocal part of J through $R(\xi + \ldots, \xi + \ldots)$. The result would have been very different from (6.8). See the proof of Lemma (5.16) which involves differentiation with respect to ξ in the original coordinates. In the derivation of (6.8) we have used the fact that $\int_0^{2\pi} \psi_l d\eta = 0$ which implies that $\int_0^{2\pi} \frac{\partial \psi_l}{\partial \xi_j} d_\eta = 0$, so that $\int_0^{2\pi} \lambda_l(\psi) \frac{\partial \psi_l}{\partial \xi_j} d_\eta = 0$ where $\lambda_l(\psi)$ is part of

$$\mathcal{S}_l(\psi) = \mathcal{H}_l(\psi) + \mathcal{A}_l(\psi) + \mathcal{B}_l(\psi) + \lambda_l(\psi), \tag{6.9}$$

and we can reach the right side of (6.8).

The expression $\mathcal{S}(\phi)$ is invariant under re-parametrization, i.e.

$$\mathcal{S}(\varphi(\cdot,\xi))(\theta) = \mathcal{S}(\psi(\cdot,\xi))(\eta). \tag{6.10}$$

Now we return to the original coordinate system and integrate with respect to θ in (6.8). Then

$$\frac{\partial J(E_1^{\psi(\cdot,\xi)}, E_2^{\psi(\cdot,\xi)})}{\partial \xi_j} = \sum_{l=1}^{2} \int_0^{2\pi} \mathcal{S}_l(\theta) \frac{\partial \psi_l(\eta(\theta,\xi),\xi)}{\partial \xi_j} \frac{\partial \eta}{\partial \theta} d\theta. \tag{6.11}$$

We recall that ψ and η are defined explicitly as functions of θ and ξ by (6.6). Let us agree that $\psi_k = \psi_k(\eta, \xi)$ is a function of η and ξ. Set $\Psi_k(\theta, \xi) = \psi_k(\eta(\theta, \xi), \xi)$. Implicit differentiation shows that, with the help of Lemmas (5.5) and (5.16),

$$\begin{aligned}
&\begin{bmatrix} \frac{\partial \eta}{\partial \theta} & \frac{\partial \eta}{\partial \xi_1} & \frac{\partial \eta}{\partial \xi_2} \\ \frac{\partial \Psi_k}{\partial \theta} & \frac{\partial \Psi_k}{\partial \xi_1} & \frac{\partial \Psi_k}{\partial \xi_2} \end{bmatrix} = - \begin{bmatrix} \sqrt{r_k^2 + 2\Psi_k} \sin\eta & -\frac{\cos\eta}{\sqrt{r_k^2+2\Psi_k}} \\ -\sqrt{r_k^2 + 2\Psi_k} \cos\eta & -\frac{\sin\eta}{\sqrt{r_k^2+2\Psi_k}} \end{bmatrix}^{-1} \\
&\times \begin{bmatrix} \frac{\cos\theta}{\sqrt{r_k^2+2\varphi_k}} \frac{\partial \varphi_k}{\partial \theta} - \sqrt{r_k^2 + 2\varphi_k} \sin\theta & 1 + \frac{\cos\theta}{\sqrt{r_k^2+2\varphi_k}} \frac{\partial \varphi_k}{\partial \xi_1} & \frac{\cos\theta}{\sqrt{r_k^2+2\varphi_k}} \frac{\partial \varphi_k}{\partial \xi_2} \\ \frac{\sin\theta}{\sqrt{r_k^2+2\varphi_k}} \frac{\partial \varphi_k}{\partial \theta} + \sqrt{r_k^2 + 2\varphi_k} \cos\theta & \frac{\sin\theta}{\sqrt{r_k^2+2\varphi_k}} \frac{\partial \varphi_k}{\partial \xi_1} & 1 + \frac{\sin\theta}{\sqrt{r_k^2+2\varphi_k}} \frac{\partial \varphi_k}{\partial \xi_2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{-\sin\eta}{\sqrt{r_k^2+2\Psi_k}} & \frac{\cos\eta}{\sqrt{r_k^2+2\Psi_k}} \\ \sqrt{r_k^2 + 2\Psi_k} \cos\eta & \sqrt{r_k^2 + 2\Psi_k} \sin\eta \end{bmatrix} \\
&\times \begin{bmatrix} -\sqrt{r_k^2 + 2\varphi_k} \sin\theta + O(|w|^{3/2}) & 1 + O(|w|^{1/2}) & O(|w|^{1/2}) \\ \sqrt{r_k^2 + 2\varphi_k} \cos\theta + O(|w|^{3/2}) & O(|w|^{1/2}) & 1 + O(|w|^{1/2}) \end{bmatrix}.
\end{aligned}$$

At $\xi = \zeta$, $n = \theta$ and $\Psi_k = \varphi_k$ and the above becomes

$$\begin{bmatrix} \frac{\partial \eta}{\partial \theta} & \frac{\partial \eta}{\partial \xi_1} & \frac{\partial \eta}{\partial \xi_2} \\ \frac{\partial \Psi_k}{\partial \theta} & \frac{\partial \Psi_k}{\partial \xi_1} & \frac{\partial \Psi_k}{\partial \xi_2} \end{bmatrix} = \begin{bmatrix} 1 + O(|w|) & -\frac{\sin\theta}{\sqrt{r_k^2+2\varphi_k}} + O(1) & \frac{\cos\theta}{\sqrt{r_k^2+2\varphi_k}} + O(1) \\ O(|w|^2) & \sqrt{r_k^2 + 2\varphi_k} \cos\theta + O(|w|) & \sqrt{r_k^2 + 2\varphi_k} \sin\theta + O(|w|) \end{bmatrix} \tag{6.12}$$

We have found that at $\xi = \zeta$,

$$\frac{\partial \Psi_k}{\partial \xi_1} = r_k \cos\theta + O(|w|), \ \frac{\partial \Psi_k}{\partial \xi_2} = r_k \sin\theta + O(|w|). \tag{6.13}$$

To compute $\frac{\partial \psi_k}{\partial \xi_j}$, we invert $\eta = \eta(\xi, \theta)$ to express $\theta = \Theta(\eta, \xi)$. Then

$$\frac{\partial \psi_k}{\partial \xi_j} = \frac{\partial \Psi_k}{\partial \xi_j} + \frac{\partial \Psi_k}{\partial \theta}\frac{\partial \Theta}{\partial \xi_j}.$$

At $\xi = \zeta$, since

$$\frac{\partial \Psi_k}{\partial \theta} = O(|w|^2),\ \frac{\partial \Theta}{\partial \xi_j} = -\frac{\frac{\partial \eta}{\partial \xi_j}}{\frac{\partial \eta}{\partial \theta}} = O(|w|^{-1/2}), \tag{6.14}$$

we deduce that

$$\frac{\partial \psi_k}{\partial \xi_1} = r_k \cos\theta + O(|w|),\ \frac{\partial \psi_k}{\partial \xi_2} = r_k \sin\theta + O(|w|). \tag{6.15}$$

Following (6.15) and the fact that $\frac{\partial \eta}{\partial \theta} = 1 + O(|w|)$ we find that at $\xi = \zeta$, (6.11) becomes

$$\frac{\partial J(E_1^\psi, E_2^\psi)}{\partial \xi_1} = \sum_{l=1}^{2} \int_0^{2\pi} \mathcal{S}_l(\varphi)(r_l \cos\theta + O(|w|))d\theta, \tag{6.16}$$

$$\frac{\partial J(E_1^\psi, E_2^\psi)}{\partial \xi_2} = \sum_{l=1}^{2} \int_0^{2\pi} \mathcal{S}_l(\varphi)(r_l \sin\theta + O(|w|))d\theta. \tag{6.17}$$

Now we combine (5.1), (6.7), (6.16) and (6.17) to derive that

$$A_1 \int_0^{2\pi} [(rr_1 + r_2)\cos^2\theta + O(|w|)]d\theta + A_2 \int_0^{2\pi} [(rr_1 + r_2)\cos\theta\sin\theta + O(|w|)]d\theta = 0$$
$$A_1 \int_0^{2\pi} [(rr_1 + r_2)\cos\theta\sin\theta + O(|w|)]d\theta + A_2 \int_0^{2\pi} [(rr_1 + r_2)\sin^2\theta + O(|w|)]d\theta = 0.$$

Writing the system in the matrix form

$$\left(\begin{bmatrix} \pi(rr_1 + r_2) & 0 \\ 0 & \pi(rr_1 + r_2) \end{bmatrix} + O(|w|) \right) \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{6.18}$$

since (6.18) is non-singular when $|w|$ is small, we deduce that $A_1 = A_2 = 0$, proving the lemma. □

Chapter 7

Conclusion and discussion

We can now conclude that $(E_1^{\varphi(\cdot,\zeta)}, E_2^{\varphi(\cdot,\zeta)})$ solves $\mathcal{S}(\phi) = 0$ and hence the equation (1.4).

[14] gives an explanation of the absence of Ω_3 in the equation (1.4). One can see that a 3-dimensional matrix is reduced to a 2-dimensional one in [14].

In this dissertation, the condition $\Gamma_{22} > \Gamma_{12}$ is important. In reality, it means that the interaction within the constituent occupying Ω_2 is stronger than the interaction between the constituent occupying Ω_1 and the constituent occupying Ω_2.

The set of this model will also work for a 3-dimensional core-shell case. However, we are not sure if the linearized operator $\Pi\mathcal{L}_1$ remains positive definite or not.

References

[1] Giovanni Alberti, Rustum Choksi, and Felix Otto. Uniform energy distribution for an isoperimetric problem with long-range interactions. *Journal of the American Mathematical Society*, 22(2):569–605, 2009.

[2] Frank S Bates and GH Fredrickson. Block copolymers-designer soft materials. *Phys. Today. February 1999, 32*, 38, 2000.

[3] Rustum Choksi and Xiaofeng Ren. Diblock copolymer/homopolymer blends: Derivation of a density functional theory. *Physica D: Nonlinear Phenomena*, 203(1):100–119, 2005.

[4] Xiaosong Kang and Xiaofeng Ren. Ring pattern solutions of a free boundary problem in diblock copolymer morphology. *Physica D: Nonlinear Phenomena*, 238(6):645–665, 2009.

[5] Xiaosong Kang and Xiaofeng Ren. The pattern of multiple rings from morphogenesis in development. *Journal of nonlinear science*, 20(6):747–779, 2010.

[6] Cyrill B Muratov. Droplet phases in non-local ginzburg-landau models with coulomb repulsion in two dimensions. *Communications in Mathematical Physics*, 299(1):45–87, 2010.

[7] Hatsumi Nakazawa and Takao Ohta. Microphase separation of abc-type triblock copolymers. *Macromolecules*, 26(20):5503–5511, 1993.

[8] Takao Ohta and Kyozi Kawasaki. Equilibrium morphology of block copolymer melts. *Macromolecules*, 19(10):2621–2632, 1986.

[9] Yoshihito Oshita. Singular limit problem for some elliptic systems. *SIAM Journal on Mathematical Analysis*, 38(6):1886–1911, 2007.

[10] Xiaofeng Ren. Shell structure as solution to a free boundary problem from block copolymer morphology. *Discrete Contin. Dyn. Syst*, 24(3):979–1003, 2009.

[11] Xiaofeng Ren and J Wei. Single droplet pattern in the cylindrical phase of diblock copolymer morphology. *Journal of Nonlinear Science*, 17(5):471–503, 2007.

[12] Xiaofeng Ren and Juncheng Wei. On the multiplicity of solutions of two nonlocal variational problems. *SIAM Journal on Mathematical Analysis*, 31(4):909–924, 2000.

[13] Xiaofeng Ren and Juncheng Wei. Concentrically layered energy equilibria of the diblock copolymer problem. *European Journal of Applied Mathematics*, 13(5):479–496, 2002.

[14] Xiaofeng Ren and Juncheng Wei. Triblock copolymer theory: Free energy, disordered phase and weak segregation. *Physica D: Nonlinear Phenomena*, 178(1):103–117, 2003.

[15] Xiaofeng Ren and Juncheng Wei. Triblock copolymer theory: Ordered abc lamellar phase. *Journal of Nonlinear Science*, 13(2):175–208, 2003.

[16] Xiaofeng Ren and Juncheng Wei. Existence and stability of spherically layered solutions of the diblock copolymer equation. *SIAM Journal on Applied Mathematics*, 66(3):1080–1099, 2006.

[17] Xiaofeng Ren and Juncheng Wei. Many droplet pattern in the cylindrical phase of diblock copolymer morphology. *Reviews in Mathematical Physics*, 19(08):879–921, 2007.

[18] Xiaofeng Ren and Juncheng Wei. Spherical solutions to a nonlocal free boundary problem from diblock copolymer morphology. *SIAM Journal on Mathematical Analysis*, 39(5):1497–1535, 2008.

[19] Xiaofeng Ren and Juncheng Wei. Oval shaped droplet solutions in the saturation process of some pattern formation problems. *SIAM Journal on Applied Mathematics*, 70(4):1120–1138, 2009.

[20] Xiaofeng Ren and Juncheng Wei. A double bubble in a ternary system with inhibitory long range interaction. *Archive for Rational Mechanics and Analysis*, 208(1):201–253, 2013.

[21] Peter Sternberg and Ihsan Topaloglu. On the global minimizers of a nonlocal isoperimetric problem in two dimensions. *Interfaces Free Bound*, 13:155–169, 2011.

[22] Yves van Gennip and Mark A Peletier. Stability of monolayers and bilayers in a copolymer-homopolymer blend model. *arXiv preprint arXiv:0710.3298*, 2007.

[23] Yves van Gennip and Mark A Peletier. Copolymer–homopolymer blends: global energy minimisation and global energy bounds. *Calculus of Variations and Partial Differential Equations*, 33(1):75–111, 2008.

[24] K. Yoshida. *Functional Analysis*. Classics in mathematics / Springer. Springer Berlin Heidelberg, 1980.

Appendix A

Derivation of the Euler Lagrange Equation

We are going to compute the Euler Lagrange equatuon of (2.2). The computation contains a local part and a nonlocal part.

The computation of the local part is the following.

Arclength of the inner circle can be written as

$$J_{local} = \int_0^{2\pi} \sqrt{r_1^2 + 2\phi_1(\theta) + \frac{(\phi_1'(\theta))^2}{r_1^2 + 2\phi_1(\theta)}} d\theta.$$

In this computation, I will write ϕ instead of $\phi_1(\theta)$ and r instead of r_1 for easier notation. $J_{local}(\phi + \epsilon\psi) = \int_0^{2\pi} \sqrt{r^2 + 2(\phi + \epsilon\psi) + \frac{(\phi' + \epsilon\psi')^2}{r^2 + 2(\phi + \epsilon\psi)}} d\theta$.

$$\begin{aligned}
\frac{dJ_{local}(\phi + \epsilon\psi)}{d\epsilon}|_{\epsilon=0} &= \int_0^{2\pi} \frac{\psi + \frac{\psi'\phi'(r^2+2\phi) - (\phi')^2\psi}{(r^2+2\phi)^2}}{\sqrt{r^2 + 2\phi + \frac{(\phi')^2}{r^2+2\phi}}} d\theta \\
&= \int_0^{2\pi} \left(\frac{\psi(1 - \frac{(\phi')^2}{(r^2+2\phi)^2})}{\sqrt{r^2 + 2\phi + \frac{(\phi')^2}{r^2+2\phi}}} + \frac{\psi' \frac{\phi'}{r^2+2\phi}}{\sqrt{r^2 + 2\phi + \frac{(\phi')^2}{r^2+2\phi}}} \right) d\theta.
\end{aligned}$$

The second term of the preceding line can be computed using integration by parts.

It can be written as

$$-\int_0^{2\pi} \psi \frac{\phi''(r^2+2\phi)^2 - 3(\phi')^2(r^2+2\phi) - \frac{(\phi')^4}{r^2+2\phi}}{(r^2+2\phi)^2(r^2+2\phi+\frac{(\phi')^2}{r^2+2\phi})^{3/2}} d\theta.$$

Hence the variation of the arclength becomes

$$\begin{aligned}
&\frac{(1-\frac{(\phi')^2}{(r^2+2\phi)^2})}{\sqrt{r^2+2\phi+\frac{(\phi')^2}{r^2+2\phi}}} - \frac{\phi''(r^2+2\phi)^2 - 3(\phi')^2(r^2+2\phi) - \frac{(\phi')^4}{r^2+2\phi}}{(r^2+2\phi)^2(r^2+2\phi+\frac{(\phi')^2}{r^2+2\phi})^{3/2}} \\
&= \frac{(1-\frac{(\phi')^2}{(r^2+2\phi)^2})(r^2+2\phi+\frac{(\phi')^2}{r^2+2\phi}) - \phi'' + \frac{3(\phi')^2}{r^2+2\phi} + \frac{(\phi')^4}{(r^2+2\phi)^3}}{(r^2+2\phi+\frac{(\phi')^2}{r^2+2\phi})^{3/2}} \\
&= \frac{r^2+2\phi+\frac{3(\phi')^2}{r^2+2\phi} - \phi''}{(r^2+2\phi+\frac{(\phi')^2}{r^2+2\phi})^{3/2}}
\end{aligned}$$

which is exactly the curvature of the inner circle.

Computation of directional derivative of the nonlocal part of the energy:

$$\begin{aligned}
J_{nonlocal} =& \frac{\gamma_{11}}{2}\int_D |\ (-\Delta)^{-\frac{1}{2}}(\chi_{\Omega_1} - w_1)\ |^2\ dx \\
&+\gamma_{12}\int_D (-\Delta)^{-\frac{1}{2}}(\chi_{\Omega_1} - w_1)(-\Delta)^{-\frac{1}{2}}(\chi_{\Omega_2} - w_2)dx \\
&+\frac{\gamma_{22}}{2}\int_D |\ (-\Delta)^{-\frac{1}{2}}(\chi_{\Omega_2} - w_2)\ |^2\ dx \\
=& J_1 + J_2 + J_3.
\end{aligned}$$

Computation of directional variation of J_2:

$$
\begin{aligned}
J_2 =&\gamma_{12}\int_D (-\Delta)^{(-\frac{1}{2})}(\chi_{\Omega_1}-w_1)(-\Delta)^{(-\frac{1}{2})}(\chi_{\Omega_2}-w_2)dx\\
=&\gamma_{12}\int_D (-\Delta)^{-1}(\chi_{\Omega_1}-w_1)(\chi_{\Omega_2}-w_2)dx\\
=&\gamma_{12}\int_D u(\chi_{\Omega_2}-w_2)dx \text{ where } u=\int_D G(\chi_{\Omega_1}-w_1)dy\\
=&\gamma_{12}\int_D\int_D G(\chi_{\omega_1}-w_1)dy(\chi_{\Omega_2}-w_2)dx\\
=&\gamma_{12}\int_{\Omega_2}\int_{\Omega_1} G(x,y)dydx\\
=&\gamma_{12}\int_0^{2\pi}d\theta\int_0^{\sqrt{r_1^2+2\phi_1(\theta)}}dr\int_0^{2\pi}d\omega\int_{\sqrt{r_1^2+2\phi_1(\omega)}}^{\sqrt{r_2^2+2\phi_2(\omega)}}G(\xi+re^{i\theta},\xi+te^{iw})rtdt.
\end{aligned}
$$

Along h_1

$$
\begin{aligned}
&J_2(\phi_1+\epsilon h_1)\\
=&\gamma_{12}\int_0^{2\pi}d\theta\int_0^{\sqrt{r_1^2+2\phi_1(\theta)+\epsilon h_1(\theta)}}dr\int_0^{2\pi}d\omega\int_{\sqrt{r_1^2+2\phi_1(\omega)+\epsilon h_1(\omega)}}^{\sqrt{r_2^2+2\phi_2(\omega)}}G(\xi+re^{i\theta},\xi+te^{iw})rtdt
\end{aligned}
$$

$$
\begin{aligned}
&\frac{J_2(\phi_1+\epsilon h_1)}{d\epsilon}\\
=&\gamma_{12}\int_0^{2\pi}d\theta\int_0^{2\pi}d\omega\int_{\sqrt{r_1^2+2\phi_1(\omega)+\epsilon h_1(\omega)}}^{\sqrt{r_2^2+2\phi_2(\omega)}}G(\xi+\sqrt{r_1^2+2\phi_1(\theta)+\epsilon h_1(\theta)}e^{i\theta},\xi+te^{iw})\cdot th_1(\theta)\\
-&\gamma_{12}\int_0^{2\pi}d\theta\int_0^{\sqrt{r_1^2+2\phi_1(\theta)+\epsilon h_1(\theta)}}dr\int_0^{2\pi}d\omega\\
&G(\xi+re^{i\theta},\xi+\sqrt{r_1^2+2\phi_1(\omega)+\epsilon h_1(\omega)})e^{i\omega}\cdot rh_1(\omega)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{J_2(\phi_1+\epsilon h_1)}{d\epsilon}|_{\epsilon=0}\\
=&\gamma_{12}\int_0^{2\pi}d\theta\int_0^{2\pi}d\omega\int_{\sqrt{r_1^2+2\phi_1(\omega)}}^{\sqrt{r_2^2+2\phi_2(\omega)}}G(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},\xi+te^{iw})\cdot th_1(\theta)\\
-&\gamma_{12}\int_0^{2\pi}d\theta\int_0^{\sqrt{r_1^2+2\phi_1(\theta)}}dr\int_0^{2\pi}d\omega G(\xi+re^{i\theta},\xi+\sqrt{r_1^2+2\phi_1(\omega)}e^{i\omega})\cdot rh_1(\omega)\\
=&\gamma_{12}\int_0^{2\pi}\int_{\Omega_2}G(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},y)h_1(\theta)dyd\theta\\
-&\gamma_{12}\int_0^{2\pi}\int_{\Omega_1}G(\xi+\sqrt{r_1^2+2\phi_1(\omega)}e^{i\omega},y)h_1(\omega)dyd\omega.
\end{aligned}
$$

Along h_2

$$
\begin{aligned}
&J_2(\phi_2+\epsilon h_2)\\
&=\gamma_{12}\int_0^{2\pi}d\theta\int_0^{\sqrt{r_1^2+2\phi_1(\theta)}}dr\int_0^{2\pi}d\omega\int_{\sqrt{r_1^2+2\phi_1(\omega)}}^{\sqrt{r_2^2+2(\phi_2(\omega)+\epsilon h_2(\omega))}}dtG(\xi+re^{i\theta},\xi+te^{iw})rt.
\end{aligned}
$$

$$
\begin{aligned}
\frac{J_2(\phi_2+\epsilon h_2)}{d\epsilon}=&\gamma_{12}\int_0^{2\pi}d\theta\int_0^{\sqrt{r_1^2+2\phi_1(\theta)}}dr\int_0^{2\pi}d\omega\\
&G(\xi+re^{i\theta},\xi+\sqrt{r_2^2+2(\phi_2(\omega)+\epsilon h_2(\omega))}e^{iw})r\cdot h_2(\omega)
\end{aligned}
$$

$$
\begin{aligned}
\frac{J_2(\phi_2+\epsilon h_2)}{d\epsilon}|_{\epsilon=0}=&\gamma_{12}\int_0^{2\pi}d\theta\int_0^{\sqrt{r_1^2+2\phi_1(\theta)}}dr\int_0^{2\pi}d\omega\\
&G(\xi+re^{i\theta},\xi+\sqrt{r_2^2+2\phi_2(\omega)}e^{iw})r\cdot h_2(\omega)\\
=&\gamma_{12}\int_0^{2\pi}\int_{\Omega_1}G(\xi+\sqrt{r_2^2+2\phi_2(\omega)}e^{i\omega},y)h_2(\omega)dyd\omega.
\end{aligned}
$$

For J_1:

$$\begin{aligned}
J_1 =& \frac{\gamma_{11}}{2} \int_D \mid (-\Delta)^{-\frac{1}{2}} (\chi_{\Omega_1} - w_1) \mid^2 dx \\
=& \frac{\gamma_{11}}{2} \int_D (-\Delta)^{-1} (\chi_{\Omega_1} - w_1)(\chi_{\Omega_1} - w_1) dx \\
=& \frac{\gamma_{11}}{2} \int_D u(\chi_{\Omega_1} - w_1) dx \\
=& \frac{\gamma_{11}}{2} \int_{\Omega_1} \int G(\chi_{\Omega_1} - w_1) dy (\chi_{\Omega_1} - w_1) dx \\
=& \frac{\gamma_{11}}{2} \int_{\Omega_1} \int_{\Omega_1} G(x,y) dx dy \\
=& \frac{\gamma_{11}}{2} \int_0^{2\pi} d\theta \int_0^{\sqrt{r_1^2 + 2\phi_1(\theta)}} dr \int_0^{2\pi} d\omega \int_0^{\sqrt{r_1^2 + 2\phi_1(\omega)}} dt G(\xi + re^{i\theta}, \xi + te^{i\omega}) rt.
\end{aligned}$$

Along h_1

$$\begin{aligned}
J_1(\phi_1 + \epsilon h_1) =& \frac{\gamma_{11}}{2} \int_0^{2\pi} d\theta \int_0^{\sqrt{r_1^2 + 2(\phi_1(\theta) + \epsilon h_1(\theta))}} dr \int_0^{2\pi} d\omega \int_0^{\sqrt{r_1^2 + 2(\phi_1(\omega) + \epsilon h_1(\omega))}} dt \\
& G(\xi + re^{i\theta}, \xi + te^{i\omega}) rt
\end{aligned}$$

$$\begin{aligned}
\frac{J_1(\phi_1 + \epsilon h_1)}{\epsilon} =& \frac{\gamma_{11}}{2} \int_0^{2\pi} d\theta \int_0^{\sqrt{r_1^2 + 2(\phi_1(\theta) + \epsilon h_1(\theta))}} dr \int_0^{2\pi} d\omega \\
& G(\xi + re^{i\theta}, \xi + \sqrt{r_1^2 + 2(\phi_1(\omega) + \epsilon h_1(\omega))} e^{i\omega}) r \cdot h_1(\omega) \\
& + \frac{\gamma_{11}}{2} \int_0^{2\pi} d\theta \int_0^{\sqrt{r_1^2 + 2(\phi_1(\omega) + \epsilon h_1(\omega))}} dt \int_0^{2\pi} d\omega \\
& G(\xi + \sqrt{r_1^2 + 2(\phi_1(\theta) + \epsilon h_1(\theta))} e^{i\theta}, \xi + te^{i\omega}) t \cdot h_1(\theta)
\end{aligned}$$

$$
\begin{aligned}
&\frac{J_1(\phi_1+\epsilon h_1)}{\epsilon}|_{\epsilon=0}= \\
&\frac{\gamma_{11}}{2}\int_0^{2\pi}d\theta\int_0^{\sqrt{r_1^2+2\phi_1(\theta)}}dr\int_0^{2\pi}d\omega G(\xi+re^{i\theta},\xi+\sqrt{r_1^2+2\phi_1(\omega)}e^{i\omega})r\cdot h_1(\omega)\\
+&\frac{\gamma_{11}}{2}\int_0^{2\pi}d\theta\int_0^{\sqrt{r_1^2+2\phi_1(\omega)}}dt\int_0^{2\pi}d\omega G(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},\xi+te^{i\omega})t\cdot h_1(\theta)\\
=&\frac{\gamma_{11}}{2}\int_0^{2\pi}\int_{\Omega_1}G(\xi+\sqrt{r_1^2+2\phi_1(\omega)}e^{i\omega},y)dyh_1(\omega)d\omega\\
+&\frac{\gamma_{11}}{2}\int_0^{2\pi}\int_{\Omega_1}G(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},y)dyh_1(\theta)d\theta\\
=&\gamma_{11}\int_0^{2\pi}\int_{\Omega_1}G(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},y)dyh_1(\theta)d\theta.
\end{aligned}
$$

Now for J_3

$$
\begin{aligned}
J_3=&\frac{\gamma_{22}}{2}\int_D|\ (-\Delta)^{-\frac{1}{2}}(\chi_{\Omega_2}-w_2)\ |^2\ dx\\
=&\frac{\gamma_{22}}{2}\int_{\Omega_2}\int_{\Omega_2}G(x,y)dxdy\\
=&\frac{\gamma_{22}}{2}\int_0^{2\pi}d\theta\int_{\sqrt{r_1^2+2\phi_1(\theta)}}^{\sqrt{r_2^2+2\phi_2(\theta)}}dr\int_0^{2\pi}d\omega\int_{\sqrt{r_1^2+2\phi_1(\omega)}}^{\sqrt{r_2^2+2\phi_2(\omega)}}dtG(\xi+re^{i\theta},\xi+te^{i\omega})rt.
\end{aligned}
$$

Along h_2

$$
\begin{aligned}
&J_3(\phi_2+\epsilon h_2)\\
=&\frac{\gamma_{22}}{2}\int_0^{2\pi}d\theta\int_{\sqrt{r_1^2+2\phi_1(\theta)}}^{\sqrt{r_2^2+2(\phi_2(\theta)+\epsilon h_2(\theta))}}dr\int_0^{2\pi}d\omega\int_{\sqrt{r_1^2+2\phi_1(\omega)}}^{\sqrt{r_2^2+2(\phi_2(\omega)+\epsilon h_2(\omega))}}dtG(\xi+re^{i\theta},\xi+te^{i\omega})rt
\end{aligned}
$$

$$
\begin{aligned}
&\frac{dJ_3(\phi_2+\epsilon h_2)}{d\epsilon}|_{\epsilon=0}\\
=&\gamma_{22}\int_0^{2\pi}d\theta\int_{\sqrt{r_1^2+2\phi_1(\theta)}}^{\sqrt{r_2^2+2\phi_2(\theta)}}dr\int_0^{2\pi}d\omega G(\xi+re^{i\theta},\xi+\sqrt{r_2^2+2\phi_2(\omega)}e^{i\omega})rh_2(\omega)\\
=&\gamma_{22}\int_0^{2\pi}\int_{\Omega_2}G(\xi+\sqrt{r_2^2+2\phi_2(\omega)}e^{i\omega},y)h_2(\omega)d\omega.
\end{aligned}
$$

Similarly,

$$\frac{dJ_3(\phi_1+\epsilon h_1)}{d\epsilon}|_{\epsilon=0} = -\gamma_{22}\int_0^{2\pi}\int_{\Omega_2} G(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},y)h_1(\theta)d\theta.$$

Putting all directional derivatives together, the nonlocal part of J gives us fours terms about h_1 and two terms about h_2.

$$\begin{aligned}
&\gamma_{12}\int_0^{2\pi}\int_{\Omega_2} G(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},y)h_1(\theta)dyd\theta\\
&-\gamma_{12}\int_0^{2\pi}\int_{\Omega_1} G(\xi+\sqrt{r_1^2+2\phi_1(\omega)}e^{i\omega},y)h_1(\omega)dyd\omega\\
&+\gamma_{11}\int_0^{2\pi}\int_{\Omega_1} G(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},y)dyh_1(\theta)d\theta\\
&-\gamma_{22}\int_0^{2\pi}\int_{\Omega_2} G(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta},y)h_1(\theta)d\theta
\end{aligned}$$

gives

$$\begin{aligned}
&\gamma_{12}(-\Delta)^{-1}(\chi_{\Omega_2}-w_2)(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta})\\
&-\gamma_{12}(-\Delta)^{-1}(\chi_{\Omega_1}-w_1)(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta})\\
&\gamma_{11}(-\Delta)^{-1}(\chi_{\Omega_1}-w_1)(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta})\\
&-\gamma_{22}(-\Delta)^{-1}(\chi_{\Omega_2}-w_2)(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta})\\
&=(\gamma_{12}-\gamma_{22})(-\Delta)^{-1}(\chi_{\Omega_2}-w_2)(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta})\\
&+(\gamma_{11}-\gamma_{12})(-\Delta)^{-1}(\chi_{\Omega_1}-w_1)(\xi+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta}).
\end{aligned}$$

$$\begin{aligned}
&\gamma_{12}\int_0^{2\pi}\int_{\Omega_1} G(\xi+\sqrt{r_2^2+2\phi_2(\omega)}e^{i\omega},y)h_2(\omega)dyd\omega\\
&\gamma_{22}\int_0^{2\pi}\int_{\Omega_2} G(\xi+\sqrt{r_2^2+2\phi_2(\omega)}e^{i\omega},y)h_2(\omega)d\omega
\end{aligned}$$

gives

$$\begin{aligned}
&\gamma_{12}(-\Delta)^{-1}(\chi_{\Omega_1}-w_1)(\xi+\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta})\\
&\gamma_{22}(-\Delta)^{-1}(\chi_{\Omega_2}-w_2)(\xi+\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta}).
\end{aligned}$$

Therefore Euler-Lagrange equation of (1) becomes

$$\mathcal{H}_1(\phi_1)(\theta) + (\gamma_{12} - \gamma_{22})(-\Delta)^{-1}(\chi_{\Omega_2} - w_2)(\xi + \sqrt{r_1^2 + 2\phi_1(\theta)}e^{i\theta}) \\ +(\gamma_{11} - \gamma_{12})(-\Delta)^{-1}(\chi_{\Omega_1} - w_1)(\xi + \sqrt{r_1^2 + 2\phi_1(\theta)}e^{i\theta}) = \lambda_1$$

$$\mathcal{H}_2(\phi_2)(\theta) + \gamma_{12}(-\Delta)^{-1}(\chi_{\Omega_1} - w_1)(\xi + \sqrt{r_2^2 + 2\phi_2(\theta)}e^{i\theta}) \\ +\gamma_{22}(-\Delta)^{-1}(\chi_{\Omega_2} - w_2)(\xi + \sqrt{r_2^2 + 2\phi_2(\theta)}e^{i\theta}) = \lambda_2$$

where $\mathcal{H}_1$ and $\mathcal{H}_2$ are the variation of the arclength.

Index

binary system, 1

characteristic function, 1

contraction mapping, 41

core-shell pattern, 5

fixed point, 38

parametrization, 46

perimeter, 6

ternary system, 4